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Mathematical Model and Surface Deviation of Helipoid Gears Cut by Shaper Cutters

Crossed-axis helical gears and hypoid gears are two conventional crossed-axis power transmission devices. Helipoid gears, a novel gear proposed herein, possess the merits of the crossed-axis helical and hypoid gears. A mathematical model of the proposed helipoid gear cut by shapers is also derived according to the cutting mechanism and the theory of gearing. The investigation shows that the tooth surface varies with the number of teeth of the shaper. Computer graphs of the helipoid gear are presented according to the developed gear mathematical model, and the tooth surface deviations due to the number of teeth of the shaper are also investigated. [DOI: 10.1115/1.1564570]

Introduction

Helical and hypoid gears are widely used as crossed-axis power transmission devices. Hypoid gears offer a high load capability and a high contact ratio, and are used for rear-axle transmission in automobiles [1]. However, hypoid gears should be manufactured by special machines with various machine-tool settings due to complex tooth surface geometries. A hypoid gear set can obtain good contact positions and contact patterns only with appropriate machine-tool settings. Accordingly, the manufacture of hypoid gear sets requires experienced and well-trained engineers. Only pairs of hypoid gears produced in the same batch mesh well. Once a pinion or gear fails, only replacements from the same batch can be used. Otherwise, the pair of hypoid gears must be replaced. Therefore, the production and maintenance costs of a hypoid gear are relatively high. The manufacture of helical gears requires only easily operated, conventional machines, and the production cost is lower. However, the load capability is generally low due to the limited contact area.

Hypoid gears and spiral bevel gears work on the same principle. A hypoid gear pair becomes a spiral bevel gear pair if the pinion offset of the gear pair is zero. Several mathematical models of, and geometric investigations into hypoid and spiral bevel gears have been made. Litvin and Gutman [2] proposed methods of synthesis and analysis for "Formate" and "Helixform" hypoid gear-drives. Huston and Coy [3] presented a mathematical model for ideal spiral bevel gears. Litvin et al. [4] discussed the determination of tool settings of a tilted head cutter to generate hypoid gears. Litvin et al. [5,6], Zhang and Litvin [7] and Lin et al. [8] proposed methodologies to minimize the surface deviation for real cut gear tooth surfaces and analyze the meshing and contact of such surfaces.

The crossed-axis helical gear is basically a typical helical gear. The crossed helical-axis gears are in point contact and the dimension of the gear contact pattern decreases as the crossed angle increases. The load capacity of the crossed-axis helical gear is therefore low [9]. The advantages of crossed-axis helical gears include insensitivity to variations in axial movement, shaft angle and center distance, and low manufacturing cost [10]. A helical

gear can be produced by hobbing or shaping machines, and mathematical modelling of helical gears generated by the hobbing method has been much studied. Chang et al. [11] proposed a mathematical model of the helical gear generated by CNC hobbing machines, and Tsay [12] proposed a model for that generated by rack cutters. Tsay et al. [13] also presented a mathematical model of spur gears generated by shapers.

A new type of gear, named the helipoid gear, is proposed herein an attempt to achieve a better balance between gear performance and manufacturing cost, than that of hypoid and crossed-axis helical gears. This study aims to develop a mathematical model of the proposed helipoid gear. The helipoid gear is a new type of crossed-axis gearing. The word "helipoid" is a blend of "helical" and "hypoid." Compared with a helical gear pair, if the helipoid gear pair is assembled accurately, the load capacity can be increased. The proposed helipoid gear can be generated by a general gear shaping machine by tilting the axis of the shaper cutter. Thus, the manufacturing cost of the helipoid gear is similar to that of the helical gear and far less than that of the hypoid gear. In addition to developing a novel mathematical model for helipoid gears cut by a shaper cutter, this study investigates the relationship between tooth surface deviations and the number of shaper teeth. This mathematical model can facilitate further investigation including tooth contact, contact ratio, and contact stress analyses.

