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Condition for the numerical range to contain an elliptic disc ☆

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Abstract

For an n-by-n matrix A and an elliptic disc E in the plane, we show that the sum of the number of common supporting lines and the number of common intersection points to E and the numerical range $W(A)$ of A should be at least $2n+1$ in order to guarantee that E be contained in $W(A)$. This generalizes previous results of Anderson and Thompson. As an application, our result is used to verify a special case of the Poncelet property conjecture.

Keywords: Numerical range; Kippenhahn curve; Poncelet property

Let A be an n-by-n (complex) matrix and let E be a closed elliptic disc in the plane. The purpose of this paper is to answer the question: How many common supporting lines with some common intersection points to E and the numerical range $W(A)$ of A are needed in order to guarantee that E be contained in $W(A)$? Recall that the *numerical range* $W(A)$ of A is the set $\{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$ in the plane, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{C}^n$. For properties of numerical ranges, a good reference is [4, Chapter 1]. The motivation of this problem comes from two results, one old and another more recent. In the early 1970s, Anderson obtained that if $W(A)$ is contained in the closed unit disc $\overline{\mathbb{D}}$ ($\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$) and $W(A) \cap \partial\overline{\mathbb{D}}$ has more than n points, then $W(A) = \overline{\mathbb{D}}$ (cf. [10, p. 507]). Through

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an affine transformation, this can be easily extended to the setting with $\overline{\mathbb{D}}$ replaced by any closed elliptic disc E (cf. [7, Theorem 4.12]). More recently, in a similar vein Thompson [12] proved that if $\partial W(A)$ contains an arc of an ellipse with positive length, then $E \subseteq W(A)$. The main result of this paper is a generalization of these two results to their sharpest form:

Theorem. *Let A be an n-by-n matrix $(n \geqslant 2)$ and let E be a closed elliptic disc in the plane. Assume that $W(A)$ and E have l $(\geqslant 1)$ common supporting lines $L_1, \ldots, L_l$ with m $(0 \leqslant m \leqslant l)$ of the intersections $L_k \cap W(A) \cap E$ nonempty. If $l + m \geqslant 2n+1$, then $E \subseteq W(A)$. In this case, the number $2n+1$ is sharp and the two foci of ∂E are the eigenvalues of A.*

Note that the result of Anderson and that of Thompson follow from this theorem by letting $l = m = n+1$.

The original idea of the proof by Anderson, though never published, is to consider $W(A)$ as the convex hull of the Kippenhahn curve $C(A)$ of A (see below) and then apply Bézout's theorem (cf. [6, Theorem 3.1]). In recent years, two more proofs for his result appeared in the literature. One is by Dritschel and Woerdeman [1, Theorem 5.8], based on their canonical decomposition and radial tuples for numerical contractions. Another is due to the second author (cf. [11, Lemma 6]); it depends on a classical theorem of Riesz and Fejér on nonnegative trigonometric polynomials. The original proof of Thompson's result is lengthy; the one in [12, Section 5] by the referee is again based on the Kippenhahn curve and Bézout's theorem. Our theorem can also be proved in this fashion. In the main text below, we adopt a proof in which the fundamental theorem of algebra is used in place of Bézout's theorem, while in the appendix we give the one based on Bézout's theorem.

