Applied Mathematics and Computation 137 (2003) 303–321

APPLIED MATHEMATICS AND COMPUTATION

www.elsevier.com/locate/amc

# The linear-exponential-quadratic-Gaussian control for discrete systems with application to reliable stabilization

Der-Cherng Liaw *, Chun-Hone Chen

*Department of Electrical and Control Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 30010, Taiwan, ROC*

**Abstract**

In this paper, we derive the discrete linear-exponential-quadratic-Gaussian (LEQG) controller which can take both the system and measurement noise covariances into consideration. Comparing with the traditional linear-quadratic-Gaussian (LQG) design, the LEQG has the wilder design freedom. The proposed discrete LEQG control scheme is then applied to the study of reliable control which can tolerate abnormal operation within some pre-specified set of actuators. This is achieved by suitable modification of the algebraic Riccati equation for the design of the controller. The bounds of gain margins for the feedback control gains of reliable stabilization are also derived. The stability of the overall system is preserved despite the abnormal operation of actuators within a pre-specified subset in the bounds of gain margins.


*Keywords:* Reliable control; Discrete linear-exponential-quadratic-Gaussian control; Kalman filter; Algebraic Riccati equation

## 1. Introduction

The linear-exponential-quadratic-Gaussian (LEQG) method was recently studied for continuous-time systems [1,2], which is mainly based on the assumption of the estimated states approaching the true ones very quickly, and

* Corresponding author.
*E-mail address:* dcliaw@cc.nctu.edu.tw (D.-C. Liaw).


PII: S0096-3003(02)00127-3

the true states in the performance index can then be replaced by the estimated ones. The corresponding state estimation procedure is similar to that of the traditional linear-quadratic-Gaussian (LQG) method using the Kalman filter, while the suboptimal control is also a linear combination of estimated states and can be obtained by solving the Hamilton–Jacobi–Bellman equation.

It is known that the control gain of LQG is deterministic since it is the same as the linear-quadratic-regulator (LQR) method where the LQR system is treated as deterministic. Thus, the LQG design has disadvantage of less responsive in the environment with significant noise. In this paper, the proposed discrete LEQG method is intended to overcome this disadvantage since the design of the control gain will take both system and measurement noise covariances into consideration. Although the derived LEQG design algorithm may be more complicated than that of the LQG method, it is more adaptive to a high-noise environment.

Reliable control is viewed as a means of ensuring system stability against the loss of control components. The basic elements of reliable control design are limited by the existing actuators of the physical system. This is somewhat different from redundant control which increases the number of actuators. In general, reliable control scheme divides the existing control components into two parts: the main controls and the auxiliary ones. For system stability, the main control part of actuators must never fail to operate. However, the auxiliary control part can provide better system stability and performance. The reliable control system is required to be stable even if the auxiliary controls operate abnormally. Although the stability and performance of a reliable control system may not be better than those obtained by standard control design, the reliable control system is guaranteed to be stable while the standard control system may lose its stability when some specified control components function abnormally. This is the tradeoff between simplicity of system integration and risking system instability when actuators may operate abnormally.

In recent years, the design of reliable control laws has attracted considerable attention. The study of reliable linear-quadratic control of nonlinear systems by employing the Hamilton–Jacobi inequality in the nonlinear case has been proposed by Liaw and Liang [3,4]. The designed controllers were shown to be able to tolerate the malfunction within a pre-specified subset of actuators. The gain margin for guaranteeing system stability and the performance bound were estimated. In [5], Veillette proposed linear-quadratic (LQ) state-feedback reliable control laws for continuous-time systems in which the actuator malfunction occurs within a pre-specified subset. In such a design, all the system states are assumed to be available for feedback design. This is generally not true for most practical applications. In many modern control systems, it is unusual to have all the states of a dynamical system available through measurements. Some system states may be impossible or too expensive to measure.

Thus, in this paper we will apply the LEQG control scheme to provide the stability of a discrete system in the presence of abnormal actuators. One of the major goals of this study is to solve the design algorithm for discrete LEQG control. The main concern is that the special structure of the LEQG controller explicitly takes both the system and measurement covariances into consideration while the LQG does not. The discrete LEQG control is then applied to the reliable control design.

The paper is organized as follows. In Section 2, the LEQG design is derived for general discrete-time systems. It is followed by the design procedure for the reliable LEQG control of discrete systems. An example is given in Section 4 to demonstrate the proposed design. Finally, Section 5 summarizes the main results.

## 2. The LEQG control for discrete systems

Consider a class of linear discrete systems as given by

$$x_{k+1} = A_k x_k + B_k u_k + \Gamma_k w_k, \tag{1}$$

with measurement process

$$z_k = H_k x_k + v_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $z_k \in \mathbb{R}^l$. Here, both $w_k$ and $v_k$ denote Gaussian white noise with zero mean. Moreover, we assume that $E\{w_k w_j^{\mathrm{T}}\} = W_k \delta_{kj}$ with $W_k \geqslant 0$, $E\{v_k v_j^{\mathrm{T}}\} = V_k \delta_{kj}$ with $V_k > 0$ and $E\{w_k v_j^{\mathrm{T}}\} = 0$, where $E\{\cdot\}$ denotes an expectation function operator and $\delta_{kj} = 1$ for $k = j$ while $\delta_{kj} = 0$ elsewhere. Before deriving the LEQG control law, the following definitions and notation are recalled and will be used in the paper.

*Definitions and notations:*

Information set [6]:

$$I_k = \{z_0, \ldots, z_k; u_0, \ldots, u_{k-1}\}. \tag{3}$$

Induced information set:

$$I_k = \{\hat{x}_0, \ldots, \hat{x}_k; P_0, \ldots, P_k\}. \tag{4}$$

Priori state estimation:

$$\bar{x}_k = E\{x_k | I_{k-1}\}. \tag{5}$$

Posteriori state estimation:

$$\hat{x}_k = E\{x_k | I_k\}. \tag{6}$$

Priori state estimation error covariance matrix:

$$M_k = E\{(x_k - \bar{x}_k)(x_k - \bar{x}_k)^{\mathrm{T}}\}. \tag{7}$$

Posteriori state estimation error covariance matrix:

$$P_k = E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^{\mathrm{T}}\}. \tag{8}$$

Applying the separation theorem, a Kalman filter can be constructed to produce the optimal estimated state from the noisy measurements. The state estimation equation can be written as

$$\hat{x}_k = \bar{x}_k + K_k s_k, \tag{9}$$

where $s_k$ is defined as the innovation

$$s_k = z_k - H_k \bar{x}_k, \tag{10}$$

with zero mean and covariance

$$S_k = E(s_k s_k^{\mathrm{T}}) = H_k M_k H_k^{\mathrm{T}} + V_k. \tag{11}$$

From Eqs. (8)–(11), we have

$$P_k = (I - K_k H_k) M_k (I - K_k H_k)^{\mathrm{T}} + K_k V_k K_k^{\mathrm{T}}. \tag{12}$$

Minimizing the trace of above estimation error covariance matrix $P_k$, i.e., trace$(P_k)$, with respect to $K_k$ [7], the Kalman gain is obtained as

$$K_k = M_k H_k^{\mathrm{T}} (H_k M_k H_k^{\mathrm{T}} + V_k)^{-1}. \tag{13}$$

Eq. (12) can then be rewritten as

$$P_k = M_k - M_k H_k^{\mathrm{T}} (H_k M_k H_k^{\mathrm{T}} + V_k)^{-1} H_k M_k. \tag{14}$$

From (5), we have

$$\bar{x}_k = A_{k-1} \hat{x}_{k-1} + B_{k-1} u_{k-1}. \tag{15}$$

Thus, the priori state estimation error covariance matrix $M_k$ can be derived as

$$M_k = A_{k-1} P_{k-1} A_{k-1}^{\mathrm{T}} + \Gamma_{k-1} W_{k-1} \Gamma_{k-1}^{\mathrm{T}}. \tag{16}$$

By giving initial values $\hat{x}_0 = \bar{x}_0$ and initial covariance matrix $M_0$, we can calculate the values for all Kalman gain $K_k$. The Kalman filter block diagram is shown in Fig. 1.

