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Robust control of non-linear affine systems

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Abstract

Issue of robust control for non-linear affine systems with uncertainties appearing in both drift and non-drift terms are presented. Based on Lyapunov function approach, control laws are proposed to guarantee uniformly asymptotic stability of the equilibrium point. To facilitate the design and simplify the checking procedure, stabilization control for the uncertain systems possess asymptotic stabilizable nominal time-invariant driftness terms are proposed for the demonstration of robust design.

Keywords: Robust control; Uncertain systems; Lyapunov functions; Matched and mismatched uncertainties

1. Introduction

In the recent years, robust stabilization of non-linear systems have been widely discussed (see e.g. [1,3–5,8,10–13,15]). For instance, Gutman [5] developed discontinuous min–max controllers to asymptotically stabilize matched-type uncertain dynamics. Corless and Leitmann [4] employed the same approach to design continuous state feedback controllers for guaranteeing uniform ultimate boundedness of matched-type uncertain system trajectories. Barmish et al. [1] introduced the concept of practical stabilizability and proposed stabilizing controllers for systems with matched-type uncertainties via Lyapunov stability. Using a differential-geometric approach, Kravaris and

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Palanki [8] designed a class of stabilizing controllers for matched-typed uncertain dynamical systems. Qu [11] introduced the concept of equivalently matched uncertainties and investigated global asymptotic stabilization of a class of non-linear dynamical systems with so-called "equivalently matched uncertainties". He also employed the backstepping design method to the robust control of the uncertain systems with *generalized matching condition* [12]. Wang et al. [15] employed differential geometric feedback linearization to deal with robust stabilization of uncertain systems with mismatched time-varying uncertainties.

The main goal of this paper is to study the robust control of non-linear affine systems. The uncertainties are supposed to appear in both drift and driftless terms of the system dynamics. Those uncertainties appearing in the drift term are assumed to be in equivalently matched-type as in [11], however, those appearing in the input-related matrix are allowed to be more general than those in [11]. Moreover, in this paper we study both local and global cases. To facilitate the proposed design, we also study the robust control of the uncertain systems without assuming the stability of the nominal drift part. This might give a guide to the construction of Lyapunov function for the implementation of control laws.

The paper is organized as follows. In Section 2, we briefly introduce the considered uncertain systems and some basic assumptions of [11]. An assumption is then introduced to relax the requirements of the equivalently matched-type uncertainties appearing in driftless part. It is followed by the design of control laws for the uncertain system having asymptotic stabilizable nominal driftless part. An illustrative example is also given to demonstrate the use of the main results. Finally, Section 3 gives the conclusions.

2. Robust stabilization of the uncertain systems

Consider a class of non-linear affine systems with uncertainties as given by

$$\dot{x}(t) = f(x(t),t) + \Delta f(x(t),q(t),t) + \{g(x(t),t) + \Delta g(x(t),q(t),t)\}u \tag{1}$$

with $x(t_0) = x_0$. Here, $t \in \mathbb{R}$ denotes time, $x(t) \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector, $q(t) \in \mathbb{R}^p$ is the uncertainty and $f(\cdot,\cdot)$, $\Delta f(\cdot,\cdot,\cdot)$, $g(\cdot,\cdot)$ and $\Delta g(\cdot,\cdot,\cdot)$, respectively, are known vectors and matrix functions with appropriate dimensions. We decompose the uncertainties into matched and mismatched parts (for definition, see e.g., [1]) as

$$\Delta f(x,q,t) = g(x,t)\Delta f_m(x,q,t) + \Delta f_{\bar{m}}(x,q,t) \tag{2}$$

and

$$\Delta g(x,q,t) = g(x,t)\Delta g_m(x,q,t) + \Delta g_{\bar{m}}(x,q,t). \tag{3}$$

For system (1), Assumptions 1 and 2 below are introduced to guarantee the existence of classical solution.

Assumption 1. The uncertainty $q(\cdot) : \mathbb{R} \to \mathbb{R}^p$, is Lebesgue measurable and its values $q(t)$ lie within a pre-specified compact set $Q \subset \mathbb{R}^p$ for all $t \in \mathbb{R}$.

