Available at
www.ComputerScienceWeb.com
POWERED BY SCIENCE @ DIRECT®

ELSEVIER

Information Processing Letters 86 (2003) 271–275

Information Processing Letters

www.elsevier.com/locate/ipl

Ring embedding in faulty pancake graphs

Chun-Nan Hung [a,*], Hong-Chun Hsu [b], Kao-Yung Liang [a], Lih-Hsing Hsu [b]

[a] *Department of Computer Science and Information Engineering, Da-Yet University, Changhua 51505, Taiwan, R.O.C.*
[b] *Department of Computer and Information Science, National Chiao Tung University, Hsinchu 300, Taiwan, R.O.C.*

Received 17 July 2002; received in revised form 29 November 2002

Communicated by M. Yamashita

Abstract

In this paper, we consider the fault hamiltonicity and the fault hamiltonian connectivity of the pancake graph P_n. Assume that $F \subseteq V(P_n) \cup E(P_n)$. For $n \geqslant 4$, we prove that $P_n - F$ is hamiltonian if $|F| \leqslant (n-3)$ and $P_n - F$ is hamiltonian connected if $|F| \leqslant (n-4)$. Moreover, all the bounds are optimal.

Keywords: Fault tolerance; Hamiltonian; Hamiltonian connected; Pancake graphs

1. Introduction

Network topology is a crucial factor for the interconnection networks since it determines the performance of the networks. Many interconnection network topologies have been proposed in the literature for the purpose of connecting hundreds or thousands of processing elements. Network topology is always represented by a graph where nodes represent processors and edges represent links between processors.

For the graph definition and notation we follow [2]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. For any vertex x of V, $\deg_G(x)$ denotes its degree in G. We use $\delta(G)$ to denote $\min\{\deg_G(x) \mid x \in V(G)\}$. Two vertices u and v are *adjacent* if $(u, v) \in E$. A *path* is represented by $\langle v_0, v_1, v_2, \ldots, v_k \rangle$. The *length* of a path Q is the number of edges in Q. We also write the path $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ as $\langle v_0, Q_1, v_i, v_{i+1}, \ldots, v_j, Q_2, v_t, \ldots, v_k \rangle$, where Q_1 is the path $\langle v_0, v_1, \ldots, v_i \rangle$ and Q_2 is the path $\langle v_j, v_{j+1}, \ldots, v_t \rangle$. Hence, it is possible to write a path as $\langle v_0, v_1, Q, v_1, v_2, \ldots, v_k \rangle$ if the length of Q is 0. Sometimes, a path is also represented by $\langle v_0, v_1, \ldots, v_i, e, v_{i+1}, \ldots, v_n \rangle$ to emphasize that e is the edge (v_i, v_{i+1}). We use $d(u, v)$ to denote the distance between u and v, i.e., the length of the shortest path joining u and v. A path is a *hamiltonian path* if its vertices span V. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last vertex.

Throughout this paper, we assume that n is a positive integer. We use $\langle n \rangle$ to denote the set $\{1, 2, \ldots, n\}$. The *n-dimensional pancake graph*, denoted by P_n, is a graph with the vertex set

* Corresponding author. Address: 112 Shan-Jiau Rd, Da-Tsuen, Changhua 51505, Taiwan, R.O.C.
E-mail address: spring@mail.dyu.edu.tw (C.-N. Hung).

0020-0190/03/$ – see front matter
doi:10.1016/S0020-0190(02)00510-0

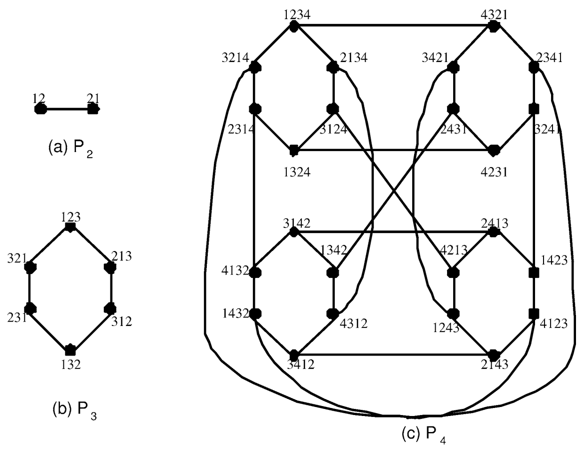

Fig. 1. The pancake graphs P_2, P_3, and P_4.

$$V(P_n) = \{u_1u_2\ldots u_n \mid u_i \in \langle n\rangle \text{ and } u_i \neq u_j \text{ for } i \neq j\}.$$

The adjacency is defined as follows: $u_1u_2\ldots u_i\ldots u_n$ is adjacency to $v_1v_2\ldots v_i\ldots v_n$ through an edge of dimension i with $2 \leqslant i \leqslant n$ if $v_j = u_{i-j+1}$ if $1 \leqslant j \leqslant i$ and $v_j = u_j$ if $i < j \leqslant n$. The pancake graphs P_2, P_3, and P_4 are illustrated in Fig. 1. By definition, P_n is an $(n-1)$-regular graph with $n!$ vertices. Moreover, it is vertex transitive.

