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Generalized honeycomb torus ☆

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Abstract

Stojmenovic introduced three different honeycomb tori by adding wraparound edges on honeycomb meshes, namely honeycomb rectangular torus, honeycomb rhombic torus, and honeycomb hexagonal torus. These honeycomb tori have been recognized as an attractive alternative to existing torus interconnection networks in parallel and distributed applications. In this paper, we propose generalized honeycomb tori. The three different honeycomb tori proposed by Stojmenovic are proved to be special cases of our proposed generalized honeycomb tori. We also discuss the Hamiltonian property of some generalized honeycomb tori.

Keywords: Honeycomb torus; Hamiltonian cycle; Interconnection networks

1. Introduction

Network topology is a crucial factor for interconnection network as it determines the performance of the network. Many interconnection network topologies have been proposed before for the purpose of connecting a large number of processing elements. Network topology is always represented by a graph, where nodes represent processors and edges represent links between processors. One of the most popular architectures is the mesh connected computers [4]. Each processor is placed in a square or a rectangular grid and is connected by a communication link to its neighbors in up to four directions.

It is well known that there are three possible tessellations of a plane with regular polygons of the same kind: square, triangular, and hexagonal, corresponding to dividing a plane into regular squares, triangles, and hexagons, respectively. Based on this observation, some computer and communication networks have been built. The square tessellation is the basis for mesh-connected computers. The triangle tessellation is the basis to define hexagonal mesh multiprocessors, studied in [3,9]. The hexagonal tessellation is the basis to define the honeycomb meshes, studied in [2,8].

Tori are meshes with wraparound connections to achieve node and edge symmetry. Meshes and tori are among the most frequent multiprocessor networks available on the market. Stojmenovic [8] introduced three different honeycomb tori by adding wraparound

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edges on honeycomb meshes, namely honeycomb rectangular torus, honeycomb rhombic torus, and honeycomb hexagonal torus. Recently, these honeycomb tori have been recognized as an attractive alternative to existing torus interconnection networks in parallel and distributed applications [5–8].

In this paper, we propose a generalized honeycomb torus. We will prove that all the honeycomb tori mentioned above are special cases of the generalized honeycomb torus. In the following section, we give some graph terms that are used in this paper and the formal definition of various honeycomb tori. Then we present our generalized honeycomb torus. In Section 3, we prove that our generalized honeycomb tori cover all honeycomb tori mentioned above. The Hamiltonian property of some generalized tori are discussed in Section 4. Finally, we give a discussion in the final section.

2. Definitions

For the graph definition and notation we follow [1]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *node set* and E is the *edge set* of G. Two nodes a and b are *adjacent* if $(a, b) \in E$. A *path* is a sequence of nodes such that two consecutive nodes are adjacent. A path P is delimited by $\langle x_0, x_1, x_2, \ldots, x_{n-1}\rangle$. We use P^{-1} to denote the path $\langle x_{n-1}, \ldots, x_2, x_1, x_0\rangle$ if P is the path $\langle x_0, x_1, x_2, \ldots, x_{n-1}\rangle$. A path is called a *Hamiltonian path* if its nodes are distinct and span V. A *cycle* is a path of at least three nodes such that the first node is the same as the last node. A cycle is called a *Hamiltonian cycle* if its nodes are distinct except for the first node and the last node and if they span V. Graph G_1 is *isomorphic* to graph G_2 if there exists a one-to-one mapping ϕ, called an *isomorphism*, from $V(G_1)$ onto $V(G_2)$ such that ϕ preserves adjacency and nonadjacency.

We use the brick drawing, proposed in [7,8], to define the honeycomb rectangular torus. Assume that m and n are positive even integers. The *honeycomb rectangular torus* $\mathrm{HReT}(m, n)$ is the graph with the node set $\{(i, j) \mid 0 \leqslant i < m,\ 0 \leqslant j < n\}$ such that (i, j) and (k, l) are adjacent if they satisfy one of the following conditions:

(1) $i = k$ and $j = l \pm 1 \pmod n$; and
(2) $j = l$ and $k = i - 1 \pmod m$ if $i + j$ is even.