Generation Concept of Helipoid Gears

A hyperboloid is a ruled surface formed by a straight line rotating about an axis. According to Fig. 1, two mating hyperboloids contact each other along a straight line, which is the so-called axis of screw motion. The two mating hyperboloids in the figure roll about and slide along the axis of screw motion with correlative angular velocities. Thus, these two hyperboloids form a conjugate, kinematic pair, such that under the ideal meshing condition, no transmission error occurs. Based on this concept, a new method for generating the conjugate helipoid gear set cut by a spur shaper cutter is proposed. The axode of the proposed helipoid gear is in a hyperboloid-like shape, which provides a higher load capability than that of the crossed-axis helical gear with a cylindrical axode.

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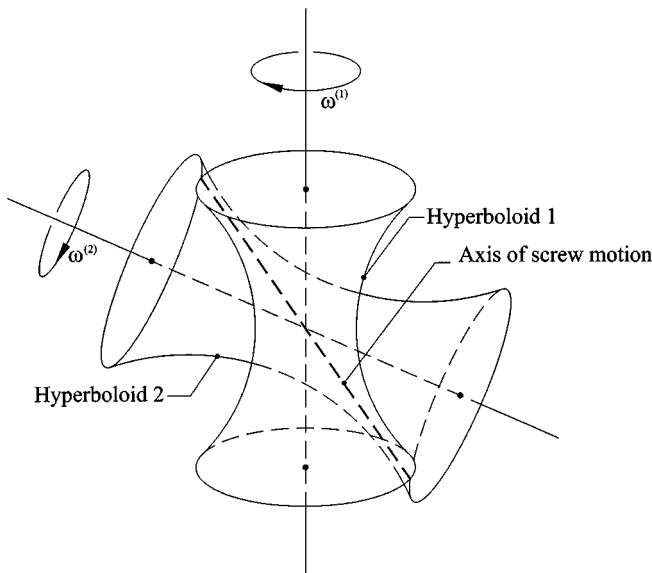


Fig. 1 Two mating hyperboloids contact each other along a straight line

A mathematical model is developed for the helipoid gear according to the aforementioned cutting mechanism to facilitate the related researches.

Generation Mechanism and Kinematic Relation. In this study, the helipoid gear is cut by a spur shaper cutter with an

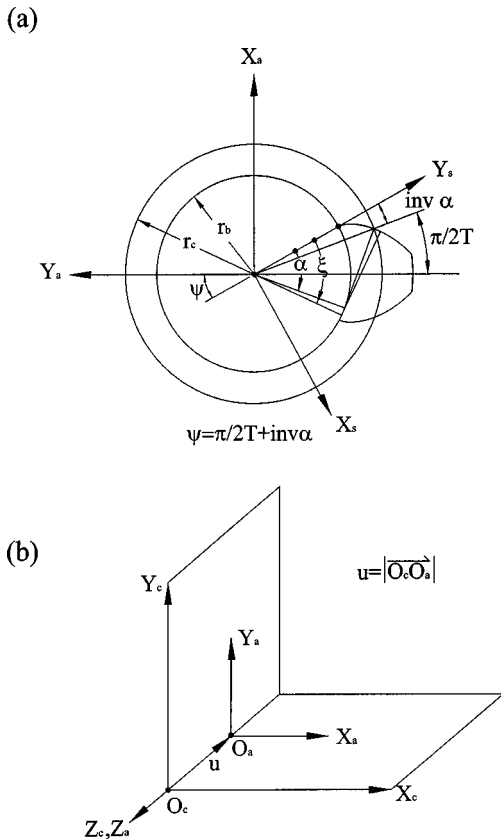


Fig. 2 (a) Cutting edge of the involute-shape shaper cutter and (b) the relationships between the shaper cutter and the normal section

