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Brother trees: A family of optimal 1_p-hamiltonian and 1-edge hamiltonian graphs

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Abstract

In this paper we propose a family of cubic bipartite planar graphs, brother trees, denoted by $BT(n)$ with $n \geqslant 2$. Any $BT(n)$ is hamiltonian. It remains hamiltonian if any edge is deleted. Moreover, it remains hamiltonian when a pair of nodes (one from each partite set) is deleted. These properties are optimal. Furthermore, the number of nodes in $BT(n)$ is $6 \cdot 2^n - 4$ and the diameter is $2n + 1$.

Keywords: Hamiltonian; Diameter; Complete binary tree; Interconnection networks

1. Introduction

An interconnection network connects the processors of the parallel computer. Its architecture can be represented as a graph in which the nodes correspond to the processors and the edges to the communication links. Hence we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology of computer networks. It is almost impossible to design a network optimum from all aspects. One has to design a suitable network satisfying the requirements. Diameter is one of the major requirements in designing the topology of network. Usually a network with smaller diameter is more preferable. The hamiltonian properties is another requirement. For example, "*Token Passing*" approach is used in some distributed operation systems. Interconnection network requires the presence of hamiltonian cycles in the structure to meet this approach. Fault tolerance is also desirable in massive parallel systems that have a relatively high probability of failure. A number of fault tolerant designs for specific multiprocessor architectures have been proposed based on graph theoretic models in which the processor-to-processor interconnection structure is represented by a graph.

For the graph definition and notation, we follow [1]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *node set* and E is the *edge set* of G. Two nodes, a and b, are *adjacent* if $(a, b) \in E$. A *path* is a sequence of consecutive adjacent nodes. A path

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is usually delimited by $\langle x_0, x_1, x_2, \ldots, x_{n-1}\rangle$. We use P^{-1} to denote the path $\langle x_{n-1}, x_{n-2}, \ldots, x_1, x_0\rangle$ if P is the path $\langle x_0, x_1, x_2, \ldots, x_{n-1}\rangle$. Let $d_G(x, y)$ denote the distance between two nodes x and y in graph G and $D(G)$ denote the diameter of G. A path is called a *hamiltonian path* if its nodes are distinct and span V.

A *cycle* is a path of at least three nodes such that the first node is the same as the last node. A cycle is called a *hamiltonian cycle* if its nodes are distinct except for the first node and last node and if they span V. A graph is *hamiltonian* if it contains a hamiltonian cycle. A graph $G = (V, E)$ is 1-*edge hamiltonian* if $G - e$ is hamiltonian for any $e \in E$. Obviously, any 1-edge hamiltonian graph is hamiltonian. A 1-edge hamiltonian graph G is *optimal* if it contains the least number of edges among all 1-edge hamiltonian graphs with the same number of nodes as G. A graph $G = (V, E)$ is 1-*node hamiltonian* if $G - v$ is hamiltonian for any $v \in V$. A 1-node hamiltonian graph G is *optimal* if it contains the least number of edges among all 1-node hamiltonian graphs with the same number of nodes as G. A graph $G = (V, E)$ is 1-*hamiltonian* if it is 1-edge hamiltonian and 1-node hamiltonian. A 1-hamiltonian graph G is *optimal* if it contains the least number of edges among all 1-hamiltonian graphs with the same number of nodes as G. This study of optimal 1-hamiltonian graphs is motivated by optimal fault tolerant token ring design in computer networks. A number of optimal 1-hamiltonian graphs have been proposed [2,5,7]. Obviously, $\deg_G(x) \geqslant 3$ for any node x in a 1-edge hamiltonian, 1-node hamiltonian, or 1-hamiltonian graph G. It has been proven that any 1-hamiltonian regular graph is optimal if and only if it is 3-regular.

Note that any cycle of a bipartite graph contains the same number of nodes in each partite set. Thus, the deletion of a node from a hamiltonian bipartite graph results in a non-hamiltonian graph. However, the fault tolerant hamiltonian property is not the only factor in designing the topology of networks. For example, the hypercube Q_n is a hamiltonian bipartite graph for $n \geqslant 2$. Hence, it is not 1-hamiltonian. When fault occurs, we are interested in the longest cycle in the faulty hypercube [6].

Let G be a bipartite graph with bipartition W and B. We use $\mathcal{F}(G)$ to denote $\{\{c, d\} \mid c \in W,\ d \in B\}$. A hamiltonian bipartite graph is 1_p-*hamiltonian* if $G - F$ remains hamiltonian for any $F \in \mathcal{F}(G)$. A 1_p-hamiltonian graph G is *optimal* if it contains the least number of edges among all 1_p-hamiltonian graphs with the same number of nodes as G. Obviously, $\deg_G(x) \geqslant 3$ for any node x in a 1_p-hamiltonian graph G. Thus any 1_p-hamiltonian graph that is 3-regular is optimal. In [3], a family of optimal 1-edge hamiltonian and 1_p-hamiltonian bipartite graphs, called *honeycomb rectangular torus*, is discussed. It can be shown that the diameter of the honeycomb rectangular torus is $\Omega(\sqrt{p})$, where p is the number of nodes.