Assume that m and n are positive integers where n is even. The *honeycomb rhombic torus* $\mathrm{HRoT}(m, n)$ is the graph with the node set $\{(i, j) \mid 0 \leqslant i < m,\ 0 \leqslant j - i < n\}$ such that (i, j) and (k, l) are adjacent if they satisfy one of the following conditions:

(1) $i = k$ and $j = l \pm 1 \pmod n$;
(2) $j = l$ and $k = i - 1$ if $i + j$ is even; and
(3) $i = 0$, $k = m - 1$, and $l = j + m$ if j is even.

Assume that n is a positive integer. The *honeycomb hexagonal mesh* $\mathrm{HM}(n)$ is the graph with the node set $\{(x_1, x_2, x_3) \mid -n + 1 \leqslant x_1, x_2, x_3 \leqslant n$ and $1 \leqslant x_1 + x_2 + x_3 \leqslant 2\}$. Two nodes (x_1^1, x_2^1, x_3^1) and (x_1^2, x_2^2, x_3^2) are adjacent if and only $|x_1^1 - x_1^2| + |x_2^1 - x_2^2| + |x_3^1 - x_3^2| = 1$. The *honeycomb hexagonal torus* $\mathrm{HT}(n)$ is the graph with the same node set as $\mathrm{HM}(n)$. The edge set is the union of $E(\mathrm{HM}(n))$ and the wraparound edge set

$$\begin{aligned}&\big\{\big((i, n - i + 1, 1 - n), (i - n, 1 - i, n)\big) \mid 1 \leqslant i \leqslant n\big\}\\ &\quad\cup \big\{\big((1 - n, i, n - i + 1), (n, i - n, 1 - i)\big) \mid \\ &\qquad\qquad 1 \leqslant i \leqslant n\big\}\\ &\quad\cup \big\{\big((i, 1 - n, n - i + 1), (i - n, n, 1 - i)\big) \mid \\ &\qquad\qquad 1 \leqslant i \leqslant n\big\}.\end{aligned}$$

Assume that m and n are positive integers where n is even. Let d be any integer such that $(m - d)$ is an even number. The *generalized honeycomb rectangular torus* $\mathrm{GHT}(m, n, d)$ is the graph with the node set $\{(i, j) \mid 0 \leqslant i < m,\ 0 \leqslant j < n\}$ such that (i, j) and (k, l) are adjacent if they satisfy one of the following conditions:

(1) $i = k$ and $j = l \pm 1 \pmod n$;
(2) $j = l$ and $k = i - 1$ if $i + j$ is even; and
(3) $i = 0$, $k = m - 1$, and $l = j + d \pmod n$ if j is even.

See Fig. 1 for various honeycomb tori. Obviously, any $\mathrm{GHT}(m, n, d)$ is a 3-regular bipartite graph. We can label those nodes (i, j) white when $i + j$ is even or black if otherwise.