The LEQG optimization problem for the discrete system (1) is to minimize the performance index:

$$\mathrm{PI} = E\left\{ \exp\left[\frac{\mu}{2}\Phi\right]\right\}, \tag{17}$$

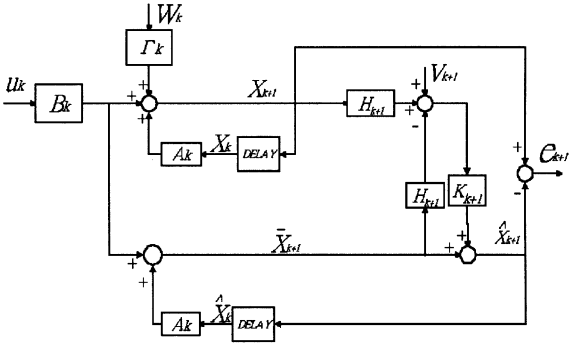


Fig. 1. The Kalman filter block diagram.

where

$$\Phi = \hat{x}_N^{\mathrm{T}} Q_N \hat{x}_N + \sum_{k=0}^{N-1} (\hat{x}_k^{\mathrm{T}} Q_k \hat{x}_k + u_k^{\mathrm{T}} R_k u_k), \tag{18}$$

with $Q_N$ and $Q_k$ being positive semi-definite matrices, $R_k$ a positive definite matrix and $\mu$ a tunable scalar.

From the definitions and derivations above, the LEQG control law can be obtained as in the following Algorithm. Note that, details of the derivation are given in Appendix A.

**Algorithm** (LEQG control law):

If $S_k^{-1} - \mu K_k^{\mathrm{T}} \Theta_k K_k > 0$ for $k = 0, \ldots, N$, then the LEQG control law for the minimization of performance index (17) can be obtained as

$$u_k^* = -C_k \hat{x}_k \quad \text{for } k = 0, \ldots, N-1, \tag{19}$$

where

$$C_k = (R_k + B_k^{\mathrm{T}} \Lambda_{k+1} B_k)^{-1} B_k^{\mathrm{T}} \Lambda_{k+1} A_k, \tag{20}$$

$$\Lambda_k = \Theta_k + \mu \Theta_k K_k (S_k^{-1} - \mu K_k^{\mathrm{T}} \Theta_k K_k)^{-1} K_k^{\mathrm{T}} \Theta_k \tag{21}$$

and

$$\Theta_{k-1} = Q_{k-1} + A_{k-1}^{\mathrm{T}} [\Lambda_k - \Lambda_k B_{k-1} (R_{k-1} + B_{k-1}^{\mathrm{T}} \Lambda_k B_{k-1})^{-1} B_{k-1}^{\mathrm{T}} \Lambda_k] A_{N-1}, \tag{22}$$

with boundary condition $\Theta_N = Q_N$.

Following the algorithm above, the performance index (17) can be simplified as

$$J^* = \alpha_0 \exp\left\{\frac{\mu}{2}\left[\bar{x}_0^{\mathrm{T}} \Lambda_0 \bar{x}_0\right]\right\}, \tag{23}$$

where $\alpha_0$ is solved by the backward recursive process:

$$\alpha_{k-1} = \alpha_k |I - \mu K_k^{\mathrm{T}} \Theta_k K_k S_k|^{-1/2}, \tag{24}$$

with boundary condition $\alpha_N = 1$.

Suppose system (1) is linear time-invariant with $N = \infty$ in (18). Employing an equality property of matrix inverses (see e.g., Appendix A.21 in [8]), Eqs. (20)–(22) can be rewritten as

$$C = (R + B^{\mathrm{T}} \Lambda B)^{-1} B^{\mathrm{T}} \Lambda A, \tag{25}$$

where

$$\Lambda = \Theta + \mu\Theta K(S^{-1} - \mu K^{\mathrm{T}} \Theta K)^{-1} K^{\mathrm{T}} \Theta = (\Theta^{-1} - \mu K S K^{\mathrm{T}})^{-1}, \tag{26}$$

$$\Theta = Q + A^{\mathrm{T}}[\Lambda - \Lambda B(R + B^{\mathrm{T}} \Lambda B)^{-1} B^{\mathrm{T}} \Lambda]A = Q + A^{\mathrm{T}}(\Lambda^{-1} + BR^{-1}B^{\mathrm{T}})^{-1} A \tag{27}$$

and the steady Kalman gain is

$$K = MH^{\mathrm{T}}(HMH^{\mathrm{T}} + V)^{-1}. \tag{28}$$

Combining Eqs. (26) and (27), we have

$$Q = \Theta - A^{\mathrm{T}}(\Theta^{-1} + BR^{-1}B^{\mathrm{T}} - \mu K S K^{\mathrm{T}})^{-1} A, \tag{29}$$

$$\text{i.e., } Q = \Theta - A^{\mathrm{T}} D^{-1} A - \mu A^{\mathrm{T}}[D^{-1} K (S^{-1} - \mu K^{\mathrm{T}} D^{-1} K) K^{\mathrm{T}} D^{-1}]^{-1} A, \tag{30}$$

where

$$D \equiv \Theta^{-1} + BR^{-1}B^{\mathrm{T}}. \tag{31}$$

Let

$$Q_{\mathrm{LEQG}} = Q + \mu A^{\mathrm{T}}[D^{-1} K (S^{-1} - \mu K^{\mathrm{T}} D^{-1} K) K^{\mathrm{T}} D^{-1}]^{-1} A. \tag{32}$$

Thus, we have

$$Q_{\mathrm{LEQG}} = \Theta - A^{\mathrm{T}} D^{-1} A = \Theta - A^{\mathrm{T}}[\Theta - \Theta B(R + B^{\mathrm{T}} \Theta B)^{-1} B^{\mathrm{T}} \Theta]A. \tag{33}$$

Let $P$ be the algebraic Riccati equation (ARE) solution of the standard LQG problem with the same weighting matrices $Q$ and $R$, i.e., $P$ solves the ARE below:

$$Q = P - A^{\mathrm{T}}[P - PB(R + B^{\mathrm{T}} PB)^{-1} B^{\mathrm{T}} P]A. \tag{34}$$

Then the LQG state-feedback control gain is known to be

$$C_{\mathrm{LQG}} = (R + B^{\mathrm{T}} PB)^{-1} B^{\mathrm{T}} PA. \tag{35}$$

Suppose $\mu \geqslant 0$. According to Lemma 4.1 of [10], we have $Q \leqslant Q_{\text{LEQG}}$ which implies $P \leqslant \Theta$. Two properties of this LEQG design can then be summarized as follows.