The pancake graphs are an important family of interconnection networks proposed by Akers and Krishnameurthy [1]. Some interesting properties of pancake graphs are studied [3,10,4–7]. In particular, Gates and Papadimitriou [6] studied the diameter of the pancake graphs. Until now, we do not know the exact value of the diameter of the pancake graphs [7]. Kanevsky and Feng [10] proved that all cycles of length l where $6 \leqslant l \leqslant n! - 2$ and $l = n!$ can be embedded in the pancake graph P_n with $n \geqslant 4$.

In this paper, we consider two important properties of the pancake graphs, fault hamiltoniancity and fault hamiltonian connectivity. These two parameters for interconnection networks are proposed by Huang et al. [8,9].

A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle. A hamiltonian graph G is *k-fault hamiltonian* if $G - F$ remains hamiltonian for every $F \subset V(G) \cup E(G)$ with $|F| \leqslant k$. The *fault hamiltonicity*, $\mathcal{H}_f(G)$, is defined to be the maximum integer k such that G is k-fault hamiltonian if G is hamiltonian and undefined if otherwise. Clearly, $\mathcal{H}_f(G) \leqslant \delta(G) - 2$ if $\mathcal{H}_f(G)$ is defined. In this paper, we prove that $\mathcal{H}_f(P_n) = n - 3$ if $n \geqslant 4$. Since $\delta(P_n) = n - 1$, the fault hamiltonicity of the pancake graph P_n is optimal if $n \geqslant 4$. In particular, the fact that $P_n - F$ is hamiltonian when F consists of only a single vertex implies the existence of a cycle of length $n! - 1$. As a simple consequence, we improve the result in [10].

To discuss the fault hamiltonicity of the pancake graphs, we need the concept of fault hamiltonian connectivity. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. All hamiltonian connected graphs except K_1 and K_2 are hamiltonian. A graph G is *k-fault hamiltonian connected* if $G - F$ remains hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \leqslant k$. The *fault hamiltonian connectivity*, $\mathcal{H}_f^\kappa(G)$, is defined to be the maximum integer k such that G is k-fault hamiltonian connected if G is hamiltonian connected and undefined if otherwise. It can be checked that $\mathcal{H}_f^\kappa(G) \leqslant \delta(G) - 3$ if $\mathcal{H}_f^\kappa(G)$ is defined and $|V(G)| \geqslant 4$. In this paper, we prove that $\mathcal{H}_f^\kappa(P_n) = n - 4$ if $n \geqslant 4$. Again, the fault hamiltonian connectivity of the pancake graph P_n is optimal if $n \geqslant 4$.

2. Some properties of the pancake graph P_n

Let $\boldsymbol{u} = u_1u_2\ldots u_n$ be any vertex of the pancake graph P_n. We say that u_i is the ith coordinate of $\boldsymbol{u}$, denoted by $(\boldsymbol{u})_i$, for $1 \leqslant i \leqslant n$. By the definition of P_n, there is exactly one neighbor $\boldsymbol{v}$ of $\boldsymbol{u}$ such that $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent through an i-dimensional edge with $2 \leqslant i \leqslant n$. For this reason, we use $i(\boldsymbol{u})$ to denote the unique i-neighbor of $\boldsymbol{u}$. Obviously, $i(i(\boldsymbol{u})) = \boldsymbol{u}$. For $1 \leqslant i \leqslant n$, let $P_n(i)$ denote the subgraph of P_n induced by those vertices $\boldsymbol{u}$ with $(\boldsymbol{u})_n = i$. Obviously, P_n can be decomposed into n subgraph $P_n(i)$, $1 \leqslant i \leqslant n$, such that each $P_n(i)$ is isomorphic to P_{n-1}. Thus, the pancake graph can be constructed recursively. Let $I \subseteq \langle n\rangle$. We use $P_n(I)$ to denote the subgraph of P_n induced by $\bigcup_{i\in I} V(P_n(i))$. For $1 \leqslant i \neq j \leqslant n$, we use $E^{i,j}$ to denote the set of edges between $P_n(i)$ and $P_n(j)$. Obviously, we have the following lemmas.

Lemma 1. $|E^{i,j}| = (n-2)!$ *for any* $1 \leqslant i \neq j \leqslant n$.

Lemma 2. *Let* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *be two distinct vertices of* P_n *with* $(\boldsymbol{u})_n = (\boldsymbol{v})_n$ *such that* $d(\boldsymbol{u}, \boldsymbol{v}) \leqslant 2$. *Then* $(n(\boldsymbol{u}))_n \neq (n(\boldsymbol{v}))_n$.