Assumption 2. The functions $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $\Delta f(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^n$ and $\Delta g(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^{n\times m}$ are all continuous.

Note that, it is known that continuous and piecewise continuous functions are two kinds of Lebesgue measurable functions (e.g., [14]). Regarding the nominal system and the uncertainties, [11] introduced the following three assumptions:

Assumption 3. The origin of the uncontrolled nominal model $\dot{x} = f(x,t)$ of system (1) is locally uniformly asymptotically stable. In particular, there exists a neighborhood Ω_x, a smooth function (i.e., continuously differentiable function) $V : \Omega_x \times \mathbb{R} \to \mathbb{R}^+$ and continuous, strictly increasing functions $\gamma_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1,2,3$, with

$$\gamma_i(0) = 0, \quad i = 1,2,3 \tag{4}$$

$$\lim_{r\to\infty} \gamma_i(r) = \infty, \quad i = 1,2 \tag{5}$$

such that for all $(x,t) \in \Omega_x \times \mathbb{R}$,

$$\gamma_1(\|x\|) \leqslant V(x,t) \leqslant \gamma_2(\|x\|) \tag{6}$$

and

$$\frac{\partial V(x,t)}{\partial t} + \nabla_x^{\mathrm{T}} V(x,t) f(x,t) \leqslant -\gamma_3(\|x\|). \tag{7}$$

Here $\|\cdot\|$ denotes the Euclidean norm and $\nabla_x^{\mathrm{T}} V(x,t)$ denotes the transpose of the column vector $\nabla_x V(x,t) = \frac{\partial}{\partial x} V(x,t)$.

Assumption 4. There exist two known, non-negative continuous functions $e_m(x,t)$ and $e_{\bar{m}}(x,t)$ such that

(i) $\|\Delta f_m(x,q,t)\| \leqslant e_m(x,t)$ and

(ii) $\|\nabla_x^{\mathrm{T}} V(x,t) \Delta f_{\bar{m}}(x,q,t)\| \leqslant e_{\bar{m}}(x,t)$ and $e_{\bar{m}}(x,t)/\|\nabla_x^{\mathrm{T}} V(x,t) g(x,t)\|$ is uniformly bounded with respect to t. Here, the two scalar functions $e_m(x,t)$ and $e_{\bar{m}}(x,t)$ are assumed to be uniformly bounded with respect to t.

Assumption 5. The uncertainties of system (1) appear in non-drift part satisfying the following condition:

$$\inf_{(x,q,t)}\left\{\lambda_{\min}\left(I+\frac{1}{2}(\Delta g_m(x,q,t)+\Delta g_m^{\mathrm{T}}(x,q,t))\right)-\frac{\|\nabla_x^{\mathrm{T}}V(x,t)\Delta g_{\bar{m}}(x,q,t)\|}{\|\nabla_x^{\mathrm{T}}V(x,t)g(x,t)\|}\right\}$$
$$\geqslant \eta > 0. \tag{8}$$

Here, I and $\lambda_{\min}(\cdot)$ denote, respectively, the identity matrix and the smallest eigenvalue of a symmetric matrix.

Note that, the uncertainties $\Delta f_{\bar{m}}(x,q,t)$ in Assumption 4 and $\Delta g_{\bar{m}}(x,q,t)$ in Assumption 5 are referred as the so-called "equivalently matched-type" uncertainties [11]. From Assumptions 4 and 5, $\nabla_x^{\mathrm{T}}V(x,t)\Delta f_{\bar{m}}(x,q,t)$ and $\nabla_x^{\mathrm{T}}V(x,t)\times\Delta g_{\bar{m}}(x,q,t)$ must vanish when $\nabla_x^{\mathrm{T}}V(x,t)g(x,q,t)=0$.

Under Assumptions 1–5, [11] proposed a class of control laws that makes the origin of the closed-loop uncertain system globally asymptotically stable. However, the stability conclusion can be derived under a more general assumption as given in Assumption 6 below. In addition, both local and global cases will also be considered in this study. Details are given as follows.