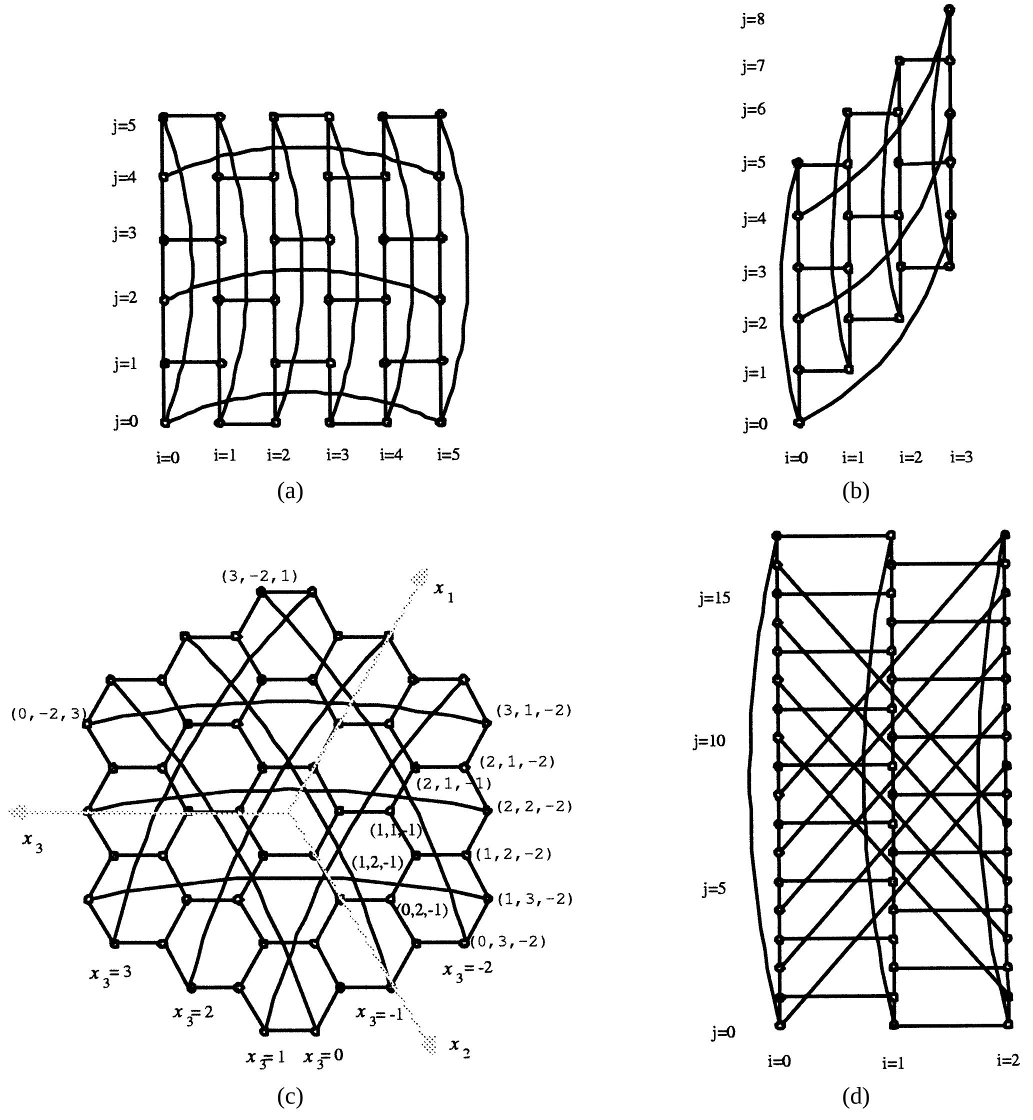


Fig. 1. (a) HReT(6, 6), (b) HRoT(4, 6), (c) HT(3), and (d) GHT(3, 18, 9).

3. Isomorphisms

By definition, we can easily prove that the honeycomb rectangular torus $\mathrm{HReT}(m,n)$ is isomorphic to $\mathrm{GHT}(m,n,0)$ and the honeycomb rhombic torus $\mathrm{HRoT}(m,n)$ is isomorphic to $\mathrm{GHT}(m,n,m \pmod n)$. With the following theorem, the honeycomb hexagonal torus $\mathrm{HT}(n)$ is isomorphic to $\mathrm{GHT}(n,6n,3n)$.

Theorem 1. $\mathrm{HT}(n)$ *is isomorphic to* $\mathrm{GHT}(n,6n,3n)$.

Proof. Let h be the function from the node set of $\mathrm{HT}(n)$ into the node set of $\mathrm{GHT}(n,6n,3n)$ by setting $h(x_1,x_2,x_3) = (x_3, x_1 - x_2 + 2n)$ if $0 \leqslant x_3 < n$, $h(x_1,x_2,x_3) = (0, x_1 - x_2 + 5n \pmod{6n})$ if $x_3 = n$, and $h(x_1,x_2,x_3) = (x_3 + n, x_1 - x_2 + 5n \pmod{6n})$ if otherwise.