**Property 1.** *Suppose the LEQG solutions $\Theta$ and $\Lambda$ satisfy Eqs.* (26) *and* (27). *Then $P \leqslant \Theta \leqslant \Lambda$* (*resp. $P \geqslant \Theta \geqslant \Lambda$*) *if the LQG solution P satisfies Eq.* (34) *with the same weighting matrices Q and R with $\mu \geqslant 0$* (*resp. $\mu \leqslant 0$*).

**Property 2.** *The LQG method is a special case of the LEQG control when $\mu$ tends to zero, i.e., $P = \Theta = \Lambda$.*

It is observed from (29) that there exists a positive upper bound $\mu_{\max}$ such that $0 \leqslant BR^{-1}B^{\mathrm{T}} - \mu KSK^{\mathrm{T}}$ for $\mu \leqslant \mu_{\max}$. From Eqs. (25) and (35), Property 1 implies that the LEQG method in general has larger control gain than that of the LQG scheme. Moreover, with the larger control gain, the LEQG control system will have greater immunity to low-frequency environment noise. For simplicity and without loss of generality, we set the LEQG tunable scalar $\mu$ as $0 \leqslant \mu \leqslant \mu_{\max}$ in the remainder of this paper.

The next result follows readily from Theorem 6.5 and Corollary 6.6 of [9], and the discussions above.

**Lemma 1.** *Let $Q = q^T q \geqslant 0$, $R > 0$, $\Gamma W \Gamma^T = FF^T \geqslant 0$ and $V > 0$. If $(A, B)$ is stabilizable, $(q, A)$ is detectable, $(H, A)$ is detectable and $(A, F)$ is stabilizable, then system* (1) *is asymptotically stabilizable by LEQG control. The corresponding control gain C is given in Eq.* (25) *and estimator gain K is in Eq.* (28).

## 3. LEQG scheme for reliable control design

In the following, we apply the LEQG control scheme derived in Section 2 to the stabilization of system (1) subject to the abnormal operation of actuators. Let the control matrix $B$ and the weighting matrix $R$ be decomposed as

$$B = \begin{bmatrix} B_{\Omega'} & B_{\Omega} \end{bmatrix}$$

and

$$R = \begin{bmatrix} R_{\Omega'} & 0 \\ 0 & R_{\Omega} \end{bmatrix}$$

respectively, where $B_{\Omega'}$ corresponds to the normally operating actuators and $B_{\Omega}$ for possible malfunctioning actuators.

First, we consider the worst case of which $B_{\Omega} = 0$, i.e., the minimum number of actuators is under operation. Let the weighting matrices of the cost function in (17) be $Q$ and $R_{\Omega'}$. Denote $\Theta$ the solution of the following ARE:

$$Q = \Theta - A^{\mathrm{T}}(\Theta^{-1} + B_{\Omega'}R_{\Omega'}^{-1}B_{\Omega'}^{\mathrm{T}} - \mu KSK^{\mathrm{T}})^{-1}A. \tag{36}$$

According to that of [11], there exists a unique and positive definite symmetric solution $\Theta$ for Eq. (36) if $(A, B_{\Omega'})$ is stabilizable. Let

$$D_{\Omega'} \equiv \Theta^{-1} + B_{\Omega'}R_{\Omega'}^{-1}B_{\Omega'}^{\mathrm{T}} - \mu KSK^{\mathrm{T}}. \tag{37}$$

Since $\Theta > 0$ and $R_{\Omega'} > 0$, we have $D_{\Omega'} = D_{\Omega'}^{\mathrm{T}} > 0$.

Next, we consider the reliable design of the estimator by checking whether the matrices $\Theta$ and $\Lambda$ can also solve for the new ARE for the system (1) with $B_{\Omega} \neq 0$. Let

$$Q_{\mathrm{rel}} \equiv \Theta - A^{\mathrm{T}}(\Theta^{-1} + BR^{-1}B^{\mathrm{T}} - \mu KSK^{\mathrm{T}})^{-1}A. \tag{38}$$

From Eqs. (36) and (37) and an equality property of matrix inverses, we have

$$Q_{\mathrm{rel}} = Q + A^{\mathrm{T}}[D_{\Omega'}^{-1}B_{\Omega}(R_{\Omega} + B_{\Omega}^{\mathrm{T}}D_{\Omega'}^{-1}B_{\Omega})B_{\Omega}^{\mathrm{T}}D_{\Omega'}^{-1}]A. \tag{39}$$

It is obvious that $Q_{\mathrm{rel}}$ is a symmetric and semi-positive definite matrix. Thus there exists a matrix $q_{\mathrm{rel}}$ such that $Q_{\mathrm{rel}} = q_{\mathrm{rel}}^{\mathrm{T}}q_{\mathrm{rel}}$. In addition, from (37) and (39) we have $Q_{\mathrm{rel}} \geqslant Q$. According to Lemma 4.1 of [10], since $Q_{\mathrm{rel}} \geqslant Q$, we have that $(q_{\mathrm{rel}}, A)$ is a detectable pair if $(q, A)$ is a detectable pair too. The reliable control gain can then be obtained as

$$\begin{aligned} C &= (R + B^{\mathrm{T}}\Lambda B)^{-1}B^{\mathrm{T}}\Lambda A \\ &= \begin{bmatrix} R_{\Omega'} + B_{\Omega'}^{\mathrm{T}}\Lambda B_{\Omega'} & B_{\Omega'}^{\mathrm{T}}\Lambda B_{\Omega} \\ B_{\Omega}^{\mathrm{T}}\Lambda B_{\Omega'} & R_{\Omega} + B_{\Omega}^{\mathrm{T}}\Lambda B_{\Omega} \end{bmatrix}^{-1} \begin{bmatrix} B_{\Omega'}^{\mathrm{T}} \\ B_{\Omega'}^{\mathrm{T}} \end{bmatrix} \Lambda A, \end{aligned} \tag{40}$$

which will also stabilize the closed-loop dynamics of system (1).

We have the next theorem.

**Theorem 1.** *Suppose the conditions of Lemma* 1 *hold. If* $(A, B_{\Omega'})$ *is stabilizable, then the closed-loop dynamics of linear discrete system* (1) *is asymptotically stabilizable in the presence of abnormal operation of actuators and satisfies the LEQG performance criterion as in* (17), *where Q is replaced by* $Q_{\mathrm{rel}}$ *as defined in* (39). *Moreover, the corresponding control gains C is given in Eq.* (40).

Now, we summarize the design procedure as follows:

*Step 1*: Suppose the conditions of Lemma 1 hold and $(A, B_{\Omega'})$ is stabilizable. Solve for the solution $\Theta$ of the ARE (36) with parameters $(A, B_{\Omega'}, Q, R_{\Omega'})$.