In this paper, we propose a family of graphs, called brother trees, denoted by $BT(n)$. The graph $BT(n)$ is planar, bipartite, 3-regular, 1-edge hamiltonian and 1_p-hamiltonian. The diameter of the brother tree is $\Theta(\log p)$, where p is the number of nodes.

2. Definitions and notation

To define brother trees, first we define brother cells. Assume that k is an integer with $k \geqslant 2$. The *kth brother cell* $BC(k)$ is the five tuple $(G_k, w_k, x_k, y_k, z_k)$, where $G_k = (V, E)$ is a bipartite graph with bipartition W (white) and B (black) and contains four distinct nodes w_k, x_k, y_k and z_k. w_k is the *white terminal*; x_k the *white root*; y_k the *black terminal* and z_k the *black root*. We can recursively define $BC(k)$ as follows:

(1) $BC(2)$ is the 5-tuple $(G_2, w_2, x_2, y_2, z_2)$ where $V(G_2) = \{w_2, x_2, y_2, z_2, s, t\}$, and $E(G_2) = \{(w_2, s), (s, x_2), (x_2, y_2), (y_2, t), (t, z_2), (w_2, z_2), (s, t)\}$.

(2) The kth brother cell $BC(k)$ with $k \geqslant 3$ is composed of two disjoint copies of $(k-1)$th brother cells

$$BC^1(k-1) = \bigl(G^1_{k-1}, w^1_{k-1}, x^1_{k-1}, y^1_{k-1}, z^1_{k-1}\bigr),$$
$$BC^2(k-1) = \bigl(G^2_{k-1}, w^2_{k-1}, x^2_{k-1}, y^2_{k-1}, z^2_{k-1}\bigr),$$

a white root x_k, and a black root z_k. To be specific,

$$V(G_k) = V\bigl(G^1_{k-1}\bigr) \cup V\bigl(G^2_{k-1}\bigr) \cup \{x_k, z_k\},$$
$$\begin{aligned} E(G_k) = {} & E\bigl(G^1_{k-1}\bigr) \cup E\bigl(G^2_{k-1}\bigr) \\ & \cup \bigl\{\bigl(z_k, x^1_{k-1}\bigr), \bigl(z_k, x^2_{k-1}\bigr), \bigl(x_k, z^1_{k-1}\bigr), \\ & \bigl(x_k, z^2_{k-1}\bigr), \bigl(y^1_{k-1}, w^2_{k-1}\bigr)\bigr\}, \end{aligned}$$
$$w_k = w^1_{k-1}, \text{ and } y_k = y^2_{k-1}.$$

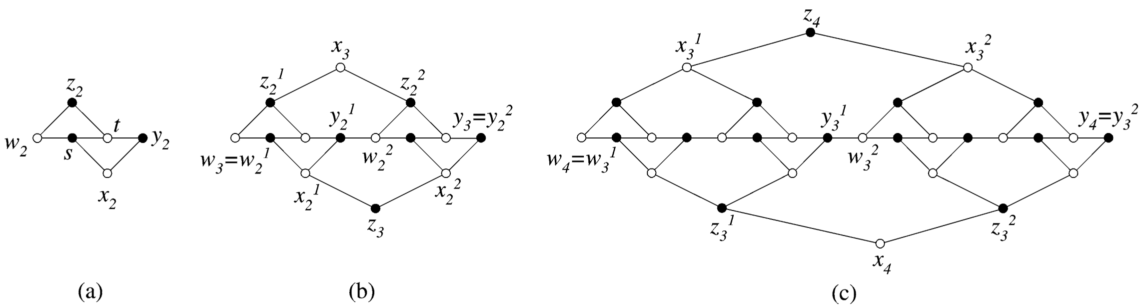

Fig. 1. (a) $BC(2)$, (b) $BC(3)$ and (c) $BC(4)$.

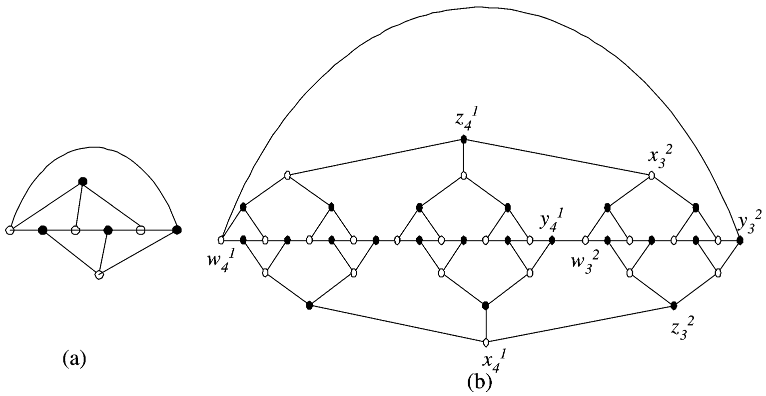

Fig. 2. (a) $BT(1)$, (b) $BT(3)$.