For any $1 - n \leqslant c \leqslant n$, we use X_c to denote the set of those nodes (x_1,x_2,x_3) in $\mathrm{HT}(n)$ with $x_3 = c$. We use Y_c to denote the set of nodes (i,j) in $\mathrm{GHT}(n,6n,3n)$ where

(1) $i = c + n$ and $j \in \{k \mid 4n - c - 3 < k < 6n\} \cup \{k \mid 0 \leqslant k < n + c\}$ if $c < 0$,
(2) $i = 0$ and $j \in \{1 \leqslant j < 4n\}$ if $c = 0$,
(3) $i = c$ and $\{j \mid c \leqslant j \leqslant 4n - c\}$ if $0 < c < n$, and
(4) $i = 0$ and $j \in \{k \mid 4n \leqslant k < 6n\} \cup \{0\}$ if $c = n$.

Let h_c denote the function of h induced by X_c. It is easy to check that h_c is a one-to-one function from X_c onto Y_c. Thus, h is one-to-one and onto.

To prove h is an isomorphism, we need to check that h preserves the adjacency. Suppose that $e = ((x_1, x_2, x_3), (x_1', x_2', x_3'))$ be an edge of $\mathrm{HT}(n)$. Without loss of generality, we assume that $x_1 + x_2 + x_3 = 2$ and $x_1' + x_2' + x_3' = 1$.

Suppose that e is an edge of $\mathrm{HM}(n)$. Then either $x_3 = x_3'$ or $x_3 - x_3' = \pm 1$.

Case 1: $x_3 = x_3'$. Obviously, either $(x_1', x_2', x_3') = (x_1 - 1, x_2, x_3)$ or $(x_1', x_2', x_3') = (x_1, x_2 - 1, x_3)$ holds.

Suppose that $0 \leqslant x_3 < n$. Then $h(x_1, x_2, x_3) = (x_3, x_1 - x_2 + 2n)$. Moreover, $h(x_1', x_2', x_3') = (x_3, x_1 - x_2 - 1 + 2n)$ if $(x_1', x_2', x_3') = (x_1 - 1, x_2, x_3)$ and $h(x_1', x_2', x_3') = (x_3, x_1 - x_2 + 1 + 2n)$ if $(x_1', x_2', x_3') = (x_1, x_2 - 1, x_3)$. Suppose that $x_3 = n$. Then $h(x_1, x_2, x_3) = (0, x_1 - x_2 + 5n \pmod{6n})$. Moreover, $h(x_1', x_2', x_3') = (x_3, x_1 - x_2 - 1 + 5n \pmod{6n})$ if $(x_1', x_2', x_3') = (x_1 - 1, x_2, x_3)$ and $h(x_1', x_2', x_3') = (x_3, x_1 - x_2 + 1 + 5n \pmod{6n})$ if $(x_1', x_2', x_3') = (x_1, x_2 - 1, x_3)$. Suppose that $x_3 < 0$. Then $h(x_1, x_2, x_3) = (x_3 + n, x_1 - x_2 + 5n \pmod{6n})$. Moreover, $h(x_1', x_2', x_3') = (x_3, x_1 - x_2 - 1 + 5n \pmod{6n})$ if $(x_1', x_2', x_3') = (x_1 - 1, x_2, x_3)$ and $h(x_1', x_2', x_3') = (x_3, x_1 - x_2 + 1 + 5n \pmod{6n})$ if $(x_1', x_2', x_3') = (x_1, x_2 - 1, x_3)$. Hence, $h(x_1, x_2, x_3)$ and $h(x_1', x_2', x_3')$ are adjacent.

Case 2: $x_3 - x_3' = \pm 1$. Since $x_1 + x_2 + x_3 = 2$ and $x_1' + x_2' + x_3' = 1$, $(x_1', x_2', x_3') = (x_1, x_2, x_3 - 1)$.