We start with a brief review of Kippenhahn's result. For any n-by-n matrix A, consider the homogeneous degree-n polynomial $p_A(x, y, z) = \det(x \operatorname{Re} A + y \operatorname{Im} A + zI_n)$, where $\operatorname{Re} A = (A + A^*)/2$ and $\operatorname{Im} A = (A - A^*)/(2\mathrm{i})$ are the real and imaginary parts of A, respectively, and I_n denotes the n-by-n identity matrix. The *Kippenhahn curve* $C(A)$ of A is the curve dual to the algebraic curve determined by $p_A(x, y, z) = 0$ in the complex projective plane $\mathbb{CP}^2$, that is, $C(A)$ consists of all points $[u, v, w]$ in $\mathbb{CP}^2$ such that $ux + vy + wz = 0$ is a tangent line to $p_A(x, y, z) = 0$. As usual, we identify the point (x, y) in $\mathbb{C}^2$ with $[x, y, 1]$ in $\mathbb{CP}^2$ and identify any point $[x, y, z]$ in $\mathbb{CP}^2$ such that $z \neq 0$ with $(x/z, y/z)$ in $\mathbb{C}^2$. Thus, in particular, the plane $\mathbb{R}^2$ (identified with $\mathbb{C}$) sits in $\mathbb{CP}^2$ by way of the identification of the point (a, b) of $\mathbb{R}^2$ (or $a + b\mathrm{i}$ of $\mathbb{C}$) with $[a, b, 1]$ in $\mathbb{CP}^2$. The algebraic curve $p(x, y, z) = 0$ in $\mathbb{CP}^2$, where p is a homogeneous polynomial, can be dehomogenized to yield the curve $p(x, y, 1) = 0$ in $\mathbb{C}^2$ and, conversely, an algebraic curve $q(x, y) = 0$ in $\mathbb{C}^2$ can be homogenized to a curve in $\mathbb{CP}^2$ with equation obtained by simplifying $q(x/z, y/z) = 0$. A result of Kippenhahn says that the numerical range $W(A)$ is the convex hull of the real points of $C(A)$ (cf. [5, p. 199]). Note that, as proved in [2, Theorem 1.3], if $x_0u + y_0v + z_0w = 0$ is a supporting line of $W(A)$, then $\det(x_0 \operatorname{Re} A + y_0 \operatorname{Im} A + z_0 I_n) = 0$. Since the dual of $C(A)$ is the original curve $p_A(x, y, z) = 0$, we infer, in particular, that every supporting line of $W(A)$ is tangent to $C(A)$.

We are now ready for the proof of our theorem.

Proof. For the ease of exposition, we first show that in our assertions the elliptic disc E may be assumed to be the unit disc $\overline{\mathbb{D}}$. Indeed, we first apply a translation followed by a rotation (with respect to the origin) to transform E to an elliptic disc with boundary given by $(u^2/a^2) + (v^2/b^2) = 1$. Another affine transformation $(u, v) \mapsto (u/a, v/b)$ then transforms the latter to the unit disc $\overline{\mathbb{D}}$. Since supporting lines are preserved under such transformations, we may apply one more rotation, if necessary, to ensure that none of the l transformed supporting lines to $\overline{\mathbb{D}}$ is horizontal.

If

$$f(u, v) = (a_1u + b_1v + c_1, a_2u + b_2v + c_2)$$

(a_j, b_j and c_j are real for $j = 1, 2$ and $a_1b_2 \neq b_1a_2$) denotes the affine transformation of $\mathbb{R}^2$ obtained from the composite of the above, then we have $f(E) = \overline{\mathbb{D}}$. Let $f(A)$ denote the matrix

$$(a_1 \operatorname{Re} A + b_1 \operatorname{Im} A + c_1 I_n) + \mathrm{i}(a_2 \operatorname{Re} A + b_2 \operatorname{Im} A + c_2 I_n).$$

It can be easily verified that $f(W(A)) = W(f(A))$. Hence in the proof below for the assertion on the supporting lines, we may replace the matrix A by $f(A)$ and the elliptic disc E by $\overline{\mathbb{D}}$, and assume that none of the supporting lines L_k to $\overline{\mathbb{D}}$ is horizontal. As for the assertion on the foci, though they are in general not preserved by affine transformations, we may argue as follows. Let B be a 2-by-2 matrix whose numerical range is E. If we can show that $p_{f(B)}$ is a factor of $p_{f(A)}$, then by a simple computation we have that p_B is a factor of p_A. Since the foci of ∂E are eigenvalues of B, they are zeros of the polynomial $p_B(-1, -\mathrm{i}, z)$. We then infer from above that the foci are also zeros of $p_A(-1, -\mathrm{i}, z)$, and hence they are eigenvalues of A. In this situation, we have

$$W(f(B)) = f(W(B)) = f(E) = \overline{\mathbb{D}}$$

and thus $f(B)$ is unitarily equivalent to

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Hence, in the following, we may assume that

$$B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Then $W(B) = \overline{\mathbb{D}}$ and $p_B(x, y, z) = z^2 - x^2 - y^2$.