*Step 2*: Substitute the solution $\Theta$ obtained from Step 1 to calculate $\Lambda$ in Eq. (26) and the control gain as in (40) with parameters $(A, B, Q, R)$.

Now, we consider the gain margin of the control gain $C$. Suppose $R$ is a diagonal matrix and denote $N^C$ the diagonal matrix corresponding to the magnitude change of the control gain as given by

$$N^C = \begin{bmatrix} N_{\Omega'} & 0 \\ 0 & N_\Omega \end{bmatrix} = \mathrm{diag}(n_{\Omega'1}, n_{\Omega'2}, \ldots, n_{\Omega'r}, n_{\Omega 1}, n_{\Omega 2}, \ldots, n_{\Omega(m-r)}). \tag{41}$$

Multiplying $C$ by $N^C$, the closed-loop dynamics of (1) becomes $x_{k+1} = (A - BN^C C)x_k$. From [12], $A - BN^C C$ is stable if $(A - BN^C C)^{\mathrm{T}} \Lambda (A - BN^C C) - \Lambda \leqslant 0$, where $\Lambda$ in (26) can be obtained by $\Theta$ from solving (36). This leads to the checking of the negative semi-definiteness for $(A - BN^C C)^{\mathrm{T}} \Lambda (A - BN^C C) - \Lambda$ to provide the stability of $A - BN^C C$. Details are given in Appendix B. We then have the following result.

**Theorem 2.** *The matrix $A - BN^C C$ is stable if the gain matrix $N^C$ satisfies either of the following two conditions*:

$$\text{(i)}\ \frac{1}{1+a} < n_{\Omega' i} < \frac{1}{1-a} \quad \textit{and} \quad b n_{\Omega i}^2 + c - (1 - n_{\Omega i})^2 - d(1 - n_{\Omega i})^2 > 0$$

*or*

$$\text{(ii)}\ \frac{1 - \sqrt{b + c - bc}}{1 - b} < n_{\Omega i} < \frac{1 + \sqrt{b + c - bc}}{1 - b} \quad \textit{and}$$
$$a^2 n_{\Omega' i}^2 - (1 - n_{\Omega' i})^2 - f(1 - n_{\Omega' i})^2 > 0,$$

*where the scalars $a$, $b$, $c$, $d$ and $f$ are defined in Eqs.* (B.10)–(B.14).

Note that, $A - BN^C C$ is guaranteed to be stable if both $n_{\Omega' i}$ and $n_{\Omega i}$ statisfy the bounds of Theorem 2. However, if either $n_{\Omega' i}$ or $n_{\Omega i}$ fail to satisfy the bounds of Theorem 2, the eigenvalues of $A - BN^C C$ must be calculated to check the system stability. According to the results of [13], we have $a \to 1$, $b \to 1$, $c \to 1$, $d \to 0$ and $f \to 0$ if the sampling time is sufficiently small. This leads to the unification of the two sufficient conditions of Theorem 2 as given in the next corollary.

**Corollary 1.** *Suppose the sampling time is sufficiently small and the matrices $R$ and $N^C$ are diagonal. Then $A - BN^C C$ is stable if* (*i*) $0.5 < n_{\Omega' i} < \infty$ *for all* $i = 1, \ldots, r$ *and* (*ii*) $0 < n_{\Omega i} < \infty$ *for all* $i = 1, \ldots, m - r$.

Note that, Theorem 2 and Corollary 1 show the stability of $A - B(N^C C)$ about the control gain variation. In fact, they can also be explained as the stability of $A - (BN^C)C$ with respect to the variation of control matrix $B$.

## 4. Illustrative example

In this section, we present the numerical results for the proposed discrete-time LEQG reliable control. Suppose

$$A = \begin{bmatrix} 1 & 0.05 \\ 0 & 1 \end{bmatrix} \tag{42}$$

and

$$B = \begin{bmatrix} B_{\Omega'} & B_{\Omega} \end{bmatrix} = \begin{bmatrix} 0.0025 & 0.0625 \\ 0.1 & 0.5 \end{bmatrix}. \tag{43}$$

Let

$$\Gamma = [0.01 \quad 0.01]^{\mathrm{T}}, \tag{44}$$

$$H = \begin{bmatrix} 0.0025 & 0.1 \\ 0.0625 & 0.5 \end{bmatrix}, \tag{45}$$

$$Q = R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{46}$$

$$W = 1 \tag{47}$$

and

$$V = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}. \tag{48}$$

In the LEQG design, the Kalman gain must be solved firstly. The solution $M$ for the ARE is determined to be

$$M = \begin{bmatrix} 0.0102 & 0.0024 \\ 0.0024 & 0.0017 \end{bmatrix}. \tag{49}$$

The Kalman filter gain can then be calculated as

$$K = \begin{bmatrix} 0.0250 & 0.1741 \\ 0.0168 & 0.0955 \end{bmatrix}. \tag{50}$$

### *4.1. Case of actuators are normal*

First, we consider all actuators function normally with $B = [B_{\Omega'} \; B_{\Omega}]$. For the design of LEQG controller with the LEQG tunable scalar $\mu = 0.02$ in this paper and Kalman gain in (50), we obtain the solutions $\Theta$ and $\Lambda$ for AREs as

$$\Theta = \begin{bmatrix} 20.2428 & 0.0904 \\ 0.0904 & 2.5243 \end{bmatrix} \tag{51}$$

and

$$\Lambda = \begin{bmatrix} 20.2455 & 0.0906 \\ 0.0906 & 2.5244 \end{bmatrix}. \tag{52}$$

The corresponding optimal state feedback control gain for the standard design can hence be calculated as

$$C = \begin{bmatrix} -0.0391 & 0.1522 \\ 0.7668 & 0.7656 \end{bmatrix}. \tag{53}$$

The eigenvalues of the uncontrolled version of system (1) are double roots of 1, which imply the instability of system (1). However, the eigenvalues of the controlled system are found to be 0.9502 and 0.6039. It is obvious that the closed-loop dynamics of system (1) is stabilized by LEQG control gain matrix $C$.

### 4.2. Case of actuators may be abnormal

Now, we study the reliable design for possible abnormal functioning of actuators. Following the design procedure in Section 3 with $B$ and $R$ are replaced by $B_{\Omega'}$ and $R_{\Omega'}$, respectively, we obtain the reliable design solutions $\Theta$ and $\Lambda$ for AREs (36) and (26) as

$$\Theta = \begin{bmatrix} 28.8645 & 10.0494 \\ 10.0494 & 14.6868 \end{bmatrix} \tag{54}$$

and

$$\Lambda = \begin{bmatrix} 28.8723 & 10.0535 \\ 10.0535 & 14.6890 \end{bmatrix}. \tag{55}$$

The corresponding optimal state feedback control gain of the proposed reliable design is attained from (40) as

$$C = \begin{bmatrix} 0.0484 & 0.2886 \\ 1.2546 & 1.4925 \end{bmatrix}. \tag{56}$$

The eigenvalues of the controlled system are found to be $\lambda(A - BC) = 0.9593$ and 0.1870. The scalars in Theorem 2 are calculated to be $a = 0.9317$, $b = 0.1847$ and $c = 0.8937$. From Theorem 2, the bounds of gain margins for $n_{\Omega' i}$ and $n_{\Omega i}$ are depicted in Fig. 2. From Fig. 2, $A - (BN)K_C$ is stable for $n_{\Omega'} = 1$ and $0.0544 < n_{\Omega} < 2.3988$.