Let $F \subset V(P_n) \cup E(P_n)$ be any faulty set of P_n. An edge $(\boldsymbol{u}, \boldsymbol{v})$ is F*-fault* if $(\boldsymbol{u}, \boldsymbol{v}) \in F$, $\boldsymbol{u} \in F$, or $\boldsymbol{v} \in F$; and $(\boldsymbol{u}, \boldsymbol{v})$ is F*-fault free* if $(\boldsymbol{u}, \boldsymbol{v})$ is not F-fault. Let $H = (V', E')$ be a subgraph of P_n. We use $F(H)$ to denote the set $(V' \cup E') \cap F$.

Lemma 3. *Assume that* $n \geqslant 5$ *and* $I = \{i_1, i_2, \ldots, i_m\}$ *is a subset of* $\langle n \rangle$ *such that* $|I| = m \geqslant 2$. *Let* $F \subset V(P_n) \cup E(P_n)$ *be any faulty set such that* $P_n(i) - F$ *is hamiltonian connected for any* $i \in I$ *and there are at least three* F*-fault free edges in* $E^{i_j, i_{j+1}}$ *for any* $1 \leqslant j < m$. *Then there exists a hamiltonian path of* $P_n(I) - F$ *joining any two vertices* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *with* $\boldsymbol{u} \in V(P_n(i_1)) - F$ *and* $\boldsymbol{v} \in V(P_n(i_m)) - F$.

Proof. Let $\boldsymbol{u}^1 = \boldsymbol{u}$ and $\boldsymbol{v}^m = \boldsymbol{v}$. Since there are at least three F-fault free edges in $E^{i_j, i_{j+1}}$ for any $1 \leqslant j < m$, we can easily choose two different vertices $\boldsymbol{u}^{i_j}$ and $\boldsymbol{v}^{i_{j+1}}$ in $P_n(i_j)$ such that $(\boldsymbol{v}^{i_j}, \boldsymbol{u}^{i_{j+1}})$ is F-fault free. Obviously, $\boldsymbol{u}^{i_j} \neq \boldsymbol{v}^{i_j}$. Since $P_n(i_j) - F$ is hamiltonian connected for all $i_j \in I$, there is a hamiltonian path Q_j of $P_n(i_j)$ joining $\boldsymbol{u}^{i_j}$ and $\boldsymbol{v}^{i_j}$. Thus, $\langle \boldsymbol{u}^{i_1}, Q_1, \boldsymbol{v}^{i_1}, \boldsymbol{u}^{i_2}, Q_2, \ldots, \boldsymbol{v}^{i_{m-1}}, \boldsymbol{u}^{i_m}, Q_m, \boldsymbol{v}^{i_m} \rangle$ forms a hamiltonian path of $P_n(I) - F$ joining $\boldsymbol{u}$ and $\boldsymbol{v}$. The lemma is proved. □

3. Fault hamiltonicity and fault hamiltonian connectivity of the pancake graphs

Lemma 4. P_4 *is* 1*-fault hamiltonian and hamiltonian connected.*

Proof. To prove P_4 is 1-fault hamiltonian we need to prove $P_4 - F$ is hamiltonian for any $F = \{f\}$ with $f \in V(P_4) \cup E(P_4)$. Without loss of generality, we may assume that $f = 1234$ if f is a vertex, or $f \in \{(1234, 2134), (1234, 3214), (1234, 4321)\}$ if f is an edge. The corresponding hamiltonian cycles of $P_4 - F$ are listed in Table 1.

To prove P_4 is hamiltonian connected, we have to find the hamiltonian path joining any two vertices u and v. By the symmetric property of P_4, we may assume that $\boldsymbol{u} = 1234$ and $\boldsymbol{v}$ is any vertex in $V(P_4) - \{\boldsymbol{u}\}$. The corresponding hamiltonian paths are listed in Table 2. Thus, the lemma is proved. □

Lemma 5. *Suppose that* $n \geqslant 5$. *If* P_{n-1} *is* $(n-4)$*-fault hamiltonian and* $(n-5)$*-fault hamiltonian connected, then* P_n *is* $(n-3)$*-fault hamiltonian.*

Table 1

⟨3214, 2314, 4132, 1432, 2341, 4321, 3421, 2431, 1342, 3142, 2413, 4213, 1243, 2143, 3412, 4312, 2134, 3124, 1324, 4231, 3241, 1423, 4123, 3214⟩
⟨1234, 3214, 2314, 1324, 3124, 2134, 4312, 1342, 3142, 4132, 1432, 3412, 2143, 4123, 1423, 2413, 4213, 1243, 3421, 2431, 4231, 3241, 2341, 4321, 1234⟩
⟨1234, 4321, 3421, 2431, 4231, 3241, 2341, 1432, 4132, 3142, 1342, 4312, 3412, 2143, 1243, 4213, 2413, 1423, 4123, 3214, 2314, 1324, 3124, 2134, 1234⟩
⟨1234, 3214, 2314, 4132, 1432, 3412, 4312, 1342, 3142, 2413, 4213, 1243, 2143, 4123, 1423, 3241, 2341, 4321, 3421, 2431, 4231, 1324, 3124, 2134, 1234⟩