First, we introduce the next assumption, which can also be found in ([13], condition 2.3) and ([2], Assumption 5) for a more general version. However, Qu only studied the global result and Chen only obtained practical stability result.

Assumption 6. $\Delta g(x,q,t)$ is uniformly bounded with respect to time and there exists an $\eta>0$ such that

$$\Delta_x^{\mathrm{T}}V(x,t)\Delta g(x,q,t)g^{\mathrm{T}}(x,t)\nabla_x V(x,t)\geqslant(\eta-1)\cdot\|\nabla_x^{\mathrm{T}}V(x,t)g(x,t)\|^2 \tag{9}$$

for all (x,q,t) with $x\in\Omega_x$ and $q\in\Omega_q$, where Ω_x is a neighborhood of $x=0$ and Ω_q denotes the region of the uncertainty parameter q.

To study the relationship between Assumption 5 and 6, we show in Lemma 1 below that condition (8) implies condition (9).

Lemma 1. *Condition* (8) *implies Condition* (9).

Proof. Suppose condition (8) holds. Then multiplying (8) by $\|\nabla_x^{\mathrm{T}}V(x,t)g(x,t)\|^2$, we have for all (x,q,t),

$$\lambda_{\min}\left(I+\tfrac{1}{2}(\Delta g_m(x,q,t)+\Delta g_m^{\mathrm{T}}(x,q,t))\right)\|\nabla_x^{\mathrm{T}}V(x,t)g(x,t)\|^2$$
$$-\eta\|\nabla_x^{\mathrm{T}}V(x,t)g(x,t)\|^2\geqslant\|\nabla_x^{\mathrm{T}}V(x,t)\Delta g_{\bar{m}}(x,q,t)\|\cdot\|\nabla_x^{\mathrm{T}}V(x,t)g(x,t)\|. \tag{10}$$

From the Cauchy–Schwartz Inequality, we have $|v_1^{\mathrm{T}}v_2|\leqslant\|v_1\|\cdot\|v_2\|$ for any vectors $v_1,v_2\in\mathbb{R}^n$. Also, it is known that $\lambda_{\min}(A)\cdot\|v\|^2\leqslant v^{\mathrm{T}}Av$ for any $A=A^{\mathrm{T}}\in\mathbb{R}^{n\times n}$ and $v\in\mathbb{R}^n$. Inequality (10) then leads to

$$\begin{aligned}&\nabla_x^{\mathrm{T}} V(x,t)g(x,t)\{I+\Delta g_m(x,q,t)\}g^{\mathrm{T}}(x,t)\nabla_x V(x,t)\\&\quad -\eta\nabla_x^{\mathrm{T}} V(x,t)g(x,t)g^{\mathrm{T}}(x,t)\nabla_x V(x,t)\\&\quad\geqslant -\nabla_x^{\mathrm{T}} V(x,t)\Delta g_{\bar{m}}(x,q,t)g^{\mathrm{T}}(x,t)\nabla_x V(x,t).\end{aligned} \tag{11}$$

Here, we have used the fact that $\frac{1}{2}[\Delta g_m(x,q,t)+\Delta g_m^{\mathrm{T}}(x,q,t)]=\Delta g_m(x,q,t)+\frac{1}{2}[\Delta g_m^{\mathrm{T}}(x,q,t)-\Delta g_m(x,q,t)]$ with $\Delta g_m^{\mathrm{T}}(x,q,t)-\Delta g_m(x,q,t)$ an antisymmetric matrix and that $x^{\mathrm{T}}Bx=0$ for all x if B is an antisymmetric matrix. Rearranging these terms, we have

$$\begin{aligned}&\nabla_x^{\mathrm{T}} V(x,t)\{g(x,t)\Delta g_m(x,q,t)+\Delta g_{\bar{m}}(x,q,t)\}g^{\mathrm{T}}(x,t)\nabla_x V(x,t)\\&\quad\geqslant(\eta-1)\nabla_x^{\mathrm{T}} V(x,t)g(x,t)g^{\mathrm{T}}(x,t)\nabla_x V(x,t).\end{aligned} \tag{12}$$

Since $\Delta g(x,q,t)=g(x,t)\Delta g_m(x,q,t)+\Delta g_{\bar{m}}(x,q,t)$, the assertion is then proved. □

Lemma 1 shows that Assumption 6 is more general than Assumption 5. It is noted that the converse of Lemma 1 is not true. An example is given in Example 1 below.