Table 1 below shows the reliable controller design with $N = 200$ in (18) and the initial value $\bar{x}_0 = [1\ 0]^{\mathrm{T}}$.

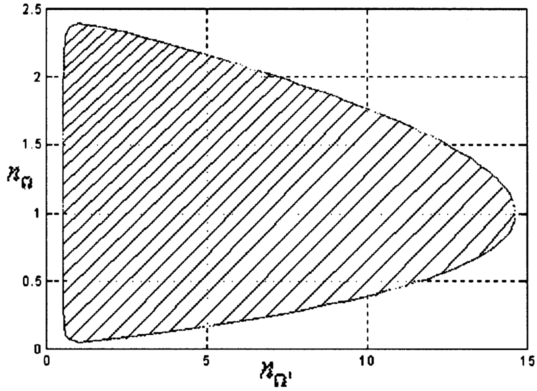


Fig. 2. The stability bounds of $(n_{\Omega'}, n_{\Omega})$.

Table 1
Comparisons of different control designs

| System characteristic | Performance index from (17) | Closed-loop eigenvalues |
|---|---|---|
| $[B_{\Omega'}\ B_{\Omega}]$ (standard design) | | |
| Normal system $[B_{\Omega'}\ B_{\Omega}]$ | 1.2407 | 0.9502, 0.6039 |
| Faulty system with $[B_{\Omega'}\ 0]$ | Not available | 1.0083, 0.9765 |
| $B_{\Omega'}$ only (reliable design) | | |
| Normal system $[B_{\Omega'}\ B_{\Omega}]$ | 1.2514 | 0.9593, 0.1870 |
| Faulty system with $[B_{\Omega'}\ 0]$ | 10.8352 | $0.9855 \pm 0.0057i$ |

From Table 1, the standard LEQG design makes the system to be unstable for $B_{\Omega} = 0$. However, the proposed reliable design still can tolerate the case for $B_{\Omega} = 0$ with the tradeoff for higher performance index.

## 5. Conclusions

In this paper, we have studied the reliable stabilization of discrete-time systems using LEQG approach. A procedure has been derived for the design of reliable discrete LEQG control. The key is to find the ARE solutions $\Theta$ and $\Lambda$ for reliable controller, which maintains system stability despite the abnormal operation of actuators. In additions, the bounds of gain margins for the feedback control is also obtained as in Theorem 2. In contrast to the traditional LQG method, LEQG control design explicitly takes both the system and measurement covariances into consideration. Moreover, the LQG control law was shown to be a special case of the LEQG control when tunable scalar $\mu$ tends to zero.

## Appendix A. Development of LEQG control algorithm

Eqs. (17) and (18) can be rewritten in terms of a nested conditional expectation as

$$\mathrm{PI} = E\Big\{E_{|D_0}\Big\{E_{|D_1}\Big\{ \cdots E_{|D_N}\mu\Big\{ \exp\Big[\frac{\mu}{2}\Phi\Big]\Big\}\cdots\Big\}\Big\}\Big\}. \tag{A.1}$$

Let

$$\Psi(D_k) = E_{|D_k}\Big\{ \exp\Big[\frac{\mu}{2}\Phi\Big]\Big\}. \tag{A.2}$$

A recursive formula for $\Psi(D_k)$ can be obtained from (A.1) and (A.2) as

$$\Psi(D_k) = E_{|D_k}\{\Psi(D_{k+1})\} \quad \text{for } k = 0,\ldots,N-1. \tag{A.3}$$

Moreover,

$$\begin{aligned}\Psi(D_N) &= E_{|D_N}\left\{ \exp\left\{ \frac{\mu}{2}\left[\hat{x}_N^{\mathrm{T}} Q_N \hat{x}_N + \sum_{k=0}^{N-1}(\hat{x}_k^{\mathrm{T}} Q_k \hat{x}_k + u_k^{\mathrm{T}} R_k u_k)\right]\right\}\right\} \\ &= \alpha_N \exp\left\{ \frac{\mu}{2}\left[\hat{x}_N^{\mathrm{T}} \Theta_N \hat{x}_N + \sum_{k=0}^{N-1}(\hat{x}_k^{\mathrm{T}} Q_k \hat{x}_k + u_k^{\mathrm{T}} R_k u_k)\right]\right\},\end{aligned} \tag{A.4}$$

where $\alpha_N \equiv 1$ and $\Theta_N \equiv Q_N$.

From Eqs. (9)–(11), (A.3) and (A.4), we then have

$$\begin{aligned}\Psi(D_{N-1}) &= E_{|D_{N-1}}\left\{ \exp\left\{ \frac{\mu}{2}\left[\hat{x}_N^{\mathrm{T}} \Theta_N \hat{x}_N + \sum_{k=0}^{N-1}(\hat{x}_k^{\mathrm{T}} Q_k \hat{x}_k + u_k^{\mathrm{T}} R_k u_k)\right]\right\}\right\} \\ &= E_{|D_{N-1}}\left\{ \exp\left\{ \frac{\mu}{2}\left[(\bar{x}_N + K_N s_N)^{\mathrm{T}} \Theta_N (\bar{x}_N + K_N s_N) \right.\right.\right. \\ &\qquad \left.\left.\left. + \sum_{k=0}^{N-1}(\hat{x}_k^{\mathrm{T}} Q_k \hat{x}_k + u_k^{\mathrm{T}} R_k u_k)\right]\right\}\right\} \\ &= \exp\left\{ \frac{\mu}{2}\sum_{k=0}^{N-1}(\hat{x}_k^{\mathrm{T}} Q_k \hat{x}_k + u_k^{\mathrm{T}} R_k u_k)\right\} \int_{-\infty}^{\infty} \exp\left\{ \frac{\mu}{2}[(\bar{x}_N + K_N s_N)^{\mathrm{T}}\right.\end{aligned}$$

$$\times \Theta_N(\bar{x}_N + K_N s_N)]\Big\} \frac{1}{[(2\pi)^n |S_N|]^{1/2}} \exp\left\{ -\frac{1}{2} s_N^T S_N^{-1} s_N \right\} \mathrm{d}s_N$$
$$= \exp\left\{ \frac{\mu}{2} \sum_{k=0}^{N-1} (\hat{x}_k^{\mathrm{T}} Q_k \hat{x}_k + u_k^{\mathrm{T}} R_k u_k) \right\} \int_{-\infty}^{\infty} \frac{1}{[(2\pi)^n |S_N|]^{1/2}}$$
$$\times \exp\left\{ -\frac{1}{2} [s_N^{\mathrm{T}} (S_N^{-1} - \mu K_N^{\mathrm{T}} \Theta_N K_N) s_N - \mu \bar{x} N^{\mathrm{T}} \Theta_N \bar{x}_N \right.$$
$$\left. - \mu (K_N s_N)^{\mathrm{T}} \Theta_N \bar{x}_N - \mu \bar{x}_N^{\mathrm{T}} \Theta_N K_N s_N] \right\} \mathrm{d}s_N. \tag{A.5}$$