Table 2

⟨1234, 3214, 2314, 1324, 4231, 3241, 2341, 4321, 3421, 2431, 1342, 3142, 4132, 1432, 3412, 4312, 2134, 3124, 4213, 2413, 1423, 4123, 2143, 1243⟩
⟨1234, 3214, 2314, 4132, 3142, 2413, 4213, 1243, 3421, 4321, 2341, 1432, 3412, 2143, 4123, 1423, 3241, 4231, 2431, 1342, 4312, 2134, 3124, 1324⟩
⟨1234, 3214, 2314, 4132, 3142, 2413, 4213, 1243, 2143, 4123, 1423, 3241, 4231, 1324, 3124, 2134, 4312, 3412, 1432, 2341, 4321, 3421, 2431, 1342⟩
⟨1234, 3214, 2314, 1324, 3124, 2134, 4312, 1342, 2431, 4231, 3241, 2341, 4321, 3421, 1243, 4213, 2413, 3142, 4231, 1432, 3412, 2143, 4123, 1423⟩
⟨1234, 3214, 2314, 1324, 3124, 2134, 4312, 3412, 2143, 4123, 1423, 2413, 4213, 1243, 3421, 4321, 2341, 2141, 4231, 2431, 1342, 3142, 4132, 1432⟩
⟨1234, 3214, 2314, 1324, 3124, 4213, 2413, 1423, 4123, 2143, 1243, 3421, 4321, 2341, 3241, 4231, 2431, 1342, 3142, 4132, 1432, 3412, 4312, 2134⟩
⟨1234, 3214, 2314, 1324, 3124, 2134, 4312, 3412, 4132, 4132, 3142, 1342, 2431, 4231, 3241, 2341, 4321, 3421, 1243, 4213, 2413, 1423, 4123, 2143⟩
⟨1234, 3214, 4123, 2143, 1243, 4213, 2413, 1423, 3241, 4231, 1324, 3124, 2134, 4312, 3412, 1432, 2341, 4321, 3421, 2431, 1342, 3142, 4132, 2314⟩
⟨1234, 3214, 2314, 1324, 3124, 2134, 4312, 1342, 2431, 4231, 3241, 1423, 4123, 2143, 3412, 1432, 4132, 3142, 2413, 4213, 1243, 3421, 4321, 2341⟩
⟨1234, 3214, 2314, 1324, 3124, 2134, 4312, 3412, 2143, 4123, 1423, 3241, 4231, 2431, 1342, 3142, 4132, 1432, 2341, 4321, 3421, 1243, 4213, 2413⟩
⟨1234, 3214, 2314, 1324, 3124, 2134, 4312, 1342, 3142, 4132, 1432, 3412, 2143, 4123, 1423, 2413, 4213, 1243, 3421, 4321, 2341, 3214, 4231, 2431⟩
⟨1234, 3214, 4123, 2143, 1243, 4213, 2413, 1423, 3241, 4231, 1324, 2314, 4132, 3142, 1342, 2431, 3421, 4321, 2341, 1432, 3412, 4312, 2134, 3124⟩
⟨1234, 3214, 2314, 1324, 3124, 2134, 4312, 1342, 2431, 4231, 3241, 2341, 4321, 3421, 1243, 4213, 2413, 1423, 4123, 2143, 3412, 1432, 4132, 3142⟩
⟨1234, 2134, 3124, 1324, 2314, 4132, 3142, 1342, 4312, 3412, 1432, 2341, 4321, 3421, 2431, 4231, 3241, 1423, 2413, 4213, 1243, 2143, 4123, 3214⟩
⟨1234, 3214, 2314, 1324, 3124, 2134, 4312, 3412, 2143, 4123, 1423, 2413, 4213, 1243, 3421, 4321, 2341, 1432, 4132, 3142, 1342, 2431, 4231, 3241⟩
⟨1234, 3214, 2314, 1324, 4231, 3241, 2341, 4321, 3421, 2431, 1342, 4312, 2134, 3124, 4213, 1243, 2143, 4123, 1423, 2413, 3142, 4132, 1432, 3412⟩
⟨1234, 3214, 2314, 4132, 1432, 3412, 4312, 2134, 3124, 1324, 4231, 2431, 1342, 3142, 2413, 4213, 1243, 2143, 4123, 1423, 3241, 2341, 4321, 3421⟩
⟨1234, 3214, 2314, 1324, 4231, 3241, 2341, 4321, 3421, 2431, 1342, 4312, 2134, 3124, 4213, 1243, 2143, 3412, 1432, 4132, 3142, 2413, 1423, 4123⟩
⟨1234, 3214, 2314, 1324, 3124, 2134, 4312, 1342, 2431, 4231, 3241, 1423, 4123, 2143, 3412, 1432, 2341, 4321, 3421, 1243, 4213, 2413, 3142, 4132⟩
⟨1234, 3214, 2314, 1324, 3124, 2134, 4312, 3412, 1432, 4132, 3142, 1342, 2431, 4231, 3241, 2341, 4321, 3421, 1243, 2143, 4123, 1423, 2413, 4213⟩
⟨1234, 4321, 3421, 2431, 1342, 3142, 4132, 2314, 3214, 4123, 2143, 1243, 4213, 2413, 1423, 3241, 2341, 1432, 3412, 4312, 2134, 3124, 1324, 4231⟩
⟨1234, 3214, 2314, 4132, 1432, 3412, 2143, 4123, 1423, 3241, 2341, 4321, 3421, 1243, 4213, 2413, 3142, 1342, 2431, 4231, 1324, 3124, 2134, 4312⟩
⟨1234, 3214, 2314, 1324, 3124, 2134, 4312, 1342, 3142, 4132, 1432, 3412, 2143, 4123, 1423, 2413, 4213, 1243, 3421, 2431, 4231, 3241, 2341, 4321⟩