Example 1. Consider system (1) with $x(t)$, $u(t)$, $q(t)\in\mathbb{R}^2$, $g(x,t)=\mathrm{diag}\{x_1,x_2^3\}$ and $f(x,t)=0$. Also, the uncertainties given by (2) and (3) are

$$\Delta f(x,q,t)=0, \tag{13}$$

$$\Delta g_m(x,q,t)=\begin{pmatrix} q_1 & 2\\ 2 & q_2\end{pmatrix}\quad\text{for } q_1,q_2\geqslant\frac{3}{4} \tag{14}$$

and

$$\|\Delta g_{\bar{m}}(x,q,t)\|\leqslant\sqrt{x_1^4+x_2^8}. \tag{15}$$

For $q_1=q_2=1$, we have

$$\lambda_{\min}\left(I+\tfrac{1}{2}[\Delta g_m(x,q,t)+\Delta g_m^{\mathrm{T}}(x,q,t)]\right)=0. \tag{16}$$

It follows that there does not exist an $\eta>0$ such that condition (8) holds. However, we can show that Assumption 6 holds for such system. To see this, choose

$$V(x_1,x_2)=\tfrac{1}{2}(x_1^2+x_2^2).$$

It follows that

$$\begin{aligned}
&\nabla_x^{\mathrm{T}} V(x,t)\Delta g(x,q,t)g^{\mathrm{T}}(x,t)\nabla_x V(x,t) \\
&= \nabla_x^{\mathrm{T}} V(x,t)g(x,t)\Delta g_m(x,q,t)g^{\mathrm{T}}(x,t)\nabla_x V(x,t) \\
&\quad + \nabla_x^{\mathrm{T}} V(x,t)\Delta g_{\bar{m}}(x,q,t)g^{\mathrm{T}}(x,t)\nabla_x V(x,t) \\
&\geqslant q_1 x_1^4 + q_2 x_2^8 + 4x_1^2 x_2^4 - \sqrt{x_1^2 + x_2^2}\cdot(x_1^4 + x_2^8) \\
&\geqslant (\eta - 1)\cdot(x_1^4 + x_2^8) \\
&= (\eta - 1)\cdot\|\nabla_x^{\mathrm{T}} V(x,t)g(x,t)\|^2
\end{aligned}$$

for all x around the neighborhood $\Omega_x = \{x|\ \|x\| \leqslant 1/4\}$ of the origin for $0 < \eta < \frac{3}{2}$, which implies that Assumption 6 holds.

To achieve certain stability performance, several control laws have been proposed (see e.g. [3,4,8,11]). Among these control laws, in order to achieve maximum control effort regarding the Lyapunov function $V(x)$ for the nominal system, they are usually chosen in the form of

$$u(x,t) = -w(x,t)g^{\mathrm{T}}(x,t)\nabla_x V(x,t), \tag{17}$$

where $w(x,t)$ is a non-negative scalar function to be determined by compensating the effect of uncertainties. Under the control law given by (17) and the decomposition of uncertainties given (2) and (3), the time derivative of $V(x)$ along the trajectories of the uncertain system (1) is calculated as

$$\begin{aligned}
\dot{V} &= \frac{\partial V(x,t)}{\partial t} + \nabla_x^{\mathrm{T}} V(x,t)[f(x,t) + g(x,t)\Delta f_m(x,q,t) + \Delta f_{\bar{m}}(x,q,t)] \\
&\quad - w(x,t)\nabla_x^{\mathrm{T}} V(x,t)[g(x,t) + \Delta g(x,q,t)]g^{\mathrm{T}}(x,t)\nabla_x V(x,t) \\
&\leqslant -\gamma_3(\|x\|) + e_m(x,t)\|\nabla_x^{\mathrm{T}} V(x,t)g(x,t)\| + e_{\bar{m}}(x,t) \\
&\quad - \eta w(x,t)\|\nabla_x^{\mathrm{T}} V(x,t)g(x,t)\|^2
\end{aligned} \tag{18}$$