Suppose $S_N^{-1} > \mu K_N^{\mathrm{T}} \Theta_N K_N$. It is clear that Eq. (A.5) can be rewritten as

$$\Psi(D_{N-1}) = \left[ \frac{|(S_N^{-1} - \mu K_N^{\mathrm{T}} \Theta_N K_N)^{-1}|}{|S_N|} \right]^{1/2} \exp\left\{ \frac{\mu}{2} \sum_{k=0}^{N-1} (\hat{x}_k^{\mathrm{T}} Q_k \hat{x}_k + u_k^{\mathrm{T}} R_k u_k) \right\}$$
$$\times \exp\left\{ -\frac{1}{2} [ - \mu \bar{x}_N^{\mathrm{T}} \Theta_N \bar{x}_N - (\mu K_N^{\mathrm{T}} \Theta_N \bar{x}_N)^{\mathrm{T}} (S_N^{-1} - \mu K_N \Theta_N K_N)^{-1} \right.$$
$$\left. \times (\mu K_N^{\mathrm{T}} \Theta_N \bar{x}_N)] \right\} \left\langle \int_{-\infty}^{\infty} \frac{1}{[(2\pi)^n |(S_N^{-1} - \mu K_N^T \Theta_N K_N)^{-1}|]^{1/2}} \right.$$
$$\times \exp\left\{ -\frac{1}{2} [s_N - (S_N^{-1} - \mu K_N^{\mathrm{T}} \Theta_N K_N)^{-1} (\mu K_N^{\mathrm{T}} \Theta_N \bar{x}_N)]^{\mathrm{T}} \right.$$
$$\left. \left. (S_N^{-1} - \mu K_N^{\mathrm{T}} \Theta_N K_N)[s_N - (S_N^{-1} - \mu K_N^{\mathrm{T}} \Theta_N K_N)^{-1} (\mu K_N^{\mathrm{T}} \Theta_N \bar{x}_N)] \right\} \mathrm{d}s_N \right\rangle. \tag{A.6}$$

Since the integral term in the bracket $\langle \cdot \rangle$ of (A.6) equals one, we then have

$$\Psi(D_{N-1}) = \alpha_N \left[ \frac{|(S_N^{-1} - \mu K_N^{\mathrm{T}} \Theta_N K_N)^{-1}|}{|S_N|} \right]^{1/2} \exp\left\{ \frac{\mu}{2} \sum_{k=0}^{N-1} (\hat{x}_k^{\mathrm{T}} Q_k \hat{x}_k + u_k^{\mathrm{T}} R_k u_k) \right\}$$
$$\times \exp\left\{ \frac{\mu}{2} \bar{x}_N^{\mathrm{T}} [\Theta_N + \mu \Theta_N K_N (S^{-1} - \mu K_N^{\mathrm{T}} \Theta_N K_N)^{-1} K_N^{\mathrm{T}} \Theta_N] \bar{x}_N \right\}. \tag{A.7}$$

Define

$$\Psi(D_{N-1}) \equiv \alpha_{N-1} \exp\left\{ \frac{\mu}{2} \left[ \bar{x}_N^{\mathrm{T}} \Lambda_N \bar{x}_N + \sum_{k=0}^{N-1} (\hat{x}_k^{\mathrm{T}} Q_k \hat{x}_k + u_k^{\mathrm{T}} R_k u_k) \right] \right\}. \tag{A.8}$$

From (A.7) and (A.8), we have

$$\Lambda_N \equiv \Theta_N + \mu \Theta_N K_N (S_N^{-1} - \mu K_N^{\mathrm{T}} \Theta_N K_N)^{-1} K_N^{\mathrm{T}} \Theta_N \tag{A.9}$$

and

$$\alpha_{N-1} \equiv \alpha_N \left[ \frac{|(S_N^{-1} - \mu K_N^{\mathrm{T}} \Theta_N K_N)^{-1}|}{|S_N|} \right]^{1/2}$$
$$= \alpha_N |I - \mu K_N^{\mathrm{T}} \Theta_N K_N S_N|^{-1/2}. \tag{A.10}$$

From Eqs. (15) and (A.8) can be rewritten as

$$\Psi(D_{N-1}) = \alpha_{N-1} \exp\left\{ \frac{\mu}{2} \left[ (A_{N-1}\hat{x}_{N-1} + B_{N-1}u_{N-1})^{\mathrm{T}} \Lambda_N (A_{N-1}\hat{x}_{N-1} + B_{N-1}u_{N-1}) + \sum_{k=0}^{N-1} (\hat{x}_k^{\mathrm{T}} Q_k \hat{x}_k + u_k^{\mathrm{T}} R_k u_k) \right] \right\}. \tag{A.11}$$

Let

$$\mathrm{PI}^* \equiv \min_{u_0,\ldots,u_{N-1}} E\{\Psi(D_{N-1})\} = \min_{u_0,\ldots,u_{N-2}} E\{\min_{u_{N-1}} \Psi(D_{N-1})\}. \tag{A.12}$$

By solving the optimal condition of (A.12) from

$$\frac{\partial \Psi(D_{N-1})}{\partial u_{N-1}^*} = 0, \tag{A.13}$$

we have the following optimal control law:

$$u_{N-1}^* = -(R_{N-1} + B_{N-1}^{\mathrm{T}} \Lambda_N B_{N-1})^{-1} B_{N-1}^{\mathrm{T}} \Lambda_N A_{N-1} \hat{x}_{N-1}. \tag{A.14}$$

Substituting (A.14) into (A.11), we have

$$\mathrm{PI}^*(D_{N-1}) \equiv \alpha_{N-1} \exp\left\{ \frac{\mu}{2} \left[ \hat{x}_{N-1}^{\mathrm{T}} \Theta_{N-1} \hat{x}_{N-1} + \sum_{k=0}^{N-2} (\hat{x}_k^{\mathrm{T}} Q_k \hat{x}_k + u_k^{\mathrm{T}} R_k u_k) \right] \right\}, \tag{A.15}$$

where

$$\Theta_{N-1} = Q_{N-1} + A_{N-1}^{\mathrm{T}} [\Lambda_N - \Lambda_N B_{N-1} (R_{N-1} + B_{N-1}^{\mathrm{T}} \Lambda_N B_{N-1})^{-1} B_{N-1}^{\mathrm{T}} \Lambda_N] A_{N-1}. \tag{A.16}$$

Applying the procedure above recursively backward in time, we will get the feedback control input $u^*$ for each stage.