$BC(2)$, $BC(3)$, and $BC(4)$ are shown in Fig. 1. We note that $BC^1(k-1)$ and $BC^2(k-1)$ are isomorphic for $k \geqslant 3$. This property is referred to as the *symmetrical property* of $BC(k)$. For this reason, we define the degenerate case, $BC(1)$, as the 5-tuple $(G_1, w_1, x_1, y_1, z_1)$ as $V(G_1) = \{w_1, y_1\}$, $E(G_1) = \{(w_1, y_1)\}$ such that $x_1 = w_1$ and $y_1 = z_1$.

We can also define the brother cell $BC(k)$ from the *complete binary tree* $B(k)$, where $V(B(k)) = \{1, 2, \ldots, 2^k - 1\}$ and $E(B(k)) = \{(i, j) \mid \lfloor j/2 \rfloor = i\}$. Assume that k is a positive integer with $k \geqslant 2$. The kth brother cell $BC(k) = (G_k, w_k, x_k, y_k, z_k)$ can be constructed by combining two $B(k)$'s, the *upper* tree $B(k)_u$ and the *lower* tree $B(k)_l$, and adding edges between their leaf nodes.

Let n be a positive integer with $n \geqslant 1$. The *brother tree*, $BT(n)$, is composed of an $(n+1)$th brother cell $BC(n+1) = (G^1_{n+1}, w^1_{n+1}, x^1_{n+1}, y^1_{n+1}, z^1_{n+1})$ and an nth brother cell $BC(n) = (G^2_n, w^2_n, x^2_n, y^2_n, z^2_n)$ with $V(G^1_{n+1}) \cap V(G^2_n) = \emptyset$. To be specific, $V(BT(n)) = V(G^1_{n+1}) \cup V(G^2_n)$ and $E(BT(n)) = E(G^1_{n+1}) \cup E(G^2_n) \cup \{(z^1_{n+1}, x^2_n), (y^1_{n+1}, w^2_n), (x^1_{n+1}, z^2_n), (w^1_{n+1}, y^2_n)\}$. $BT(1)$ and $BT(3)$ are shown in Fig. 2. Obviously, $BT(n)$ is a 3-regular bipartite planar graph with $6 \cdot 2^n - 4$ nodes. Because the $(n+1)$th brother cell is composed of two disjoint nth brother cells and two terminals, the nth brother tree $BT(n)$ is composed of three disjoint nth brother cells, $BC^1(n)$, $BC^2(n)$, $BC^3(n)$ and two terminals, $\{x^1_{n+1}, z^1_{n+1}\}$. Moreover, $BC^1(n)$, $BC^2(n)$ and $BC^3(n)$ are arranged in a cyclic order in $BT(n)$. Thus any two nodes of $BT(n)$ are in the union of two in the node set of $BC^1(n)$, $BC^2(n)$, $BC^3(n)$ and $\{x^1_{n+1}, z^1_{n+1}\}$. For this reason, we can assume without loss of generality that any two nodes of $BT(n)$ are in G^1_{n+1} and any edge of $BT(n)$ is in G^1_{n+1}. This property is referred to as the *symmetrical property* of $BT(n)$.

3. Diameter

Theorem 3.1. *$D(BT(n)) = 2n + 1$ for any positive integer n with $n \geqslant 1$.*

Proof. It is easy to prove by induction that

$$d_{BT(n)}\left(x^1_{n+1}, z^1_{n+1}\right) = 2n + 1.$$

Let u and v be any two nodes of $BT(n)$. We will prove that $d_{BT(n)}(u, v) \leqslant 2n + 1$. Using the symmetrical property of brother trees, we may assume that u and v are in the brother cell G^1_{n+1}. Note that the brother cell G^1_{n+1} is composed of two complete binary trees $B(n + 1)_u$ and $B(n + 1)_l$. Thus,

(1) both u and v are in $V(B(n + 1)_u)$, or both u and v are in $V(B(n + 1)_l)$, or
(2) $u \in V(B(n + 1)_u)$ and $v \in V(B(n + 1)_l)$.

Now, we introduce some notations before our proof. Let $V(B(n+1)_u) = \{1, 2, \ldots, 2^{n+1} - 1\}$ and $V(B(n+1)_l) = \{1', 2', \ldots, (2^{n+1} - 1)'\}$. Now, join $B(n + 1)_u$ and $B(n + 1)_l$ with the edge set $\{(2^n + i, (2^n + i)'), ((2^n + i)', 2^n + i + 1) \mid 0 \leqslant i \leqslant 2^n - 2\} \cup \{(2^{n+1} - 1, (2^{n+1} - 1)')\}$ to obtain the brother cell $(G^1_{n+1}, w^1_{n+1}, x^1_{n+1}, y^1_{n+1}, z^1_{n+1})$, where $x_{n+1} = 1$ and $z_{n+1} = 1'$ if n is even and $x_{n+1} = 1'$ and $z_{n+1} = 1$ if otherwise. Moreover, $w_{n+1} = 2^n$ and $y_{n+1} = (2^{n+1} - 1)'$.