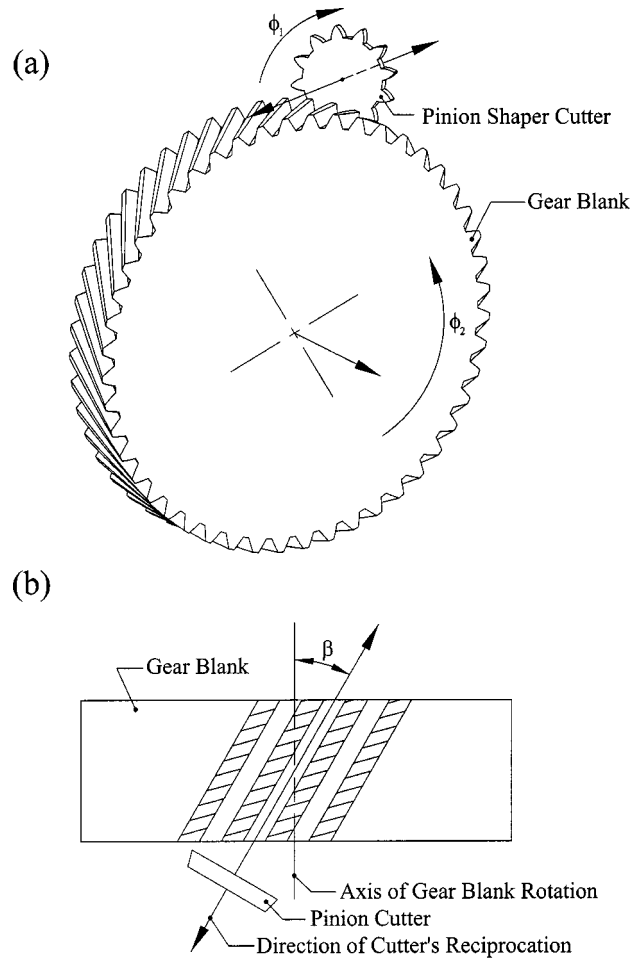


Fig. 3 Cutting mechanism and relationship between the shaper cutter and the gear blank

involute tooth profile, as depicted in Fig. 2(a). Figures 2 and 3 show the cutting mechanism and the relationship between the shaper cutter and the gear blank. The shaper cutting involves a reciprocation along the cutter axis and then a rotation through a small angle about the gear blank axis. While reciprocating the shaper cutter, chips are removed in the forward direction only, and no material is cut in the return stroke. The shaper forward direction and the gear blank rotation axis form an angle β that equal to the gear's helix-angle. After a reciprocating cutting motion, the cutter axis and the gear blank axis rotate through a small, and specific angle to perform the next reciprocating cutting motion. The rotation angle of the gear blank must relate to that of the cutter, and the ratio of the rotation angles is the gear ratio of the gear blank and the shaper cutter. Smaller rotation angles of the cutter and the gear blank yield a more accurate surface of the generated helipoid gear. Referring to Figs. 2(b) and 4, coordinate systems $S_c(X_c, Y_c, Z_c)$, $S_1(X_1, Y_1, Z_1)$ and $S_f(X_f, Y_f, Z_f)$ are attached to the shaper cutter, gear blank and machine housing, respectively. Coordinate system $S_a(X_a, Y_a, Z_a)$ is attached to the cutting edge of the shaper cutter and is shown in Fig. 2(a). During reciprocation, the cutting edge of the shaper cutter moves along the Z_c -axis and forms the shaper cutter surface as u varies.