Here, we have used Assumption 6 in the last inequality above. Choose $w(x,t)$ to satisfy

$$w(x,t) \geqslant \frac{e_m(x,t)\cdot\|\nabla_x^{\mathrm{T}} V(x,t)g(x,t)\| + e_{\bar{m}}(x,t)}{\eta\cdot\|\nabla_x^{\mathrm{T}} V(x,t)g(x,t)\|^2} \tag{19}$$

It follows that

$$\dot{V} \leqslant -\gamma_3(\|x\|) \tag{20}$$

We hence have the next results.

Theorem 1. *Consider the uncertain system* (1) *satisfying Assumptions* 1–4 *and* 6. *Then the origin is locally* (*resp*., *globally if* $\Omega_x = \mathbb{R}^n$) *uniformly asymptotically stable if the control law is chosen in the form* (17) *with* $w(x,t)$ *satisfying* (19).

To relax the possible discontinuity of the control law, [11] used the relation $\dot{V} \leqslant -\gamma_3(\|x\|) + 2\epsilon e^{-\beta t}$ instead of Inequality (20) for controller design. With a slight modification of Qu's approach [11], we propose the control law (17) with $w(x,t)$ being given by

$$w(x,t) = \begin{cases} \dfrac{e_m^2(x,t)}{\eta(e_m(x,t)\|\nabla_x^{\mathrm{T}} V(x,t)g(x,t)\| + \epsilon e^{-\beta t})} \\ \quad + \dfrac{e_{\bar{m}}^2(x,t)}{\eta(e_{\bar{m}}(x,t)\epsilon e^{-\beta t}) \cdot \|\nabla_x^{\mathrm{T}} V(x,t)g(x,t)\|^2} & \text{if } x \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

where ϵ and β are two positive constants determined by the designer to achieve desired stability performance. Note that, by Assumption 4, $e_{\bar{m}}^2(x,t)/\|\nabla_x^{\mathrm{T}} V(x,t)g(x,t)\|^2$ is uniformly bounded with respect to t and $\nabla_x^{\mathrm{T}} V(0,t) = 0$ since $V(x,t)$ is a locally positive definite function. These imply that $u(x,t)$ is continuous everywhere and

$$\|u(x,t)\| \leqslant \frac{e_m(x,t)}{\eta} + \frac{e_{\bar{m}}(x,t)}{\eta\|\nabla_x^{\mathrm{T}} V(x,t)g(x,t)\|}. \tag{22}$$

We then have the next result

Theorem 2. *Suppose the uncertain system* (1) *satisfying Assumption* 1–4 *and* 6. *Then the origin is locally* (*resp., globally if* $\Omega_x = \mathbb{R}^n$) *asymptotically stable if the control laws* $u(x,t)$ *are chosen in the form of* (17) *with* $w(x,t)$ *in* (21) *and* ϵ *satisfying condition* (A.2).

Proof. The proof is analogous to those of [11] with a slight modification. Details are given in Appendix A. □

It is observed from Theorems 1 and 2 that the designed stabilizing control laws strongly depend on the given Lyapunov function $V(x,t)$ for the nominal drift part $f(x,t)$. However, in general, there is no guideline for the construction of Lyapunov function. To facilitate the design of control laws and simplify the checking procedure, we can rewrite the uncertain system (1) as

$$\dot{x}(t) = f(x,t) + \Delta f(x,q,t) + \{g(x) + \Delta\tilde{g}(x,q,t)\}u, \tag{23}$$

where $\Delta\tilde{g}(x,q,t) = g(x,t) + \Delta g(x,q,t) - g(x)$. That is, we extract time-invariant part $g(x)$ from $g(x,t)$ and put the remaining time-varying part into the uncertain term. In the following, instead of requiring the uniformly asymptotic stability assumption on the nominal drift part $f(x,t)$, we assume that

$$\dot{x} = g(x)u \tag{24}$$

is locally asymptotically stabilizable at the origin. As given in [9], such an assumption is equivalent to Assumption 7 below.