Case 1: By symmetry, we may assume that both u and v are in $V(B(n + 1)_u)$. Suppose that u is labeled i and v is labeled j. Obviously, $\max\{\log_2(i + 1), \log_2(j + 1)\} \leqslant n + 1$. Since $d_{B(n+1)}(i, 1) = \lceil\log_2(i + 1)\rceil - 1$, there exists a path P_1 of length $\lceil\log_2(i + 1)\rceil - 1$ joining u to the root of $B(n + 1)_u$. Similarly, there exists a path P_2 of length $\lceil\log_2(j + 1)\rceil - 1$ joining v to 1. Thus, $\langle u, P_1, 1, P_2^{-1}, v\rangle$ forms a path joining u to v in $B(n + 1)_u$. Thus, $d_{BT(n)}(u, v) \leqslant \lceil\log_2(i + 1)\rceil + \lceil\log_2(j + 1)\rceil - 2 \leqslant 2n + 1$.

Case 2: We may assume that u is labeled i and v is labeled j'. Without loss of generality, we assume that $i \geqslant j$. There then exists a path P_1 from i to some leaf node h of $B(n + 1)_u$ of length $n - (\lceil\log_2(i + 1)\rceil - 1)$. Let h' be a neighborhood of h of $BT(n)$ in $B(n + 1)_l$. Obviously, there exists a path of length n from h' to the root $1'$ of $B(n + 1)_l$. Moreover, there exists a path P_3 of length $\lceil\log_2(j + 1)\rceil - 1$ joining v to $1'$. Obviously, $\langle u, P_1, h, h', P_2, 1', P_3^{-1}, v\rangle$ forms a path joining u to v in G^1_{n+1}. Thus, $d_{BT(n)}(u, v) \leqslant 2n + 1 - \lceil\log_2(i + 1)\rceil + \lceil\log_2(j + 1)\rceil$. Since, $i \geqslant j$, $d_{BT(n)}(u, v) \leqslant 2n + 1$.

The theorem is proven. □

4. 1-edge hamiltonian

Lemma 4.1. *Assume that $BC(n) = (G_n, w_n, x_n, y_n, z_n)$ for some integer $n \geqslant 2$.*

(1) *There exists a hamiltonian path P^1_n of G_n joining w_n to y_n.*
(2) *There exists a hamiltonian path P^2_n of G_n joining w_n to z_n.*
(3) *There exists a hamiltonian path P^3_n of G_n joining x_n to y_n.*
(4) *There exists a hamiltonian path P^4_n of G_n joining x_n to z_n.*
(5) *There exist two disjoint paths P^5_n and P^6_n such that* (i) *they span G_n,* (ii) *P^5_n joins w_n and x_n, and* (iii) *P^6_n joins y_n and z_n.*
(6) *There exist two disjoint paths P^7_n and P^8_n such that* (i) *they span G_n,* (ii) *P^7_n joins w_n and z_n, and* (iii) *P^8_n joins x_n and y_n.*

Proof. We prove this lemma by induction. It is easy to check that the lemma holds in $BC(2)$. Assume that the lemma holds for $BC(n)$. By definition, $BC(n + 1)$ is composed of two disjoint copies of nth brother cells $BC^1(n)$ and $BC^2(n)$, a white root x_{n+1}, and a black root z_{n+1}. By induction, $\{P^{i,1}_n\}^8_{i=1}$ exists satisfying the lemma for $BC^1(n)$ and $\{P^{i,2}_n\}^8_{i=1}$ exists satisfying the lemma for $BC^2(n)$.

We then set

$$\begin{aligned}
P^1_{n+1} &= \langle w_{n+1} = w^1_n, P^{5,1}_n, x^1_n, z_{n+1}, x^2_n, (P^{5,2}_n)^{-1}, w^2_n, \\
&\qquad y^1_n, P^{6,1}_n, z^1_n, x_{n+1}, z^2_n, (P^{6,2}_n)^{-1}, y^2_n = y_{n+1}\rangle; \\
P^2_{n+1} &= \langle w_{n+1} = w^1_n, P^{2,1}_n, z^1_n, x_{n+1}, z^2_n, \\
&\qquad (P^{4,2}_n)^{-1}, x^2_n, z_{n+1}\rangle; \\
P^3_{n+1} &= \langle x_{n+1}, z^2_n, (P^{7,2}_n)^{-1}, w^2_n, y^1_n, (P^{3,1}_n)^{-1}, \\
&\qquad x^1_n, z_{n+1}, x^2_n, P^{8,2}_n, y^2_n = y_{n+1}\rangle; \\
P^4_{n+1} &= \langle x_{n+1}, z^2_n, (P^{2,2}_n)^{-1}, w^2_n, y^1_n, (P^{3,1}_n)^{-1}, \\
&\qquad x^1_n, z_{n+1}\rangle; \\
P^5_{n+1} &= \langle w_{n+1} = w^1_n, P^{1,1}_n, y^1_n, w^2_n, P^{7,2}_n, z^2_n, x_{n+1}\rangle; \\
P^6_{n+1} &= \langle y_{n+1} = y^2_n, (P^{8,2}_n)^{-1}, x^2_n, z_{n+1}\rangle; \\
P^7_{n+1} &= \langle w_{n+1} = w^1_n, P^{5,1}_n, x^1_n, z_{n+1}\rangle; \\
P^8_{n+1} &= \langle x_{n+1}, z^1_n, (P^{6,1}_n)^{-1}, y^1_n, w^2_n, P^{1,2}_n, y^2_n = y_{n+1}\rangle.
\end{aligned}$$