Equation of the Shaper Cutter

The cutter used to generate helipoid gears is an involute-shape shaper cutter, as depicted in Figs. 2(a) and 3(a). Figure 2(a) shows the cutting edge of the involute-shape shaper cutter which can be expressed in coordinate system $S_s(X_s, Y_s, Z_s)$ by the equation:

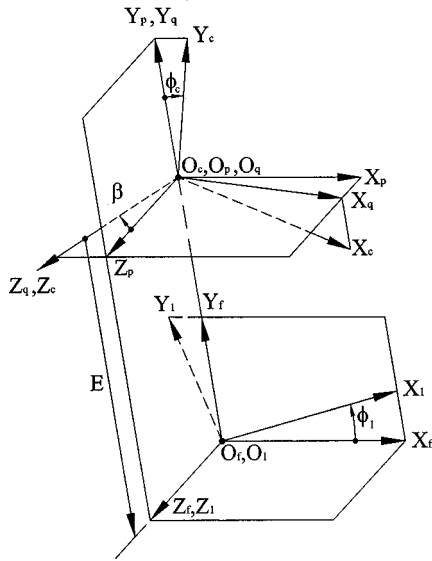


Fig. 4 Coordinate systems between the shaper cutter and generated gear

$$\mathbf{R}_s = \begin{bmatrix} r_b \sin \xi - r_b \xi \cos \xi \\ r_b \cos \xi + r_b \xi \sin \xi \\ 0 \\ 1 \end{bmatrix}, \quad (1)$$

where
$$\psi = \frac{\pi}{2T} + \sin \nu \alpha, \quad (2)$$

$$r_b = r_c \cos \alpha, \quad (3)$$

and r_b and r_c are the radii of the base circle and the pitch circle of the shaper cutter, respectively, ξ is the involute profile parameter, α is the pressure angle of the shaper cutter, and T is the number of teeth on the shaper cutter.

The shaper cutter moves and cuts the gear blank along its forward direction during generation of the helipoid gear. However, only the cutting edge of the shaper cutter cuts the gear blank. Therefore, the shaper cutter used for the generation of helipoid gears can be considered to have a two-dimensional tooth profile which is identical to the cross-section of the spur gear. According to Fig. 2, the equation of the involute-shape shaper cutter represented in coordinate system $S_c(X_c, Y_c, Z_c)$ can be obtained by [10]:

$$\mathbf{R}_c = \mathbf{M}_{ca} \mathbf{M}_{as} \mathbf{R}_s, \quad (4)$$

where
$$\mathbf{M}_{as} = \begin{bmatrix} -\cos \psi & +\sin \psi & 0 & 0 \\ -\sin \psi & -\cos \psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (5)$$

$$\mathbf{M}_{ca} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (6)$$

and \mathbf{R}_s is expressed in Eq. (1). Parameter $\psi = \pi/2T + \sin \nu \alpha$ and $u = |\mathbf{O}_c \mathbf{O}_a|$, where T is the number of teeth of the shaper. Therefore, the position vector of the shaper cutter can be expressed as follows:

$$\mathbf{R}_c = \begin{bmatrix} -r_b \sin(\xi - \psi) + r_b \xi \cos(\xi - \psi) \\ -r_b \cos(\xi - \psi) - r_b \xi \sin(\xi - \psi) \\ u \\ 1 \end{bmatrix}. \quad (7)$$

Meshing Equation of the Pinion Cutter and Gear

According to the cutting mechanism displayed in Fig. 3, the coordinate systems between the shaper cutter and the generated gear can be depicted in Fig. 4. In Fig. 4, coordinate systems $S_c(X_c, Y_c, Z_c)$, $S_1(X_1, Y_1, Z_1)$ and $S_f(X_f, Y_f, Z_f)$ are attached to the shaper cutter, gear blank and machine housing, respectively. Coordinate systems $S_p(X_p, Y_p, Z_p)$ and $S_q(X_q, Y_q, Z_q)$ and reference coordinate systems for simulating the movement of the shaper cutter. ϕ_c and ϕ_1 are rotational angles of the shaper cutter and the gear blank, respectively, during gear generation; β is the helix angle of the helipoid gear; and E is the shortest distance between the shaper cutter and the gear blank rotation axes.