Assumption 7. There exists a neighborhood Ω_x of the origin $x = 0$ and a smooth locally positive definite function $V(x)$ satisfying

$$\nabla_x^{\mathrm{T}} V(x) g(x) \neq 0 \quad \text{for all } x \in \Omega_x \setminus \{0\}. \tag{25}$$

Remark 1. Note that, although Assumption 7 is a special case for (ii) of Assumption 4, it might provide a guideline for the construction of Lyapunov function $V(x)$ for some special cases. One of the results concerning the existence of a quadratic positive definite function $V(x)$ satisfying condition (25) can be found in [9]. The results are obtained from Taylor's series expansion of $g(x)$ and the determination of the local definiteness of a defined scalar function. It was shown that, for instance, there exists a $V(x)$ satisfying condition (25) for the linear driftless system $\dot{x} = Bu$ (i.e., $g(x) = B$) if B is a square non-singular matrix. Also, condition (25) is satisfied for the bilinear driftless system $\dot{x} = \sum_{i=1}^{m} u_i B_i x$ with $\sum_{i=1}^{m} (x^{\mathrm{T}} B_i x)^2 > 0$ for all $x \neq 0$, where $x \in \mathbb{R}^n$, $u_i \in \mathbb{R}$ and $B_i \in \mathbb{R}^{n \times n}$. In particular, the condition of $\sum_{i=1}^{m} (x^{\mathrm{T}} B_i x)^2 > 0$ is guaranteed when one of the matrices B_i is definite. For details, please refer to [9].

Motivated by Assumption 7, Eq. (23) can then be rewritten as

$$\begin{aligned} \dot{x} = {} & -g(x) g^{\mathrm{T}}(x) \nabla_x V(x) + \{ f(x,t) + \Delta f(x,q,t) + g(x) g^{\mathrm{T}}(x) \nabla_x V(x) \} \\ & + \{ g(x) + \Delta \tilde{g}(x,q,t) \} u. \end{aligned} \tag{26}$$

Here, $f(x,t)$ does not require to satisfy Assumption 3. Let

$$f_0(x) = -g(x) g^{\mathrm{T}}(x) \nabla_x V(x), \tag{27}$$

then the origin is an asymptotic stable equilibrium point for $\dot{x} = f_0(x)$ if Assumption 7 holds [9]. By decomposing the uncertainty $\Delta f(x,q,t)$ and $f(x,t)$ into matched and mismatched parts, we can rewrite Eq. (26) as

$$\dot{x}(t) = f_0(x) + g(x) \Delta f_m(x,q,t) + \Delta f_{\bar{m}}(x,q,t) + \{ g(x) + \Delta \tilde{g}(x,q,t) \} u. \tag{28}$$

From the above derivation, we then have the next result from Theorem 2.

Lemma 2. *Suppose the uncertain system* (26) *satisfying Assumptions* 1, 2, 4, 6 *and* 7 *with* $\Delta g(x,q,t)$ *being replaced by* $\Delta \tilde{g}(x,q,t)$. *Then the origin is locally* (*resp., globally if* $\Omega_x = \mathbb{R}^n$) *asymptotically stable if the control laws* $u(x,t)$ *are chosen in the form of* (17) *with* $w(x,t)$ *as in* (21) *and* ϵ *satisfying condition* (A.2).

For the demonstration of the proposed robust stabilization design, numerical results for Example 1 are obtained as depicted in Fig. 1. In these simula-