To prove that p_B is a factor of p_A, let $p_A = p_B q + r$, where q is a homogeneous polynomial of degree $n-2$ and $r(x, y, z) = p_1(x, y)z + p_2(x, y)$ with

$$p_1(x, y) = \sum_{j=0}^{n-1} a_j x^{n-j-1} y^j = x^n \left(\frac{1}{x} \sum_{j=0}^{n-1} a_j \left(\frac{y}{x}\right)^j \right) \tag{1}$$

and

$$p_2(x, y) = \sum_{j=0}^{n} b_j x^{n-j} y^j = x^n \left(\sum_{j=0}^{n} b_j \left(\frac{y}{x}\right)^j \right). \tag{2}$$

If $x_0 u + y_0 v = 1$ is a common nonhorizontal supporting line to $W(A)$ and E $(= \overline{\mathbb{D}})$, then $p_A(x_0, y_0, -1) = p_B(x_0, y_0, -1) = 0$ and hence $r(x_0, y_0, -1) = 0$. This yields $p_1(x_0, y_0) = p_2(x_0, y_0)$ or

$$\frac{1}{x_0} \sum_{j=0}^{n-1} a_j \left(\frac{y_0}{x_0}\right)^j = \sum_{j=0}^{n} b_j \left(\frac{y_0}{x_0}\right)^j. \tag{3}$$

Here $x_0 \neq 0$ since $x_0 u + y_0 v = 1$ is assumed to be nonhorizontal. If, moreover, $x_0 u + y_0 v = 1$ intersects $W(A)$ and E at some common point (u_0, v_0), then either (u_0, v_0) lies on $C(A)$ or there are two distinct points, say, (u_1, v_1) and (u_2, v_2) of $C(A)$ that lie on $x_0 u + y_0 v = 1$. In the former case, $u_0 x + v_0 y = 1$ is tangent to the curve $p_A(x, y, -1) = 0$ at (x_0, y_0) by the definition of the Kippenhahn curve. Since the equation of the tangent line is also given by

$$\frac{\partial p_A}{\partial x}(x_0, y_0, -1)(x - x_0) + \frac{\partial p_A}{\partial y}(x_0, y_0, -1)(y - y_0) = 0,$$

we infer that

$$u_0 \frac{\partial p_A}{\partial y}(x_0, y_0, -1) = v_0 \frac{\partial p_A}{\partial x}(x_0, y_0, -1). \tag{4}$$

In the latter case, the two lines $u_j x + v_j y = 1$ $(j = 1, 2)$ are both tangent to $p_A(x, y, -1) = 0$ at (x_0, y_0). The nonuniqueness of the tangent line implies that

$$\frac{\partial p_A}{\partial x}(x_0, y_0, -1) = \frac{\partial p_A}{\partial y}(x_0, y_0, -1) = 0$$

and therefore (4) also holds. A similar (even easier) argument shows that

$$u_0 \frac{\partial p_B}{\partial y}(x_0, y_0, -1) = v_0 \frac{\partial p_B}{\partial x}(x_0, y_0, -1). \tag{5}$$

Some computations with (4) and (5) which take into account that $p_B(x_0, y_0, -1) = 0$ yield

$$u_0 \Big(-\frac{\partial p_1}{\partial y} + \frac{\partial p_2}{\partial y} \Big) = v_0 \Big(-\frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial x} \Big) \quad \text{at } (x_0, y_0).$$

Letting $\psi(y) = (1 - v_0 y)/u_0$ (here $u_0 \neq 0$ by the nonhorizontality of $x_0 u + y_0 v = 1$), we may consider the polynomial $\tilde{p}_j(y) = p_j(\psi(y), y)$ in y, $j = 1, 2$. Then we have $\tilde{p}_1(y_0) = \tilde{p}_2(y_0)$ and $\tilde{p}_1'(y_0) = \tilde{p}_2'(y_0)$. The quotient rule gives

$$\left(\frac{\tilde{p}_1(y)}{\psi(y)^n}\right)'(y_0) = \left(\frac{\tilde{p}_2(y)}{\psi(y)^n}\right)'(y_0).$$

Carrying out the calculations by making use of the second expressions for p_1 and p_2 in (1) and (2), we obtain

$$-\frac{a}{x_0^2}\sum_{j=0}^{n-1} a_j \left(\frac{y_0}{x_0}\right)^j + \frac{b}{x_0}\sum_{j=0}^{n-1} j a_j \left(\frac{y_0}{x_0}\right)^{j-1} = b\sum_{j=0}^{n} j b_j \left(\frac{y_0}{x_0}\right)^{j-1}, \tag{6}$$

where

$$a = \psi'(y_0) = -\frac{v_0}{u_0} \quad \text{and} \quad b = \left(\frac{y}{\psi(y)}\right)'(y_0) = \frac{u_0}{(1 - v_0 y_0)^2}.$$