## Appendix B

Let

$$X \equiv R + B^{\mathrm{T}} \Lambda B \equiv \begin{bmatrix} X_1 & X_2 \\ X_2^{\mathrm{T}} & X_3 \end{bmatrix}, \tag{B.1}$$

where $X_1 = R_{\Omega'} + B_{\Omega'}^{\mathrm{T}} \Lambda B_{\Omega'}$, $X_2 = B_{\Omega'}^{\mathrm{T}} \Lambda B_{\Omega}$ and $X_3 = R_{\Omega} + B_{\Omega}^{\mathrm{T}} \Lambda B_{\Omega}$. We have

$$\begin{aligned}
&(A - BN^C C)^{\mathrm{T}} \Lambda (A - BN^C C) - \Lambda \\
&\quad = -Q + \Theta - \Lambda + A^{\mathrm{T}} \Lambda B_{\Omega'} (R_{\Omega'} + B_{\Omega'}^{\mathrm{T}} \Lambda B_{\Omega'})^{-1} B_{\Omega'}^{\mathrm{T}} \Lambda A \\
&\qquad - A^{\mathrm{T}} \Lambda B N^C C - C^{\mathrm{T}} N^C B^{\mathrm{T}} \Lambda A + C^{\mathrm{T}} N^C B^{\mathrm{T}} \Lambda B N^C C \\
&\quad = -Q - \mu \Theta K (S^{-1} - \mu K^{\mathrm{T}} S K)^{-1} K^{\mathrm{T}} \Theta + A^{\mathrm{T}} \Lambda B \begin{bmatrix} I \\ 0 \end{bmatrix} \\
&\qquad \times \left\{ [I \ 0](R + B^{\mathrm{T}} \Lambda B) \begin{bmatrix} I \\ 0 \end{bmatrix} \right\}^{-1} [I \ 0] B^{\mathrm{T}} \Lambda A - C^{\mathrm{T}} (R + B^{\mathrm{T}} \Lambda B) N^C C \\
&\qquad - C^{\mathrm{T}} N^C (R + B^{\mathrm{T}} \Lambda B) C + C^{\mathrm{T}} N^C (R + B^{\mathrm{T}} \Lambda B) N^C C - C^{\mathrm{T}} N^C R N^C C \\
&\quad \equiv -Q^C - C^{\mathrm{T}} Y C,
\end{aligned} \tag{B.2}$$

where

$$Q^C \equiv Q + \mu \Theta K (S^{-1} - \mu K^{\mathrm{T}} S K)^{-1} K^{\mathrm{T}} \Theta \geqslant 0, \tag{B.3}$$

$$Y \equiv \begin{bmatrix} Y_1 & Y_2 \\ Y_2^{\mathrm{T}} & Y_3 \end{bmatrix} \tag{B.4}$$

and

$$Y_1 \equiv N_{\Omega'} R_{\Omega'} N_{\Omega'} - (I - N_{\Omega'}) X_1 (I - N_{\Omega'}), \tag{B.5}$$

$$Y_2 \equiv -(I - N_{\Omega'}) X_2 (I - N_{\Omega}), \tag{B.6}$$

$$Y_3 \equiv X_3 - X_2^{\mathrm{T}} X_1^{-1} X_2 + N_{\Omega} R_{\Omega} N_{\Omega} - (I - N_{\Omega}) X_3 (I - N_{\Omega}). \tag{B.7}$$

It is known (e.g. Gajic and Qureshi [12]) that $A - BN^C C$ is stable if $(A - BN^C C)^{\mathrm{T}} \Lambda (A - BN^C C) - \Lambda \leqslant 0$. Thus, from (B.2) the matrix $A - BN^C C$ is stable if $Y > 0$.

By employing elementary row and column operations on matrix $Y$, we have

$$\begin{bmatrix} I & -Y_2 Y_3^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_2^{\mathrm{T}} & Y_3 \end{bmatrix} \begin{bmatrix} I & 0 \\ -Y_3^{-1} Y_2^{\mathrm{T}} & I \end{bmatrix} = \begin{bmatrix} Y_1 - Y_2 Y_3^{-1} Y_2^{\mathrm{T}} & 0 \\ 0 & Y_3 \end{bmatrix}. \tag{B.8}$$

It is obvious from (B.8) that the matrix $Y > 0$ if $Y_3 > 0$ and $Y_1 - Y_2 Y_3^{-1} Y_2^{\mathrm{T}} > 0$. Similarly, we have

$$\begin{bmatrix} I & 0 \\ -Y_2^{\mathrm{T}} Y_1^{-1} & I \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_2^{\mathrm{T}} & Y_3 \end{bmatrix} \begin{bmatrix} I & -Y_1^{-1} Y_2 \\ 0 & I \end{bmatrix} = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_3 - Y_2^{\mathrm{T}} Y_1^{-1} Y_2 \end{bmatrix}. \tag{B.9}$$

This leads to the result of $Y > 0$ if $Y_1 > 0$ and $Y_3 - Y_2^{\mathrm{T}} Y_1^{-1} Y_2 > 0$. Now, we try to realize the two sufficient conditions above for providing $Y > 0$. Let

$$a = [\lambda_{\min}(R_{\Omega'})/\lambda_{\max}(X_1)]^{1/2}, \tag{B.10}$$

$$b = \lambda_{\min}(R_{\Omega})/\lambda_{\max}(X_3), \tag{B.11}$$

$$c = \lambda_{\min}(X_3 - X_2^{\mathrm{T}} X_1^{-1} X_2)/\lambda_{\max}(X_3), \tag{B.12}$$

$$d = \lambda_{\max}\{X_2^{\mathrm{T}}[R_{\Omega'}N_{\Omega'}^2 - X_1(I - N_{\Omega'})^2]^{-1} X_2 (I - N_{\Omega'})^2\}/\lambda_{\max}(X_3) \tag{B.13}$$

and

$$f = \lambda_{\max}\{X_2[(X_3 - X_2^{\mathrm{T}} X_1^{-1} X_2) + R_{\Omega}N_{\Omega}^2 - X_3(I - N_{\Omega})^2]^{-1} X_2^{\mathrm{T}} (I - N_{\Omega})^2\} /\lambda_{\max}(X_1). \tag{B.14}$$

Here $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the maximum and minimum eigenvalues of a matrix, respectively.

*Case* 1: (Conditions for $Y_1 > 0$ and $Y_3 - Y_2^{\mathrm{T}} Y_1^{-1} Y_2 > 0$)

First, we consider the condition for $Y_1 > 0$. Since $0 < a < 1$, from (B.5) we then have $Y_1 > 0$ if

$$\lambda_{\min}(R_{\Omega'})N_{\Omega'}^2 > \lambda_{\max}(X_1)(I - N_{\Omega'})^2.$$

Using (B.10), we have $Y_1 > 0$ if

$$[(1 - a)N_{\Omega'} - I][(1 + a)N_{\Omega'} - I] < 0.$$

This leads to the result of $Y_1 > 0$ if

$$\frac{1}{1+a} I < N_{\Omega'} < \frac{1}{1-a} I,$$

$$\text{i.e., } \frac{1}{1+a} < n_{\Omega' i} < \frac{1}{1-a} \quad \text{for all } i = 1, \ldots, r.$$

Next, we solve for the condition $Y_3 - Y_2^{\mathrm{T}} Y_1^{-1} Y_2 > 0$.