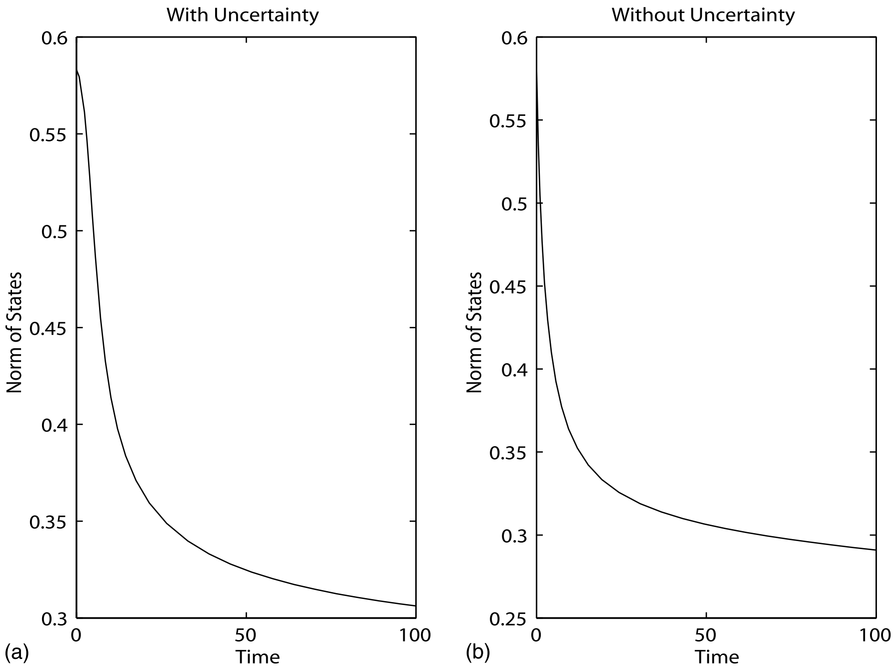

Fig. 1. Norm of system states.

tions, the initial state and the positive definite function $V(x)$ are, respectively, chosen to be $x_0 = (0.5, -0.3)^T$ and $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ to make $\nabla_x^T V(x)g(x) \neq 0$ in a neighborhood of the origin. The system parameters and the uncertainties are considered as $q_1 = q_2 = 1$, $e_{\bar{m}}(x,t) = 0$, $e_m(x,t) = \sqrt{x_1^6 + x_2^{14}}$ and

$$\Delta g_{\bar{m}}(x,q,t) = \frac{1}{2}\begin{pmatrix} x_2^4 \sin x_1 & x_1^2 \cos x_2 \\ x_1^2 \cos x_1 & x_2^4 \sin x_2 \end{pmatrix} \tag{29}$$

For the stabilization design, we choose $\eta = \epsilon = \beta = 1$. Fig. 1(a) and (b), respectively, show the time evolution of the norm of the system state with and without uncertainities. These show that all the system states converge to the origin, which agree with the results of Theorem 2. However, since the closed loop system behaves like a polynomial system with order greater than one, the convergent rate is getting smaller as system states get closer to the origin.

3. Conclusions

This paper has studied the robust stabilization of uncertain non-linear affine systems. The uncertainties considered in this study are more general than that of the so-called "equivalently matched-type". Since control laws strongly depend on the given Lyapunov function, stabilization design is also proposed for uncertain system with asymptotic stabilizable nominal driftless time-invariant terms but not the stabilizability of the nominal drift part to provide a means of the construction of Lyapunov function for fulfilling the design.

Appendix A (Proof of Theorem 2)

By direct calculation, we have from (17), (18) and (21) that

$$\dot{V} \leqslant -\gamma_3(\|x\|) + 2\epsilon e^{-\beta t}. \tag{A.1}$$

The constant ϵ in (A.1) will be determined later to guarantee the uniformly asymptotic stability of the origin. Choose $r_0 > 0$ and $\rho > 0$ such that $B_{r_0} \subseteq \Omega_x$ and $\rho < \min_{\|x\|=r_0} \gamma_1(\|x\|)$, where B_{r_0} denote the open ball with radius r_0. It implies that $\Omega_1 \stackrel{\Delta}{=} \{x \in B_{r_0}\colon \gamma_1(\|x\|) \leqslant \rho\} \subseteq B_{r_0}$. Define $\Omega_{t,\rho} \stackrel{\Delta}{=} \{x \in B_{r_0} | V(x,t) \leqslant \rho\}$. This leads to $\Omega_2 \stackrel{\Delta}{=} \{x \in B_{r_0}\colon \gamma_2(\|x\|) \leqslant \rho\} \leqslant \Omega_{t,\rho} \subseteq \Omega_1$. Choose ϵ such that

$$\epsilon < \frac{1}{2} \inf_{x \in \Omega_1 \setminus \Omega_2} \gamma_3(\|x\|). \tag{A.2}$$