Let the l common supporting lines L_k to $W(A)$ and $E(= \overline{\mathbb{D}})$ be $u\cos\theta_k + v\sin\theta_k = 1$, where $0 \leqslant \theta_k < 2\pi$ for $k = 1, \ldots, l$, with the m common intersection points $(\cos\theta_k, \sin\theta_k)$, $k = 1, \ldots, m$. Since L_k is not horizontal, we have $\cos\theta_k \neq 0$. In this case, $x_0 = \cos\theta_k$, $y_0 = \sin\theta_k$ and $y_0/x_0 = \tan\theta_k$ for each $k = 1, \ldots, l$, and $u_0 = \cos\theta_k$, $v_0 = \sin\theta_k$, $a = -\tan\theta_k$ and $b = \sec^3\theta_k$ for $k = 1, \ldots, m$. Thus (3) and (6) become

$$\sec\theta_k \sum_{j=0}^{n-1} a_j \tan^j\theta_k = \sum_{j=0}^{n} b_j \tan^j\theta_k \tag{7}$$

and

$$\begin{aligned} &\tan\theta_k \sum_{j=0}^{n-1} a_j \tan^j\theta_k + \sec^2\theta_k \sum_{j=0}^{n-1} j a_j \tan^{j-1}\theta_k \\ &\quad = \sec\theta_k \sum_{j=0}^{n} j b_j \tan^{j-1}\theta_k, \end{aligned} \tag{8}$$

respectively. If we let

$$p(\lambda) = (1 + \lambda^2)\left(\sum_{j=0}^{n-1} a_j \lambda^j\right)^2 - \left(\sum_{j=0}^{n} b_j \lambda^j\right)^2,$$

then $p(\tan\theta_k) = 0$ and

$$\begin{aligned} p'(\tan\theta_k) &= 2\tan\theta_k \left(\sum_{j=0}^{n-1} a_j \tan^j\theta_k\right)^2 + 2(1+\tan^2\theta_k)\left(\sum_{j=0}^{n-1} a_j \tan^j\theta_k\right) \\ &\quad \times \left(\sum_{j=0}^{n-1} j a_j \tan^{j-1}\theta_k\right) - 2\left(\sum_{j=0}^{n} b_j \tan^j\theta_k\right)\left(\sum_{j=0}^{n} j b_j \tan^{j-1}\theta_k\right) \\ &= 2\left(\sum_{j=0}^{n-1} a_j \tan^j\theta_k\right)\sec\theta_k\left(\sum_{j=0}^{n} j b_j \tan^{j-1}\theta_k\right) \\ &\quad - 2\left(\sum_{j=0}^{n} b_j \tan^j\theta_k\right)\left(\sum_{j=0}^{n} j b_j \tan^{j-1}\theta_k\right) = 0, \end{aligned}$$

where the second equality follows from (8) and the third from (7). This shows that each $\tan\theta_k$, $k = 1, \ldots, l$, is a zero of p with the first m having multiplicity at least two. Assume first that all the $\tan\theta_k$ are distinct. Since the degree of p is at most $2n$ and $l + m \geqslant 2n + 1$, this implies, by the fundamental theorem of algebra, that p is identically zero. Hence $(1+\lambda^2)(\sum_{j=0}^{n-1} a_j\lambda^j)^2$ and $(\sum_{j=0}^{n} b_j\lambda^j)^2$ are equal polynomials. If $\sum_j a_j\lambda^j$ and $\sum_j b_j\lambda^j$ are not identically zero, then i is a zero of $(1+\lambda^2)(\sum_j a_j\lambda^j)^2$ with odd multiplicity and hence of $(\sum_j b_j\lambda^j)^2$. But this is impossible since the latter has no such zeros. Hence we must have $p_1 = 0$ and $p_2 = 0$, and therefore $r = 0$ or p_B is a factor of p_A.

We still have to deal with the case when some of the $\tan\theta_k$ are equal. If $\tan\theta_{i_0} = \tan\theta_{i_1}$, then $u\cos\theta_{i_0} + v\sin\theta_{i_0} = \pm 1$ are both common supporting lines to $W(A)$ and E. We will proceed as in the preceding paragraphs and only give a sketch of the arguments instead of going into all the details. Letting $x_0 = \cos\theta_{i_0}$ and $y_0 = \sin\theta_{i_0}$, we obtain $p_A(x_0, y_0, \pm 1) = p_B(x_0, y_0, \pm 1) = 0$ and hence $r(x_0, y_0, \pm 1) = 0$, which yields $p_1(x_0, y_0) = \pm p_2(x_0, y_0)$. This shows that $p_1(x_0, y_0) = p_2(x_0, y_0) = 0$ or

$$\sum_{j=0}^{n-1} a_j \tan^j\theta_{i_0} = \sum_{j=0}^{n} b_j \tan^j\theta_{i_0} = 0.$$