The lemma is proven. □

Lemma 4.2. *Assume that $BC(n) = (G_n, w_n, x_n, y_n, z_n)$ for some integer $n \geqslant 2$. Let e be any edge of $BC(n)$. Then at least one of the following properties holds.*

(1) *There exists a hamiltonian path $Q_n^1(e)$ of $G_n - e$ joining w_n to y_n.*
(2) *There exists a hamiltonian path $Q_n^2(e)$ of $G_n - e$ joining w_n to z_n.*
(3) *There exists a hamiltonian path $Q_n^3(e)$ of $G_n - e$ joining x_n to y_n.*
(4) *There exists a hamiltonian path $Q_n^4(e)$ of $G_n - e$ joining x_n to z_n.*
(5) *There exist two disjoint paths $Q_n^5(e)$ and $Q_n^6(e)$ such that* (i) *they span $G_n - e$,* (ii) *$Q_n^5(e)$ joins w_n and x_n, and* (iii) *$Q_n^6(e)$ joins y_n and z_n.*
(6) *There exist two disjoint paths $Q_n^7(e)$ and $Q_n^8(e)$ such that* (i) *they span $G_n - e$,* (ii) *$Q_n^7(e)$ joins w_n and z_n, and* (iii) *$Q_n^8(e)$ joins x_n and y_n.*

Proof. We prove this lemma by induction. It is easy to check that the lemma holds for $BC(2)$. Assume that the lemma holds for $BC(n)$. By definition, $BC(n+1)$ is composed of two disjoint copies of nth brother cells $BC^1(n)$ and $BC^2(n)$, a white root x_{n+1}, and a black root z_{n+1}. Let e be any edge of $BC(n+1)$. Using the symmetrical property of $BC(n+1)$, we may assume that e is (z_{n+1}, x_n^1), (z_n^1, x_{n+1}), (y_n^1, w_n^2), or an edge in $BC^1(n)$. By induction, there exists $\{P_n^{i,1}\}_{i=1}^8$ satisfying the lemma for $BC^1(n)$ if $e \notin E(BC^1(n))$ and there exists $\{P_n^{i,2}\}_{i=1}^8$ satisfying the lemma for $BC^2(n)$ if $e \notin E(BC^2(n))$.

Case 1: $e = (z_{n+1}, x_n^1)$. We set $Q_{n+1}^7(e)$ as $\langle w_{n+1} = w_n^1, P_n^{1,1}, y_n^1, w_n^2, P_n^{5,2}, x_n^2, z_{n+1}\rangle$ and $Q_{n+1}^8(e)$ as $\langle x_{n+1}, z_n^2, (P_n^{6,2})^{-1}, y_n^2 = y_{n+1}\rangle$.

Case 2: $e = (z_n^1, x_{n+1})$. We set $Q_{n+1}^5(e)$ as $\langle w_{n+1} = w_n^1, P_n^{1,1}, y_n^1, w_n^2, P_n^{7,2}, z_n^2, x_{n+1}\rangle$ and $Q_{n+1}^6(e)$ as $\langle y_{n+1} = y_n^2, (P_n^{8,2})^{-1}, x_n^2, z_{n+1}\rangle$.

Case 3: $e = (y_n^1, w_n^2)$. We set $Q_{n+1}^3(e)$ as $\langle x_{n+1}, z_n^1, (P_n^{4,1})^{-1}, x_n^1, z_{n+1}, x_n^2, P_n^{3,2}, y_n^2 = y_{n+1}\rangle$.

Case 4: e is in $BC^1(n)$. By induction hypothesis, one of the six properties of the lemma holds for $BC^1(n)$. In the following, we find the corresponding paths that satisfy the lemma for $BC(n+1)$.