According to the theory of gearing [10], when two surfaces are in point contact, their relative velocity is perpendicular to the common normal vector. Therefore, the following equation must be observed:

$$\mathbf{N}_c \cdot \mathbf{V}_c^{(c1)} = 0 \quad (8)$$

Equation (8) is known as the equation of meshing in the theory of gearing, where \mathbf{N} is the common normal vector, and $\mathbf{V}^{(c1)}$ is the relative velocity of the two mating surfaces at their instantaneous contact point, represented in coordinate system S_c , respectively.

According to the coordinate systems depicted in Fig. 4, the relative velocity can be represented in coordinate system S_c by the following equation:

$$\mathbf{V}_c^{(c1)} = (\boldsymbol{\omega}_c^{(c)} - \boldsymbol{\omega}_c^{(1)}) \times \mathbf{R}_c - \mathbf{E}_c \times \boldsymbol{\omega}_c^{(1)}, \quad (9)$$

where
$$\boldsymbol{\omega}_c^{(1)} = \begin{bmatrix} \omega^{(1)} \sin \beta \cos \phi_c \\ \omega^{(1)} \sin \beta \sin \phi_c \\ \omega^{(1)} \cos \beta \end{bmatrix}, \quad (10)$$

$$\boldsymbol{\omega}_c^{(c)} = \begin{bmatrix} 0 \\ 0 \\ -\omega^{(c)} \end{bmatrix}, \quad (11)$$

$$\mathbf{E}_c = \begin{bmatrix} E \sin \phi_c \\ -E \cos \phi_c \\ 0 \end{bmatrix}, \quad (12)$$

and $\boldsymbol{\omega}_c^{(c)}$ and $\boldsymbol{\omega}_c^{(1)}$ are the angular velocities of the cutter and the gear blank, respectively, in coordinate system S_c . $\omega^{(1)}$ and $\omega^{(c)}$ are the magnitudes of $\boldsymbol{\omega}_c^{(1)}$ and $\boldsymbol{\omega}_c^{(c)}$. \mathbf{E}_c is the vector that describes the shortest distance between the cutter and the gear blank axes, and \mathbf{R}_c is the position vector of the shaper cutter, expressed in coordinate system S_c . Some mathematical manipulation yields the relative velocity:

$$\mathbf{V}_c^{(c1)} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}, \quad (13)$$

where
$$v_x = -u \omega^{(1)} \sin \beta \sin \phi_c - (\omega^{(1)} \cos \beta + \omega^{(c)}) [r_b \cos(\xi - \psi) + r_b \xi \sin(\xi - \psi)] + E \omega^{(1)} \cos \beta \cos \phi_c, \quad (14)$$

$$v_y = u \omega^{(1)} \sin \beta \cos \phi_c - (\omega^{(1)} \cos \beta + \omega^{(c)}) [-r_b \sin(\xi - \psi) + r_b \xi \cos(\xi - \psi)] + E \omega^{(1)} \cos \beta \sin \phi_c, \quad (15)$$

and
$$v_z = \omega^{(1)} \sin \beta \cos \phi_c [r_b \cos(\xi - \psi) + r_b \xi \sin(\xi - \psi)] + \omega^{(1)} \sin \beta \sin \phi_c [-r_b \sin(\xi - \psi) + r_b \xi \cos(\xi - \psi)] - E \omega^{(1)} \sin \beta. \quad (16)$$

The normal vector of the shaper cutter, represented in coordinate system S_c , can be obtained by applying the following equation:

$$\mathbf{N}_c = \frac{\partial \mathbf{R}_c}{\partial u} \times \frac{\partial \mathbf{R}_c}{\partial \xi}. \quad (17)$$

Substituting Eq. (7) into Eq. (17) yields

$$\mathbf{N}_c = \begin{bmatrix} -r_b \xi \cos(\xi - \psi) \\ r_b \xi \sin(\xi - \psi) \\ 0 \end{bmatrix}. \quad (18)$$