Therefore, $\tan\theta_{i_0}$ is a zero of the polynomials $a(\lambda) \equiv \sum_{j=0}^{n-1} a_j\lambda^j$ and $b(\lambda) \equiv \sum_{j=0}^{n} b_j\lambda^j$ and hence also a zero of p of multiplicity at least two. If, in addition, $x_0u + y_0v = 1$ and $x_0u + y_0v = -1$ intersect $W(A)$ and E at the common point $(u_0, v_0) \equiv (\cos\theta_{i_0}, \sin\theta_{i_0})$ and $(-u_0, -v_0)$, respectively, then, as before, we would obtain $\tilde{p}_1'(y_0) = \pm\tilde{p}_2'(y_0)$ and, therefore, $\tilde{p}_1'(y_0) = \tilde{p}_2'(y_0) = 0$. Thus both sides of (6) are zero. Since $\sum_j a_j \tan^j\theta_{i_0} = 0$, we derive from (8) that

$$\sum_{j=0}^{n-1} j a_j \tan^{j-1}\theta_{i_0} = \sum_{j=0}^{n} j b_j \tan^{j-1}\theta_{i_0} = 0.$$

So $\tan\theta_{i_0}$ is a zero of both $a(\lambda)$ and $b(\lambda)$ with multiplicity at least two, and hence a zero of p of multiplicity at least four. Finally, assume that only one of the lines $x_0u + y_0v = \pm 1$ intersects $W(A)$ and E at a common point, say, $x_0u + y_0v = 1$ at the point (u_0, v_0). Since $p(\lambda) = (1 + \lambda^2)a(\lambda)^2 - b(\lambda)^2$, a direct calculation shows that

$$\begin{aligned} p''(\lambda) = {} & 2a(\lambda)^2 + 8\lambda a(\lambda)a'(\lambda) + 2(1 + \lambda^2)a(\lambda)a''(\lambda) + 2(1 + \lambda^2)a'(\lambda)^2 \\ & - 2b(\lambda)b''(\lambda) - 2b'(\lambda)^2. \end{aligned}$$

From our assumptions, we have already had $a(\tan\theta_{i_0}) = b(\tan\theta_{i_0}) = 0$. Moreover, (8) and $a(\tan\theta_{i_0}) = 0$ yield

$$\sec^2\theta_{i_0} \sum_{j=0}^{n-1} j a_j \tan^{j-1}\theta_{i_0} = \sec\theta_{i_0} \sum_{j=0}^{n} j b_j \tan^{j-1}\theta_{i_0}.$$

From these, we obtain $p''(\tan\theta_{i_0}) = 0$. Thus $\tan\theta_{i_0}$ is a zero of p of multiplicity at least three. We conclude that in all cases p has at least $l + m$ zeros, counting multiplicity, and may infer as in the previous case when all $\tan\theta_k$ are distinct that p_B is a factor of p_A. Passing to dual curves, we have $C(B) \subseteq C(A)$. Hence $E = W(B) \subseteq W(A)$ as desired.

To show that the number $2n + 1$ is sharp, let $E = \overline{\mathbb{D}}$ and, for $n \geqslant 2$ and $n \leqslant l \leqslant 2n$, let P be an l-gon with vertices $a_1, \ldots, a_l$ and the l sides $[a_j, a_{j+1}]$, $j = 1, \ldots, l$ $(a_{l+1} \equiv a_1)$, tangent to E at b_j. Here if $l = 2$, then P is interpreted as two parallel lines tangent to E. Let $m = 2n - l$ and let A be the n-by-n diagonal matrix $\mathrm{diag}(b_1, \ldots, b_m, a_{m+2}, a_{m+4}, \ldots, a_{m+2(n-m)})$. Then $W(A)$ and E have l common supporting lines, namely, the ones determined by the l sides of P, and m common intersection points $b_1, \ldots, b_m$. But obviously, E is not contained in $W(A)$. □

As an application of our theorem, we use it to verify a special case of the Poncelet property conjecture proposed in [3, Conjecture 5.1]. Recall that an n-by-n matrix A is said to be in class S_n if A is a contraction ($\|A\| \leqslant 1$), has no eigenvalue with unit modulus and satisfies $\mathrm{rank}(I_n - A^*A) = 1$. In [3], we initiated the study of the numerical ranges of matrices in S_n and conjectured that such numerical ranges can be characterized by the so-called "Poncelet property". Our next corollary confirms this conjecture for ellipses. Such numerical ranges have also been studied by Mirman [8,9]. In his proof of the Poncelet theorem [8, Theorems 10a and 10b], he also verified the assertion in our corollary. Our present proof based on Anderson's theorem seems more concise.