Proof. Assume that F is any faulty set of P_n with $|F| \leqslant n-3$. Since $n \geqslant 5$, $|E^{i,j} - F| \geqslant (n-2)! - (n-3) \geqslant 4$ for any $1 \leqslant i, j \leqslant n$. Thus, there are at least four F-fault free edges between $P_n(i)$ and $P_n(j)$ for any $1 \leqslant i \neq j \leqslant n$. We may assume that

$$\left|F\left(P_n(i_1)\right)\right| \geqslant \left|F\left(P_n(i_2)\right)\right| \geqslant \cdots \geqslant \left|F\left(P_n(i_n)\right)\right|.$$

Case 1: $|F(P_n(i_1))| = n-3$. Thus, $F \subset P_n(i_1)$. Choose any element f in $F(P_n(i_1))$. By the assumption of this lemma, there exists a hamiltonian cycle Q of $P_n(i_1) - F + \{f\}$. We may write Q as $\langle \boldsymbol{u}, Q_1, \boldsymbol{v}, f', \boldsymbol{u}\rangle$ where $f' = f$ if f is incident with Q or f' is any edge of Q if otherwise. Obviously, $d(\boldsymbol{u}, \boldsymbol{v}) \leqslant 2$. By Lemma 2, $(n(\boldsymbol{u}))_n \neq (n(\boldsymbol{v}))_n$. Then by Lemma 3, there exists a hamiltonian path Q_2 of $P_n(\langle n\rangle - \{i_1\})$ joining $n(\boldsymbol{u})$ and $n(\boldsymbol{v})$. Then $\langle \boldsymbol{u}, n(\boldsymbol{u}), Q_2, n(\boldsymbol{v}), \boldsymbol{v}, Q_1, \boldsymbol{u}\rangle$ forms a hamiltonian cycle of $P_n - F$.

Case 2: $|F(P_n(i_1))| = n-4$. Thus, $|F - F(P_n(i_1))| \leqslant 1$. Hence, there exists an index i_2 such that

$$\left|F\left(P_n(\langle n\rangle - \{i_1, i_2\})\right)\right| = 0.$$

Since $|E^{i_1,i_2} - F| \geqslant (n-2)! - (n-3)$, there exists an F-fault free edge $(\boldsymbol{u}, \boldsymbol{v})$ in E^{i_1,i_2} such that $\boldsymbol{u} \in V(P_n(i_1))$ and $\boldsymbol{v} \in V(P_n(i_2))$. By the assumption of this lemma, there exists a hamiltonian cycle C_1 of $P_n(i_1) - F$ and there exists a hamiltonian cycle C_2 of $P_n(i_2) - F$. We may write C_1 as $\langle \boldsymbol{u}, \boldsymbol{w}, Q_1, \boldsymbol{z}, \boldsymbol{u}\rangle$ and C_2 as $\langle \boldsymbol{v}, \boldsymbol{y}, Q_2, \boldsymbol{x}, \boldsymbol{v}\rangle$. Since $d(\boldsymbol{x}, \boldsymbol{y}) \leqslant 2$ and $d(\boldsymbol{w}, \boldsymbol{z}) \leqslant 2$, by Lemma 2

$$\left(n(\boldsymbol{x})\right)_n \neq \left(n(\boldsymbol{y})\right)_n \quad \text{and} \quad \left(n(\boldsymbol{w})\right)_n \neq \left(n(\boldsymbol{z})\right)_n.$$