(1) $Q_{n+1}^7(e) = \langle w_{n+1} = w_n^1, Q_n^{1,1}(e), y_n^1, w_n^2, P_n^{5,2}, x_n^2, z_{n+1}\rangle$; and

$$Q_{n+1}^8(e) = \langle x_{n+1}, z_n^2, (P_n^{6,2})^{-1}, y_n^2 = y_{n+1}\rangle;$$

(2) $Q_{n+1}^2(e) = \langle w_{n+1} = w_n^1, Q_n^{2,1}(e), z_n^1, x_{n+1}, z_n^2, (P_n^{4,2})^{-1}, x_n^2, z_{n+1}\rangle$;

(3) $Q_{n+1}^4(e) = \langle x_{n+1}, z_n^2, (P_n^{2,2})^{-1}, w_n^2, y_n^1, (Q_n^{3,1}(e))^{-1}, x_n^1, z_{n+1}\rangle$;

(4) $Q_{n+1}^3(e) = \langle x_{n+1}, z_n^1, (Q_n^{4,1}(e))^{-1}, x_n^1, z_{n+1}, x_n^2, P_n^{3,2}, y_n^2 = y_{n+1}\rangle$;

(5) $Q_{n+1}^7(e) = \langle w_{n+1} = w_n^1, Q_n^{5,1}(e), x_n^1, z_{n+1}\rangle$; and

$$Q_{n+1}^8(e) = \langle x_{n+1}, z_n^1, (Q_n^{6,1}(e))^{-1}, y_n^1, w_n^2, P_n^{1,2}, y_n^2 = y_{n+1}\rangle;$$ and

(6) $Q_{n+1}^2(e) = \langle w_{n+1} = w_n^1, Q_n^{7,1}(e), z_n^1, x_{n+1}, z_n^2, (P_n^{2,2})^{-1}, w_n^2, y_n^1, (Q_n^{8,1}(e))^{-1}, x_n^1, z_{n+1}\rangle$.

The lemma is proven. □

Theorem 4.1. *The brother tree $BT(n)$ is 1-edge hamiltonian for any positive integer n with $n \geqslant 1$.*

Proof. Note that $BT(1)$ is isomorphic to the hypercube Q_3. Since Q_3 is hamiltonian and edge symmetric, $BT(1)$ is 1-edge hamiltonian. Now we consider $n \geqslant 2$. By definition, $BT(n)$ is composed of an $(n+1)$th brother cell, denoted by $BC^1(n+1) = (G_{n+1}^1, w_{n+1}^1, x_{n+1}^1, y_{n+1}^1, z_{n+1}^1)$ and an nth brother cell, denoted by $BC^2(n) = (G_n^2, w_n^2, x_n^2, y_n^2, z_n^2)$. Let e be any edge of $BT(n)$. By the symmetrical property of $BT(n)$, we assume that e is an edge in G_{n+1}^1. Using Lemma 4.1, $\{P_n^{i,2}\}_{i=1}^8$ exists satisfying the lemma for $BC^2(n)$. Using Lemma 4.2, one of the six properties of Lemma 4.2 holds for $BC^1(n+1)$. In the following, we find the corresponding hamiltonian cycle H_e of $BT(n) - e$.

(1) $H_e = \langle w_{n+1}^1, Q_{n+1}^{1,1}(e), y_{n+1}^1, w_n^2, P_n^{1,2}, y_n^2, w_{n+1}^1\rangle$;

(2) $H_e = \langle w_{n+1}^1, Q_{n+1}^{2,1}(e), z_{n+1}^1, x_n^2, P_n^{3,2}, y_n^2, w_{n+1}^1\rangle$;

(3) $H_e = \langle x_{n+1}^1, Q_{n+1}^{3,1}(e), y_{n+1}^1, w_n^2, P_n^{2,2}, z_n^2, x_{n+1}^1\rangle$;

(4) $H_e = \langle x_{n+1}^1, Q_{n+1}^{4,1}(e), z_{n+1}^1, x_n^2, P_n^{4,2}, z_n^2, x_{n+1}^1\rangle$;