Suppose that $1 \leqslant x_3 < n$. Then $h(x_1, x_2, x_3) = (x_3, x_1 - x_2 + 2n)$ and $h(x_1', x_2', x_3') = (x_3 - 1, x_1 - x_2 + 2n)$. Suppose that $x_3 = 0$. Then $h(x_1, x_2, x_3) = (0, x_1 - x_2 + 2n)$ and $h(x_1', x_2', x_3') = (n - 1, x_1 - x_2 + 5n \pmod{6n})$. Suppose that $x_3 = n$. Then $h(x_1, x_2, x_3) = (0, x_1 - x_2 + 5n \pmod{6n})$ and $h(x_1', x_2', x_3') = (n - 1, x_1 - x_2 + 2n)$. Suppose that $2 - n \leqslant x_3 \leqslant -1$. Then $h(x_1, x_2, x_3) = (x_3 + n, x_1 - x_2 + 5n \pmod{6n})$ and $h(x_1', x_2', x_3') = (x_3 + n - 1, x_1 - x_2 + 5n \pmod{6n})$. Hence, $h(x_1, x_2, x_3)$ and $h(x_1', x_2', x_3')$ are adjacent.

Suppose that e is an wraparound edge of $\mathrm{HM}(n)$. Then, we have the following three cases.

Case 3: $e \in \{((i, n - i + 1, 1 - n), (i - n, 1 - i, n)) \mid 1 \leqslant i \leqslant n\}$. Then $(x_1, x_2, x_3) = (i, n - i + 1, 1 - n)$ and $(x_1', x_2', x_3') = (i - n, 1 - i, n)$. Obviously, $h(x_1, x_2, x_3)$ is $(1, 4n + 2i - 1)$ and $h(x_1', x_2', x_3')$ is $(0, 4n + 2i - 1)$. Hence, $h(x_1, x_2, x_3)$ and $h(x_1', x_2', x_3')$ are adjacent.

Case 4: $e \in \{((1 - n, i, n - i + 1), (n, i - n, 1 - i)) \mid 1 \leqslant i \leqslant n\}$. Hence $(x_1, x_2, x_3) = (1 - n, i, n - i + 1)$ and $(x_1', x_2', x_3') = (n, i - n, 1 - i)$. Obviously, $h(x_1, x_2, x_3)$ is $(0, 4n)$ if $i = 1$ and $(n - i + 1, n - i + 1)$ if $1 < i \leqslant n$. Similarly, $h(x_1', x_2', x_3')$ is $(0, 4n - 1)$ if $i = 1$ and $(n - i + 1, n - i)$ if $1 < i \leqslant n$. Thus, $h(x_1, x_2, x_3)$ and $h(x_1', x_2', x_3')$ are adjacent.

Case 5: $e \in \{((i, 1 - n, n - i + 1), (i - n, n, 1 - i)) \mid 1 \leqslant i \leqslant n\}$. Thus $(x_1, x_2, x_3) = (i, 1 - n, n - i + 1)$ and $(x_1', x_2', x_3') = (i - n, n, 1 - i)$. Obviously, $h(x_1, x_2, x_3)$ is $(0, 0)$ if $i = 1$ and $(n - i + 1, 3n + i - 1)$ if $1 < i \leqslant n$. Similarly, $h(x_1', x_2', x_3')$ is $(0, 1)$ if $i = 1$ and $(n - i + 1, 3n + i)$ if $1 < i \leqslant n$. Again, $h(x_1, x_2, x_3)$ and $h(x_1', x_2', x_3')$ are adjacent.

Thus, the theorem is proved. □

For example, the honeycomb torus shown in Fig. 1(c) is actually isomorphic to the generalized honeycomb torus shown in Fig. 1(d).