Substituting Eqs. (13) and (18) into Eq. (8), gives the equation of meshing of the shaper cutter and the generated helipoid gear as follows:

$$u = \frac{-r_b(\cos \beta + m_{c1}) + E \cos \beta \cos(\xi - \psi + \phi_c)}{\sin \beta \sin(\xi - \psi + \phi_c)}, \quad (19)$$

where
$$m_{c1} = \frac{\omega^{(c)}}{\omega^{(1)}}. \quad (20)$$

Mathematical Model of the Generated Helipoid Gear

According to the theory of gearing, the equation of the generated helipoid gear surface can be attained by simultaneously considering the locus of the shaper cutter and the equation of meshing. Based on the coordinate system relationship shown in Fig. 4, the locus of the shaper cutter can be represented in the gear coordinate system, S_1 , by applying the following homogeneous coordinate transformation matrix equation:

$$\mathbf{R}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \mathbf{M}_{1f} \mathbf{M}_{fp} \mathbf{M}_{pq} \mathbf{M}_{qc} \mathbf{R}_c, \quad (21)$$

where
$$\mathbf{M}_{qc} = \begin{bmatrix} \cos \phi_c & \sin \phi_c & 0 & 0 \\ -\sin \phi_c & \cos \phi_c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (22)$$

$$\mathbf{M}_{pq} = \begin{bmatrix} \cos \beta & 0 & -\sin \beta & 0 \\ 0 & 1 & 0 & 0 \\ \sin \beta & 0 & \cos \beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (23)$$

$$\mathbf{M}_{fp} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & E \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (24)$$

and
$$\mathbf{M}_{1f} = \begin{bmatrix} \cos \phi_1 & \sin \phi_1 & 0 & 0 \\ -\sin \phi_1 & \cos \phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (25)$$

Substituting Eq. (7) into Eqs. (21)–(25) allows us to represent the locus of the shaper cutter in coordinate system S_1 :

$$\begin{aligned} x_1 = & -r_b \sin(\xi - \psi + \phi_c) \cos \beta \cos \phi_1 + r_b \xi \cos(\xi - \psi \\ & + \phi_c) \cos \beta \cos \phi_1 - u \sin \beta \cos \phi_1 - r_b \cos(\xi - \psi \\ & + \phi_c) \sin \phi_1 - r_b \xi \sin(\xi - \psi + \phi_c) \sin \phi_1 + E \sin \phi_1, \end{aligned} \quad (26)$$

Table 1 Some major design parameters of the proposed helipoid gear

	Shaper cutter	Helipoid
Normal modual	3	3
Number of teeth	24	48
Pressure angle	20°	20°
Normal pitch radius	18	72
Helix angle	—	45°
Face width	—	30 mm

$$\begin{aligned} y_1 = & r_b \sin(\xi - \psi + \phi_c) \cos \beta \sin \phi_1 - r_b \xi \cos(\xi - \psi \\ & + \phi_c) \cos \beta \sin \phi_1 + u \sin \beta \sin \phi_1 - r_b \cos(\xi - \psi \\ & + \phi_c) \cos \phi_1 - r_b \xi \sin(\xi - \psi + \phi_c) \cos \phi_1 + E \cos \phi_1, \end{aligned} \quad (27)$$

$$\text{and } z_1 = -r_b \sin(\xi - \psi + \phi_c) \sin \beta + r_b \xi \cos(\xi - \psi + \phi_c) \sin \beta + u \cos \beta. \quad (28)$$

Substituting the equation of meshing, Eq. (19), into Eqs. (26), (27) and (28) yields the equation of the proposed helipoid gear.

Computer Graphs of the Helipoid Gear

Since the mathematical model of the proposed helipoid gear has been developed, the profile of the helipoid gear can be plotted by applying the developed mathematical model and the computer graphing. An example is given here to illustrate the profile of helipoid gears. Table 1 lists some major design parameters of the helipoid gear. Figure 5 presents a three-dimensional computer graph of the helipoid gear, according to the developed mathematical model.