It follows that $\dot{V} < 0$ for all $x \in \Omega_1 \setminus \Omega_2$ and the state will remain inside Ω_1 if it starts inside Ω_2. This shows uniformly stability of the origin. To show the attractive property of the origin, it is noted that

$$\begin{aligned} 0 &\leqslant V(x(t), t) \\ &= V(x(t_0), t_0) + \textstyle\int_{t_0}^{t} \dot{V}(x, (\tau), \tau)\, d\tau \\ &\leqslant V(x(t_0), t_0) + \textstyle\int_{t_0}^{t} (-\gamma_3(\|x(\tau)\|) + 2\epsilon e^{-\beta\tau})\, d\tau. \end{aligned} \tag{A.3}$$

This implies that

$$\lim_{t\to\infty} \int_{t_0}^{t} \gamma_3(\|x(\tau)\|)\, d\tau \leqslant V(x(t_0), t_0) + \frac{2\epsilon}{\beta} e^{-\beta t_0} < \infty. \tag{A.4}$$

Moreover, the state trajectory $x(t)$ is continuous (see e.g., [6]) and bounded since the origin is uniformly stable. The boundedness property of $x(t)$ together with Assumptions 1, 2 and 6 imply that $x(t)$ is uniformly continuous. Thus, the function $\gamma_3(\|x(t)\|)$ is also uniformly continuous. Then, from (A.4) and the use of Babalat's Lemma (see e.g., [7]), we have $\gamma_3(\|x(t)\|) \to 0$ as $t \to \infty$. It follows that $x(t) \to 0$ and the results of theorem is hence implied. □

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References

[1] B.R. Barmish, M. Corless, G. Leitmann, A new class of stabilizing controller for uncertain dynamical systems, SIAM J. Control Optim. 21 (1983) 246–255.
[2] Y.H. Chen, On the robustness of mismatched uncertain dynamical systems, ASME J. Dyn. Syst. Meas. Control 109 (1987) 29–35.
[3] Y.H. Chen, Robust control design for a class of mismatched uncertain non-linear systems, J. Optim. Theory Appl. 90 (1996) 605–626.
[4] M. Corless, G. Leitmann, Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamical systems, IEEE Trans. Autom. Control AC-26 (1981) 1139–1144.
[5] S. Gutman, Uncertain dynamical system—a Lypunov min–max approach, IEEE Trans. Autom. Control 24 (1979) 437–443.
[6] J.K. Hale, Ordinary Differential Equation, R.E. Krieger, Malabar, FL, 1980.
[7] H.K. Khalil, Non-linear Systems, Prentice-Hall Inc., Englewood Cliffs, NJ, 1996.
[8] C. Kravaris, S. Palanki, A Lyapunov approach for robust non-linear state feedback synthesis, IEEE Trans. Autom. Control 33 (1988) 1188–1191.
[9] D.-C. Liaw, Y.-W. Liang, Asymptotic stabilization of driftless systems, Int. J. Control 72 (1999) 206–214.
[10] I.R. Petersen, C.V. Hollot, A Riccati equation approach to the stabilization of uncertain linear systems, Automatica 22 (1986) 397–411.
[11] Z. Qu, Global stabilization of non-linear systems with a class of unmatched uncertainties, Syst. Control Lett. 18 (1992) 301–307.
[12] Z. Qu, Robust control of non-linear uncertain systems under generalized matching conditions, Automatica 29 (1993) 985–998.
[13] Z. Qu, Robust Control of Non-linear Uncertain Systems, John Wiley & Sons Inc., New York, 1998.
[14] H.L. Royden, Real Analysis, second ed., The Macmillan Company, New York, 1968.
[15] W.H. Wang, C.B. Soh, T.Y. Chai, Robust stabilization of MIMO non-linear time-varying mismatched uncertain systems, Automatica 33 (1997) 2197–2202.