4. Hamiltonian properties of some generalized honeycomb tori

It is easy to prove that any honeycomb rectangular torus and any honeycomb rhombic torus are Hamiltonian. In [5], it is proved that any honeycomb hexagonal torus is Hamiltonian. We reprove this result with the following theorem.

Theorem 2. *Any generalized honeycomb torus* $\mathrm{GHT}(m, 2k, k)$ *is Hamiltonian.*

Proof. In $\mathrm{GHT}(m, 2k, k)$, let $P(i, j, s)$ denote the path $\langle (i, j), (i, j + 1 \pmod{2k}), (i, j + 2 \pmod{2k}), \ldots, (i, s) \rangle$ and $Q(i, s, j)$ denote the path $P^{-1}(i, j, s)$.

Assume that m is even. By the definition of $\mathrm{GHT}(m, 2k, k)$, k is even. Thus $k = 2r$ for some positive integer r.

Let R denote the path from $(0, 0)$ to $(m - 1, 0)$ defined by:

$$\begin{aligned}
&\big\langle (0,0) \xrightarrow{P(0,0,2r-1)} (0,2r-1),\\
&(1,2r-1) \xrightarrow{Q(1,2r-1,0)} (1,0), \ldots,\\
&(m-3,2r-1) \xrightarrow{Q(m-3,2r-1,0)} (m-3,0),\\
&(m-2,0) \xrightarrow{P(m-2,0,2r-1)} (m-2,2r-1),\\
&(m-1,2r-1) \xrightarrow{Q(m-1,2r-1,0)} (m-1,0)\big\rangle.
\end{aligned}$$

Let S denote the path from $(0,2r)$ to $(m-1,2r)$ defined by:

$$\begin{aligned}
&\big\langle (0,2r) \xrightarrow{P(0,2r,4r-1)} (0,4r-1),\\
&(1,4r-1) \xrightarrow{Q(1,4r-1,2r)} (1,2r), \ldots,\\
&(m-3,4r-1) \xrightarrow{Q(m-3,4r-1,2r)} (m-3,2r),\\
&(m-2,2r) \xrightarrow{P(m-2,2r,4r-1)} (m-2,4r-1),\\
&(m-1,4r-1) \xrightarrow{Q(m-1,4r-1,2r)} (m-1,2r)\big\rangle.
\end{aligned}$$

Obviously, $\langle (0,0) \xrightarrow{R} (m-1,0), (0,2r) \xrightarrow{S} (m-1,2r), (0,0)\rangle$ forms a Hamiltonian cycle for $\mathrm{GHT}(m,2k,k)$. See Fig. 2(a) for illustration.