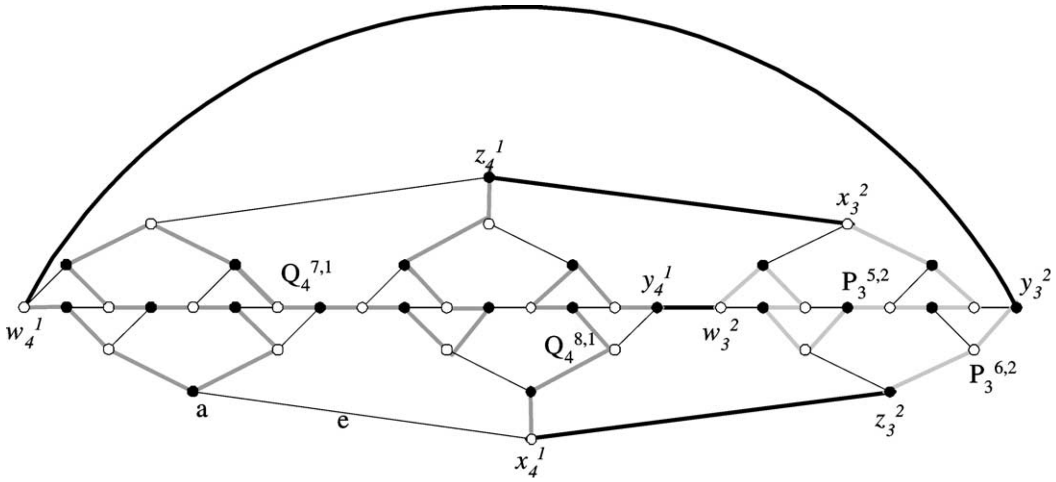

Fig. 3. Illustration of case (6) of Theorem 4.1, where $e=(a,x_4^1)$ is the faulty edge. $Q_4^{7,1}$ is the path joining w_4^1 to z_4^1, $Q_4^{8,1}$ is the path joining x_4^1 to y_4^1, $P_3^{5,2}$ is the path joining w_3^2 to x_3^2, $P_3^{6,2}$ is the path joining y_3^2 to z_3^2.

(5) $H_e=\langle w_{n+1}^1, Q_{n+1}^{5,1}(e), x_{n+1}^1, z_n^2, (P_n^{7,2})^{-1}, w_n^2, y_{n+1}^1, Q_{n+1}^{6,1}(e), z_{n+1}^1, x_n^2, P_n^{8,2}, y_n^2, w_{n+1}^1\rangle$;

(6) $H_e=\langle w_{n+1}^1, Q_{n+1}^{7,1}(e), z_{n+1}^1, x_n^2, (P_n^{5,2})^{-1}, w_n^2, y_{n+1}^1, \left(Q_{n+1}^{8,1}(e)\right)^{-1}, x_{n+1}^1, z_n^2, (P_n^{6,2})^{-1}, y_n^2, w_{n+1}^1\rangle$.

The theorem is proven. An illustration is shown in Fig. 3. □

5. 1_p-hamiltonian

Lemma 5.1. *Assume that n is an integer with $n \geqslant 2$. Let $BC(n)=(G_n, w_n, x_n, y_n, z_n)$. Suppose that c is any node of G_n. There then exists a hamiltonian path $R_n(c)$ of G_n-c such that $R_n(c)$ joins y_n to z_n if c is a white node, and $R_n(c)$ joins w_n to x_n if c is a black node.*

Proof. We prove this lemma by induction. It is easy to check that the lemma holds for $BC(2)$. Assume that the lemma holds for $BC(n)$. By definition, $BC(n+1)$ is composed of two disjoint copies of nth brother cells $BC^1(n)=(G_n^1, w_n^1, x_n^1, y_n^1, z_n^1)$ and $BC^2(n)=(G_n^2, w_n^2, x_n^2, y_n^2, z_n^2)$, a white root x_{n+1}, and a black root z_{n+1}. We only prove the case that c is a black node. Using the symmetrical property of $BC(n+1)$, we may assume that c is a node in $BC^1(n)$ or $c=z_{n+1}$. Using Lemma 4.1, there exists $\{P_n^{i,1}\}_{i=1}^8$ for $BC^1(n)$ if $c \notin V(BC^1(n))$ and $\{P_n^{i,2}\}_{i=1}^8$ for $BC^2(n)$ if $c \notin V(BC^2(n))$.

Suppose that c is in $BC^1(n)$. By induction, there exists a hamiltonian path $R_n^1(c)$ of G_n^1-c that joins w_n^1 to x_n^1. Then, we set $R_{n+1}(c)$ as $\langle w_{n+1}=w_n^1, R_n^1(c), x_n^1, z_{n+1}, x_n^2, P_n^{4,2}, z_n^2, x_{n+1}\rangle$.

Suppose that $c=z_{n+1}$. We set $R_{n+1}(c)$ as $\langle w_{n+1}=w_n^1, P_n^{1,1}, y_n^1, w_n^2, P_n^{2,2}, z_n^2, x_{n+1}\rangle$.

The lemma is proven. □

Lemma 5.2. *Assume that n is an integer with $n \geqslant 2$. Let $BC(n)=(G_n, w_n, x_n, y_n, z_n)$. Let c be a white node of G_n and d be a black node of G_n. Then at least one of the following properties holds.*

(1) *There exists a hamiltonian path $S_n^1(c,d)$ of $G_n-\{c,d\}$ joining w_n to y_n.*
(2) *There exists a hamiltonian path $S_n^2(c,d)$ of $G_n-\{c,d\}$ joining w_n to z_n.*
(3) *There exists a hamiltonian path $S_n^3(c,d)$ of $G_n-\{c,d\}$ joining x_n to y_n.*
(4) *There exists a hamiltonian path $S_n^4(c,d)$ of $G_n-\{c,d\}$ joining x_n to z_n.*
(5) *There exist two disjoint paths $S_n^5(c,d)$ and $S_n^6(c,d)$ such that* (i) *they span $G_n-\{c,d\}$,* (ii) *$S_n^5(c,d)$ joins w_n and x_n, and* (iii) *$S_n^6(c,d)$ joins y_n and z_n.*

(6) *There exist two disjoint paths* $S_n^7(c,d)$ *and* $S_n^8(c,d)$ *such that* (i) *they span* $G_n - \{c,d\}$, (ii) $S_n^7(c,d)$ *joins* w_n *and* z_n, *and* (iii) $S_n^8(c,d)$ *joins* x_n *and* y_n.