Thus, we can choose a vertex from $\boldsymbol{x}$ and $\boldsymbol{y}$, say $\boldsymbol{x}$, and we can choose a vertex from $\boldsymbol{w}$ and $\boldsymbol{z}$, say $\boldsymbol{w}$, such that $(n(\boldsymbol{w}))_n \neq (n(\boldsymbol{x}))_n$ and $(\boldsymbol{w}, n(\boldsymbol{w}))$ and $(\boldsymbol{x}, n(\boldsymbol{x}))$ are F-fault free. By Lemma 3, there exists a hamiltonian path Q_3 of $P_n(\langle n\rangle - \{i_1, i_2\}) - F$ joining $n(\boldsymbol{w})$ and $n(\boldsymbol{x})$. Hence, $\langle \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{y}, Q_2, \boldsymbol{x}, n(\boldsymbol{x}), Q_3, n(\boldsymbol{w}), \boldsymbol{w}, Q_1, \boldsymbol{u}\rangle$ forms a hamiltonian cycle of $P_n - F$.

Case 3: $|F(P_n(i_1))| \leqslant n-5$. We can choose any F-fault free edge $(\boldsymbol{u}, \boldsymbol{v})$ in E^{i_1,i_2} such that $\boldsymbol{u} \in V(P_n(i_1))$ and $\boldsymbol{v} \in V(P_n(i_2))$. By the assumption of this lemma, any $P_n(i) - F$ is hamiltonian connected for $i \in \langle n\rangle$. Then by Lemma 3, there exists a hamiltonian path Q_1 of $P_n - F$ joining $\boldsymbol{u}$ and $\boldsymbol{v}$. Then $\langle \boldsymbol{u}, Q_1, \boldsymbol{v}, \boldsymbol{u}\rangle$ forms a hamiltonian cycle of $P_n - F$. □

Lemma 6. *Suppose that $n \geqslant 5$. If P_{n-1} is $(n-4)$-fault hamiltonian and $(n-5)$-fault hamiltonian connected, then P_n is $(n-4)$-fault hamiltonian connected.*

Proof. Assume that F is any faulty set of P_n with $|F| \leqslant n-4$. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be any two arbitrary vertices of $P_n - F$. We want to construct a hamiltonian path of $P_n - F$ joining $\boldsymbol{u}$ and $\boldsymbol{v}$. Obviously, $|E^{i,j} - F| \geqslant (n-2)! - (n-4) \geqslant 5$ for any $1 \leqslant i, j \leqslant n$ with $n \geqslant 5$. Thus, there are at least five F-fault free edges between $P_n(i)$ and $P_n(j)$ for any $1 \leqslant i \neq j \leqslant n$. We assume that $|F(P_n(i_1))| \geqslant |F(P_n(i_2))| \geqslant \cdots \geqslant |F(P_n(i_n))|$.

Case 1: $|F(P_n(i_1))| = n-4$. Hence, $F \subset P_n(i_1)$.

Subcase 1.1: $(\boldsymbol{u})_n = (\boldsymbol{v})_n = i_1$. Choose any element f in $F(P_n(i_1))$. By the assumption of this lemma, there exists a hamiltonian path Q of $P_n(i_1) - F + \{f\}$ joining $\boldsymbol{u}$ and $\boldsymbol{v}$. We may write Q as $\langle \boldsymbol{u}, Q_1, \boldsymbol{x}, f', \boldsymbol{y}, Q_2, \boldsymbol{v}\rangle$ where $f' = f$ if f is incident with Q or f' is any edge of Q if otherwise. Obviously, $d(\boldsymbol{x}, \boldsymbol{y}) \leqslant 2$. By Lemma 2, $(n(\boldsymbol{x}))_n \neq (n(\boldsymbol{y}))_n$. By Lemma 3, there exists a hamiltonian path Q_3 of $P_n(\langle n\rangle - \{i_1\})$ joining $n(\boldsymbol{x})$ and $n(\boldsymbol{y})$. Then $\langle \boldsymbol{u}, Q_1, \boldsymbol{x}, n(\boldsymbol{x}), Q_3, n(\boldsymbol{y}), \boldsymbol{y}, Q_2, \boldsymbol{v}\rangle$ forms a hamiltonian path of $P_n - F$ joining $\boldsymbol{u}$ to $\boldsymbol{v}$.