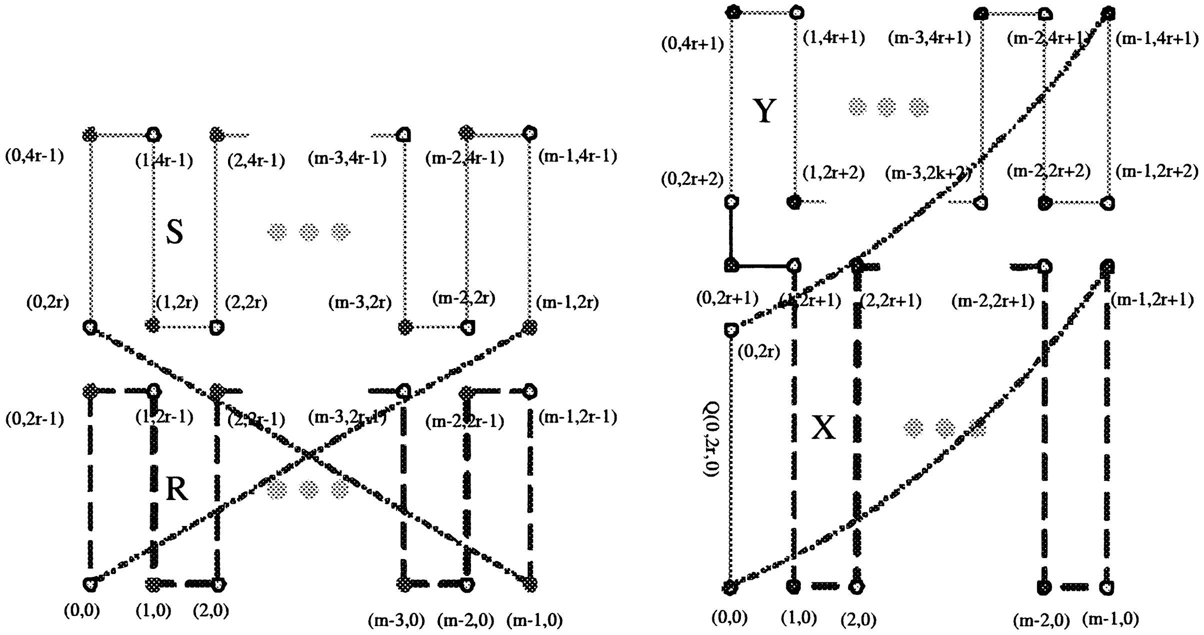
Assume that m is odd. By the definition of $\mathrm{GHT}(m,2k,k)$, k is odd. Suppose that $m=1$. Obviously, $\langle (0,0) \xrightarrow{P(0,0,2k-1)} (0,2k-1), (0,0)\rangle$ forms a Hamiltonian cycle for $\mathrm{GHT}(m,2k,k)$. Thus, we assume that $m>1$ and $k=2r+1$ for some nonnegative integer r.

Let X denote the path from $(m-1,2r+1)$ to $(1,2r+1)$ defined by:

$$\begin{aligned}
&\big\langle (m-1,2r+1) \xrightarrow{Q(m-1,2r+1,0)} (m-1,0),\\
&(m-2,0) \xrightarrow{P(m-2,0,2r+1)} (m-2,2r+1), \ldots,\\
&(3,0) \xrightarrow{Q(3,0,2r+1)} (3,2r+1),\\
&(2,2r+1) \xrightarrow{Q(2,2r+1,0)} (2,0),\\
&(1,0) \xrightarrow{P(1,0,2r+1)} (1,2r+1)\big\rangle.
\end{aligned}$$

Let Y denote the path from $(0,2r+2)$ to $(m-1,4r+1)$ defined by:

$$\begin{aligned}
&\big\langle (0,2r+2) \xrightarrow{P(0,2r+2,4r+1)} (0,4r+1),\\
&(1,4r+1) \xrightarrow{Q(1,4r+1,2r+2)} (1,2r+2), \ldots,\\
&(m-3,2r+2) \xrightarrow{P(m-3,2r+2,4r+1)} (m-3,4r+1),\\
&(m-2,4r+1) \xrightarrow{Q(m-2,4r+1,2r+2)} (m-2,2r+2),\\
&(m-1,2r+2) \xrightarrow{P(m-1,2r+2,4r+1)} (m-1,4r+1)\big\rangle.
\end{aligned}$$



(a) (b)

Fig. 2. Illustrations for Theorem 2.

Obviously,

$$\big\langle(0,0),(m-1,2r+1)\xrightarrow{X}(1,2r+1),(0,2r+1),$$
$$(0,2r+2)\xrightarrow{Y}(m-1,4r+1),(0,2r)\xrightarrow{Q(0,2r,0)}(0,0)\big\rangle$$

forms a Hamiltonian cycle for $\mathrm{GHT}(m,2k,k)$. See Fig. 2(b) for illustration.

The theorem is proved. □

By Theorems 1 and 2, any honeycomb hexagonal torus $\mathrm{HT}(n)$ is Hamiltonian.

5. Discussion

In this paper, we introduced the generalized honeycomb tori. The generalized honeycomb tori include honeycomb rectangular tori, honeycomb rhombic tori, and honeycomb hexagonal tori. We also discussed the Hamiltonian properties of some generalized honeycomb tori. We believe that all generalized honeycomb tori are Hamiltonian.

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