Proof. We prove this lemma by induction. It is easy to check that the lemma holds for $BC(2)$. Assume that the lemma holds for $BC(n)$. By definition, $BC(n+1)$ is composed of two disjoint copies of nth brother cells $BC^1(n)$ and $BC^2(n)$, a white root x_{n+1}, and a black root z_{n+1}. Using the symmetrical property of $BC(n+1)$, we may assume that one of the following cases holds:

(1) $c = x_{n+1}$ and $d = z_{n+1}$,
(2) $c = x_{n+1}$ and $d \in V(BC^1(n))$,
(3) $c \in V(BC^1(n))$ and $d = z_{n+1}$,
(4) $c \in V(BC^1(n))$, $d \in V(BC^2(n))$ or
(5) $\{c,d\} \subset V(BC^1(n))$.

In the following, we find the corresponding path(s) for each case.

Case 1: We set $S_{n+1}^1(c,d)$ as $\langle w_{n+1} = w_n^1, P_n^{1,1}, y_n^1, w_n^2, P_n^{1,2}, y_n^2\rangle$.

Case 2: With Lemma 5.1, we set $S_{n+1}^1(c,d)$ as $\langle w_{n+1} = w_n^1, R_n^1(d), x_n^1, z_{n+1}, x_n^2, P_n^{3,2}, y_n^2\rangle$.

Case 3: With Lemma 5.1, we set $S_{n+1}^3(c,d)$ as $\langle x_{n+1}, z_n^1, (R_n^1(c))^{-1}, y_n^1, w_n^2, P_n^{1,2}, y_n^2\rangle$.

Case 4: With Lemma 5.1, we set $S_{n+1}^4(c,d)$ as $\langle x_{n+1}, z_n^1, (R_n^1(c))^{-1}, y_n^1, w_n^2, R_n^2(d), x_n^2, z_{n+1}\rangle$.

Case 5: By induction hypothesis, one of the six properties of the lemma holds for $BC^1(n)$. Note that the endpoints of $S_n^i(c,d)$ are the same as the endpoints of $Q_n^i(e)$ stated in Lemma 4.2. With a similar argument for the case (4) in Lemma 4.2, we can prove that the lemma is true for this case.

Hence, the lemma is proven. □

Theorem 5.1. *The brother tree* $BT(n)$ *is* 1_p*-hamiltonian for any positive integer* n *with* $n \geqslant 1$.

Proof. This theorem can be obtained with a similar argument of Theorem 4.1. □

6. Conclusion

In this paper, we propose a family of bipartite graphs called brother trees, denoted by $BT(n)$. $BT(n)$ is a planar, bipartite, 3-regular graph, and the number of nodes in $BT(n)$ is $6 \cdot 2^n - 4$. Moreover, we prove that $BT(n)$ is optimal 1-edge hamiltonian and 1_p-hamiltonian, and the diameter of $BT(n)$ is $2n+1$.

Let G be a graph with p nodes and with maximal degree d. The famous Moore bound [2] states that the diameter $D(G)$ is at least $\log_{(d-1)} p - 2/d$. Thus the diameter of $BT(n)$ is about 2 times the Moore bound. It is interesting to find other optimal 1-edge hamiltonian and 1_p-hamiltonian bipartite graphs with smaller diameters.

We also note that the complete binary tree is one of the most important architectures for interconnection networks [4]. A lot of complete binary tree variations have been proposed. Because the brother tree is composed of several complete binary trees, we believe that the brother tree is another candidate for interconnection networks.

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References

[1] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1972.
[2] C.N. Hung, L.H. Hsu, T.Y. Sung, Christmas tree: A versatile 1-fault tolerant design for token rings, Inform. Process. Lett. 72 (1999) 55–63.
[3] H.J. Cho, L.Y. Hsu, Ring embedding in faulty honeycomb rectangular torus, Inform. Process. Lett. 84 (5) (2002) 277–284.
[4] F.T. Leighton, Introduction to Parallel Algorithms and Architectures: Arrays · Trees · Hypercubes, Morgan Kaufmann, San Mateo, CA, 1992.
[5] K. Mukhopadhyaya, B.P. Sinha, Hamiltonian graphs with minimum number of edges for fault-tolerant topologies, Inform. Process. Lett. 44 (1992) 95–99.
[6] Y.C. Tseng, Embedding a ring in a hypercube with both faulty links and faulty nodes, Inform. Process. Lett. 59 (1996) 217–222.
[7] J.J. Wang, C.N. Hung, L.H. Hsu, Optimal 1-hamiltonian graphs, Inform. Process. Lett. 65 (1998) 157–161.