Subcase 1.2: $(\boldsymbol{u})_n = i_1$ and $(\boldsymbol{v})_n = i_j$ with $j \neq 1$. By the assumption of this lemma, there exists a hamiltonian cycle C_1 of $P_n(i_1) - F$. We may write C_1 as $\langle \boldsymbol{u}, \boldsymbol{y}, Q_1, \boldsymbol{x}, \boldsymbol{u}\rangle$. Since $d(\boldsymbol{x}, \boldsymbol{y}) \leqslant 2$, by Lemma 2 $(n(\boldsymbol{x}))_n \neq (n(\boldsymbol{y}))_n$. Thus, we can choose a vertex from $\boldsymbol{x}$ and $\boldsymbol{y}$, say $\boldsymbol{x}$, such that $(n(\boldsymbol{x}))_n \neq (\boldsymbol{v})_n$. By Lemma 3, there exists a hamiltonian path Q_2 of $P_n(\langle n\rangle - \{i_1\})$ joining $\boldsymbol{v}$ and $n(\boldsymbol{x})$. Then $\langle \boldsymbol{u}, \boldsymbol{y}, Q_1, \boldsymbol{x}, n(\boldsymbol{x}), Q_2, \boldsymbol{v}\rangle$ forms a hamiltonian path of $P_n - F$ joining $\boldsymbol{u}$ to $\boldsymbol{v}$.

Subcase 1.3: $(\boldsymbol{u})_n = (\boldsymbol{v})_n = i_j$ with $j \neq 1$. Since there are at least five F-fault free edges in E^{i_1,i_j}, there exists an F-fault free edge $(\boldsymbol{w}, \boldsymbol{x})$ in E^{i_1,i_j} such that $(\boldsymbol{w})_n = i_1$, $(\boldsymbol{x})_n = i_j$, and $\boldsymbol{x} \neq \boldsymbol{v}$. By the assumption of this lemma, there exists a hamiltonian cycle C_1 of $P_n(i_1) - F$ and a hamiltonian path Q_1 of $P_n(i_j)$ joining $\boldsymbol{u}$ and $\boldsymbol{v}$. We may write Q_1 as $\langle \boldsymbol{u}, Q_2, \boldsymbol{x}, \boldsymbol{y}, Q_3, \boldsymbol{v}\rangle$ and C_1 as $\langle \boldsymbol{w}, \boldsymbol{z}', Q_4, \boldsymbol{z}, \boldsymbol{w}\rangle$. Since $d(\boldsymbol{z}', \boldsymbol{z}) \leqslant 2$, by Lemma 2 $(n(\boldsymbol{z}))_n \neq (n(\boldsymbol{z}'))_n$. Thus, we can choose a vertex from $\boldsymbol{z}$ and $\boldsymbol{z}'$, say $\boldsymbol{z}$, such that $(n(\boldsymbol{z}))_n \neq (n(\boldsymbol{y}))_n$. By Lemma 3, there exists a hamiltonian path Q_5 of $P_n(\langle n\rangle - \{i_1, i_j\})$ joining $n(\boldsymbol{y})$ and $n(\boldsymbol{z})$. Then $\langle \boldsymbol{u}, Q_2, \boldsymbol{x}, \boldsymbol{w}, \boldsymbol{z}', Q_4, \boldsymbol{z}, n(\boldsymbol{z}), Q_5, n(\boldsymbol{y}), \boldsymbol{y}, Q_3, \boldsymbol{v}\rangle$ forms a hamiltonian path of $P_n - F$ joining $\boldsymbol{u}$ and $\boldsymbol{v}$.

Subcase 1.4: $(\boldsymbol{u})_n = i_j$ and $(\boldsymbol{v})_n = i_k$ with i_j, i_k and i_1 are all distinct. Since there are at least five F-fault free edges in E^{i_1,i_j}, there exists an F-fault free edge $(\boldsymbol{w}, \boldsymbol{x})$ in E^{i_1,i_j} such that $(\boldsymbol{w})_n = i_1$, $(\boldsymbol{x})_n = i_j$, and $\boldsymbol{x} \neq \boldsymbol{u}$. By the assumption of this lemma, there exists a hamiltonian cycle C_1 of $P_n(i_1) - F$ and a hamiltonian path Q_1 of $P_n(i_j)$ joining $\boldsymbol{u}$ and $\boldsymbol{x}$. We may write C_1 as $\langle \boldsymbol{w}, z, Q_2, \boldsymbol{y}, \boldsymbol{w}\rangle$. Since $d(\boldsymbol{y}, z) \leqslant 2$, by Lemma 2 $(n(\boldsymbol{y}))_n \neq (n(z))_n$. Thus, we can choose a vertex from $\boldsymbol{y}$ and z, say $\boldsymbol{y}$, such that $(n(\boldsymbol{y}))_n \neq (\boldsymbol{v})_n$. By Lemma 3, there exists a hamiltonian path Q_3 of $P_n(\langle n\rangle - \{i_1, i_j\}$ joining $n(\boldsymbol{y})$ and $\boldsymbol{v}$. Thus, $\langle \boldsymbol{u}, Q_1, \boldsymbol{x}, \boldsymbol{w}, z, Q_2, \boldsymbol{y}, n(\boldsymbol{y}), Q_3, \boldsymbol{v}\rangle$ forms a hamiltonian path of $P_n - F$ joining $\boldsymbol{u}$ and $\boldsymbol{v}$.