It is observed from (B.5)–(B.7) that

$$Y_3 - Y_2^{\mathrm{T}} Y_1^{-1} Y_2 = R_{\Omega}N_{\Omega}^2 + (X_3 - X_2^{\mathrm{T}} X_1^{-1} X_2) - X_3(I - N_{\Omega}^2) - (I - N_{\Omega}) \times X_2^{\mathrm{T}}(I - N_{\Omega'})[R_{\Omega'}N_{\Omega'}^2 - X_1(I - N_{\Omega'})^2]^{-1}(I - N_{\Omega'})X_2(I - N_{\Omega}).$$

We then have $Y_3 - Y_2^{\mathrm{T}} Y_1^{-1} Y_2 > 0$ if

$$\lambda_{\min}(R_{\Omega})N_{\Omega}^2 + \lambda_{\min}(X_3 - X_2^{\mathrm{T}} X_1^{-1} X_2) - \lambda_{\max}(X_3)(I - N_{\Omega}^2) - \lambda_{\max}\{X_2^{\mathrm{T}}[R_{\Omega'}N_{\Omega'}^2 - X_1(I - N_{\Omega'})^2]^{-1} X_2(I - N_{\Omega'})^2\}(I - N_{\Omega})^2 > 0.$$

Using the notation defined in (B.11)–(B.13), we have $Y_3 - Y_2^{\mathrm{T}} Y_1^{-1} Y_2 > 0$ if

$$bN_{\Omega}^2 + cI - (I - N_{\Omega})^2 - d(I - N_{\Omega})^2 > 0.$$

From (41), the sufficient condition for $Y_3 - Y_2^{\mathrm{T}} Y_1^{-1} Y_2 > 0$ can be rewritten as

$$bn_{\Omega i}^2 + c - (1 - n_{\Omega i})^2 - d(1 - n_{\Omega i})^2 > 0 \quad \text{for all } i = 1, \ldots, m - r.$$

*Case* 2: (Conditions for $Y_3 > 0$ and $Y_1 - Y_2 Y_3^{-1} Y_2^{\mathrm{T}} > 0$)

Similarly, since $0 < b < 1$ and $0 < c < 1$, we have $Y_3 > 0$ if

$$\lambda_{\min}(R_\Omega) N_\Omega^2 + \lambda_{\min}(X_3 - X_2^{\mathrm{T}} X_1^{-1} X_2) > \lambda_{\max}(X_3)(I - N_\Omega)^2.$$

That is, we have $Y_3 > 0$ if

$$[(1 - b)N_\Omega - (1 - \sqrt{b + c - bc})I][(1 - b)N_\Omega - (1 + \sqrt{b + c - bc})I] < 0.$$

The sufficient condition for $Y_3 > 0$ can also be rewritten as

$$\frac{1 - \sqrt{b + c - bc}}{1 - b} I < N_\Omega < \frac{1 + \sqrt{b + c - bc}}{1 - b} I,$$

$$\text{i.e., } \frac{1 - \sqrt{b + c - bc}}{1 - b} < n_{\Omega i} < \frac{1 + \sqrt{b + c - bc}}{1 - b} \quad \text{for all } i = 1, \ldots, m - r.$$

From (B.5)–(B.7),

$$\begin{aligned} Y_1 - Y_2 Y_3^{-1} Y_2^{\mathrm{T}} &= R_{\Omega'} N_{\Omega'}^2 - X_1 (I - N_{\Omega'})^2 - (I - N_{\Omega'}) X_2 (I - N_\Omega) \\ &\quad \times [(X_3 - X_2^{\mathrm{T}} X_1^{-1} X_2) + R_\Omega N_\Omega^2 - X_3 (I - N_\Omega)^2]^{-1} (I - N_\Omega) X_2^{\mathrm{T}} (I - N_\Omega). \end{aligned}$$

Thus, we have $Y_1 - Y_2 Y_3^{-1} Y_2^{\mathrm{T}} > 0$ if

$$\begin{aligned} &\lambda_{\min}(R_{\Omega'}) N_{\Omega'}^2 - \lambda_{\max}(X_1)(I - N_{\Omega'})^2 - \lambda_{\max}\{X_2[(X_3 - X_2^{\mathrm{T}} X_1^{-1} X_2) + (R_\Omega) N_\Omega^2 \\ &\quad - X_3 (I - N_\Omega)^2]^{-1} X_2^{\mathrm{T}} (I - N_\Omega)^2\}(I - N_{\Omega'})^2 > 0. \end{aligned}$$

From (B.10) and (B.14), the above condition can be simplified as

$$a^2 N_{\Omega'}^2 - (I - N_{\Omega'})^2 - f(I - N_{\Omega'})^2 > 0.$$

$$\text{i.e., } a^2 n_{\Omega' i}^2 - (1 - n_{\Omega' i})^2 - f(1 - n_{\Omega' i})^2 > 0 \quad \text{for all } i = 1, \ldots, r.$$

Results of Theorem 2 are hence implied.

**Acknowledgements**

This research was supported by the National Science Council, Taiwan, R.O.C. under Grants NSC 90-2212-E-009-067, NSC 90-2213-E-009-102, and by the Ministry of Education, Taiwan, R.O.C. under Grant EX-91-E-FA06-4-4.

## References


[1] J.-M. Lin, S.-W. Lee, Bank-to-turn optimal guidance with linear exponential quadratic Gaussian performance criterion, AIAA J. Guidance, Control Dyn. 18 (1995) 951–958.

[2] J.-M. Lin, System designed by linear exponential quadratic Gaussian and loop transfer recovery methodology, AIAA J. Guidance Control Dyn. 18 (1995) 189–192.

[3] D.-C. Liaw, Y.-W. Liang, T.-C. Lee, Reliable control of nonlinear systems, IEEE Trans. Auto. Contr. 45 (4) (2000) 706–710.

[4] D.-C. Liaw, Y.-W. Liang, Quadratic polynomial solutions of the Hamilton–Jacobi inequality in reliable control design, IEICE Trans. Fundam. Electron., Commun. Comput. Sci. E81-A (9) (1998) 153–160.

[5] R.J. Veillette, Reliable linear-quadratic state-feedback control, Automatica 31 (1995) 137–143.

[6] R.F. Stengel, Stochastic Optimal Control, John Wiley and Sons, New York, 1986.

[7] R.G. Brown, P.Y.C. Hwang, Introduction to Random Signals and Applied Kalman Filtering, John Wiley and Sons, New York, 1997.

[8] T. Kailath, Linear System, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[9] G. Chen, G. Chen, S.H. Hsu, Linear Stochastic Control System, CRC Press, Boca Raton, FL, 1995.

[10] R.R. Bitmead, M. Gevers, V. Wertz, Adaptive Optimal Control, Prentice-Hall, Englewood Cliff, NJ, Australia Pty. Ltd., 1990 (Chapter 4).

[11] S. Bittanti, A.J. Laub, J.C. Willems, The Riccati Equation, Springer-Verlag, Berlin, Heidelberg, 1991 (Chapter 10).

[12] Z. Gajic, M.T.J. Qureshi, Lyapunov Matrix Equation in System Stability and Control, Academic Press, San Diego, CA, 1995 (Chapter 1).

[13] M.G. Safanov, Stability and Robustness of Multivariable Feedback Systems, MIT Press, Cambridge, MA, 1980 (Chapter 4).