Surface Deviation of Helipoid Gear

The main difference between the proposed helipoid gear and the conventional helical gear generated by shaper cutters is that the tooth surface of the proposed helipoid gear depends on the number of teeth of the shaper cutters. Conventionally, the rotational axis of the helical gear and the movement of the shaper cutter are parallel to each other during the cutting process, and the generated helical gear-tooth surface is independent of the number of teeth of the shaper cutter. For example, a tooth profile with a module of 3 mm/teeth generated by a 24-tooth shaper cutter is the same as that generated by a 48-tooth shaper cutter. However, the generated helipoid gear-tooth surfaces are not identical when the rotational axis of the proposed helipoid gear and the movement direction of the shaper cutter are crossed. The generated helipoid gear-tooth surfaces vary with the number of teeth of the shaper cutters. Figure 6 shows the profile deviations of the proposed helipoid gears generated by 24-tooth, 48-tooth and 960-tooth shaper cutters at cross-sections of $Z = -15$ mm, 0 mm and 15 mm, respectively. This figure shows that the tooth surface stretches out as the number of teeth of the shaper decreases and

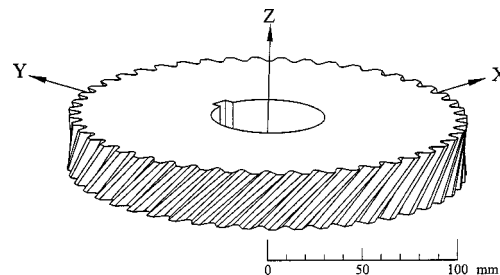


Fig. 5 Computer graph of the helipoid gear

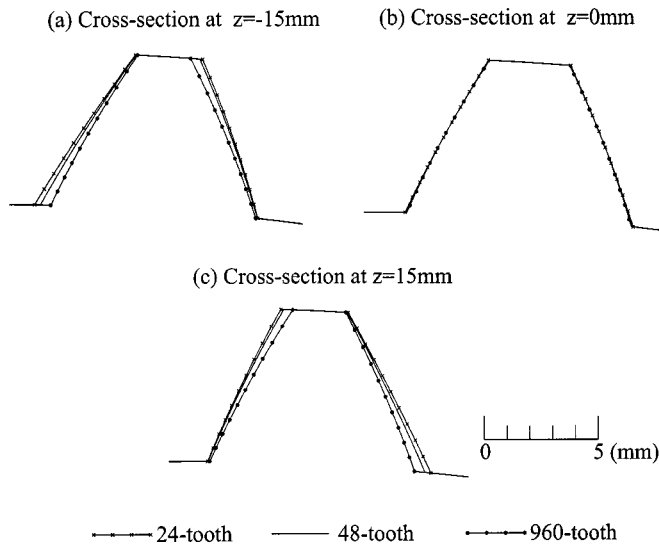


Fig. 6 Different tooth profiles of the helipoid gears generated by 24-tooth, 48-tooth and 960-tooth shaper cutters

that the deviations are large at the cross-sections, $Z = -15$ mm and $Z = 15$ mm. Notably, the tooth profiles generated by shaper cutters with different numbers of teeth are almost identical at the cross-section, $Z = 0$ mm.

Conclusion

This study developed a mathematical model of the proposed helipoid gear cut by a pinion-type shaper cutter based on the gear cutting mechanism and the theory of gearing. The cutting edges of the shaper cutter are identical to involute-shape pinion teeth. While shaping the proposed helipoid gear, the shaper cutter moves and cuts the gear blank along the helix direction, enabling the generated helipoid gear pair to mesh with point contacts. The mathematical model and a computer graph of the helipoid gears are presented. The tooth surface deviations due to the number of teeth of the shaper cutters are also discussed. The results show

that the surface deviation increases as the number of teeth on the shaper cutter decreases, and the deviation is almost zero at the cross-section, $Z = 0$ mm. Some further research into the proposed helipoid gear, such as tooth contact, sensitivity and contact stress analyses can be performed by using the developed mathematical model.

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