Case 2: $|F(P_n(i_1))| \leqslant n - 5$. By the assumption of this lemma, $P_n(i)$ is hamiltonian connected for every $1 \leqslant i \leqslant n$.

Subcase 2.1: $(\boldsymbol{u})_n = (\boldsymbol{v})_n = i_j$. By the assumption of this lemma, there exists a hamiltonian path Q_1 of $P_n(i_j) - F$ joining $\boldsymbol{u}$ to $\boldsymbol{v}$. We claim that there exists an F-fault free edge $(\boldsymbol{x}, \boldsymbol{y})$ in Q_1 such that $(\boldsymbol{x}, n(\boldsymbol{x}))$ and $(\boldsymbol{y}, n(\boldsymbol{y}))$ are F-fault free. Suppose there is no such edge, $|F| \geqslant |V(F(P_n(i_j)))| + |(V(P_n(i_j)) - V(F(P_n(i_j)))|/2 \geqslant (n-1)!/2 > n-3$ for $n \geqslant 5$. However, $|F| \leqslant n - 3$. We get a contradiction. Hence, such edge exists.

Write Q_1 as $\langle \boldsymbol{u}, Q_2, \boldsymbol{x}, \boldsymbol{y}, Q_3, \boldsymbol{v}\rangle$. Since $d(\boldsymbol{x}, \boldsymbol{y}) = 1$, by Lemma 2 $(n(\boldsymbol{x}))_n \neq (n(\boldsymbol{y}))_n$. By Lemma 3, there exists a hamiltonian path Q_4 of $P_n(\langle n\rangle - \{i_j\})$ joining $n(\boldsymbol{x})$ and $n(\boldsymbol{y})$. Then $\langle \boldsymbol{u}, Q_2, \boldsymbol{x}, n(\boldsymbol{x}), Q_4, n(\boldsymbol{y}), \boldsymbol{y}, Q_3, \boldsymbol{v}\rangle$ forms a hamiltonian path of $P_n - F$ joining $\boldsymbol{u}$ and $\boldsymbol{v}$.

Subcase 2.2: $(\boldsymbol{u})_n \neq (\boldsymbol{v})_n$. By Lemma 3, there exists a hamiltonian path of $P_n - F$ joining $\boldsymbol{u}$ and $\boldsymbol{v}$. □

Theorem 1. *Let n be a positive integers with $n \geqslant 4$. Then P_n is $(n-3)$-fault hamiltonian and $(n-4)$-fault hamiltonian connected.*

Proof. We prove this theorem by induction. The induction base, $n = 4$, is proved in Lemma 4. With Lemmas 5 and 6, we prove the induction step. □

Since $\delta(P_n) = n - 1$, we have the following corollary.

Corollary 1. *$\mathcal{H}_f(P_n) = n - 3$ and $\mathcal{H}^{\kappa}_f(P_n) = n - 4$ for any positive integer n with $n \geqslant 4$.*

Acknowledgements

The authors would like to thank the anonymous referees for their comments and suggestions. These comments and suggestions are helpful for improve the quality of this paper.

References

[1] S.B. Akers, B. Krishnameurthy, A group-theoretic model for symmetric interconnection networks, IEEE Trans. Comput. (1989) 555–566.

[2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, North-Holland, New York, 1980.

[3] A. Bouabdallah, M.C. Heydemann, J. Opatrný, D. Sotteau, Embedding complete binary trees into star and pancake graphs, Theory Comput. Syst. (1998) 279–305.

[4] W.C. Fang, C.C. Hsu, On the fault-tolerant embedding of complete binary tree in the pancake graph interconnection network, Inform. Sci. (2000) 191–204.

[5] L. Gargano, U. Vaccaro, A. Vozella, Fault tolerant routing in the star and pancake interconnection networks, Inform. Process. Lett. (1993) 315–320.

[6] W.H. Gates, C.H. Papadimitriou, Bounds for sorting by prefix reversal, Discrete Math. 27 (1979) 47–57.

[7] M.H. Heydari, I.H. Sudborough, On the diameter of the pancake network, J. Algorithms (1997) 67–94.

[8] W.T. Huang, J.M. Tan, C.N. Hung, L.H. Hsu, Fault-tolerant hamiltonicity of twisted cubes, to appear in J. Parallel Distributed Comput.

[9] W.T. Huang, Y.C. Chuang, L.H. Hsu, J.M. Tan, On the fault-tolerant hamiltonicity of crossed cubes, to appear in IEICE Trans. Fundamentals.

[10] A. Kanevsky, C. Feng, On the embedding of cycles in pancake graphs, Parallel Comput. (1995) 923–936.