Corollary. *Let E be a closed elliptic disc contained in $\mathbb{D}$. Then E is the numerical range of some matrix A in S_n if and only if it has the property that for any point λ in $\partial\mathbb{D}$ there is an $(n + 1)$-gon interscribing between $\partial\mathbb{D}$ and ∂E and having λ as a vertex. In this case, A is unique up to unitary equivalence.*

Proof. The necessity was proved in [3, Theorem 2.1]. To prove the sufficiency, let P be any of the asserted $(n+1)$-gons with $v_1, \ldots, v_{n+1}$ as its tangent points to ∂E. By [3, Theorem 3.1], there is a matrix A in S_n such that $W(A)$ is circumscribed about by P with $v_1, \ldots, v_n$ as tangent points. Thus E and $W(A)$ have $n+1$ common supporting lines with n common intersection points. Hence $E \subseteq W(A)$ by our theorem. To prove that $E = W(A)$, let $\lambda_1 = \lambda'_1$ be an arbitrary point in $\partial\mathbb{D}$. We draw successively from λ_j (resp., λ'_j), $j = 1, \ldots, n+1$, a supporting line to ∂E (resp., $\partial W(A)$), which is to intersect $\partial\mathbb{D}$ at λ_{j+1} (resp., λ'_{j+1}). Let $\lambda_j = \exp(\mathrm{i}\theta_j)$ (resp., $\lambda'_j = \exp(\mathrm{i}\theta'_j)$) with $\theta_1 = \theta'_1$ and $0 \leqslant \theta_1 < \theta_2 < \cdots < \theta_{n+2} = \theta_1 + 2\pi$ (resp., $0 \leqslant \theta'_1 < \theta'_2 < \cdots < \theta'_{n+2} = \theta'_1 + 2\pi$). Since $E \subseteq W(A)$, it is easily seen that $\theta_j \geqslant \theta'_j$ for all j and, moreover, if $\theta_{j_0} > \theta'_{j_0}$ for any j_0, then $\theta_k > \theta'_k$ for all $k \geqslant j_0$. We infer from $\theta_{n+2} = \theta'_{n+2}$ that $\theta_j = \theta'_j$ for all j. Thus ∂E and $\partial W(A)$ have the same circumscribing $(n+1)$-gons with vertices on $\partial\mathbb{D}$. Since E and $W(A)$ are both the intersection of the polygonal regions determined by such polygons, we conclude that $E = W(A)$. The uniqueness of A follows from [3, Theorem 3.2]. □

Appendix

In this appendix, we give a proof of our theorem based on Bézout's theorem. Although the argument is shorter, it does require some basic knowledge of algebraic curve theory on the readers' part. Our presentation of Bézout's theorem and its relevant notions is based on [6, Chapter 3]. Let us begin by quoting:

Bézout's theorem ([6, Theorem 3.1]). *If C and D are two algebraic curves of degrees m and n, respectively, in the complex projective plane $\mathbb{CP}^2$ which have no common component, then the total sum of the intersection multiplicity $I_P(C, D)$ over all intersection points P of C and D is equal to mn.*

Here, the *degree* of an algebraic curve $p(x, y, z) = 0$ is just the degree of the homogeneous polynomial p. The *intersection multiplicity* $I_P(C, D)$ of the curves C and D at a point P is defined to be infinity if P lies on a common component of C and D, and a nonnegative integer otherwise, which is nonzero precisely when P is in $C \cap D$. The properties for $I_P(C, D)$ are given in [6, Theorem 3.18].

Another needed result is the following proposition.

Proposition ([6, Proposition 3.22]). *Let C and D be algebraic curves and P a point in $\mathbb{CP}^2$. Then $I_P(C, D) = 1$ if and only if P is a nonsingular point of C and D and the tangent lines to C and D at P are distinct.*

The point $[a, b, c]$ of the curve $p(x, y, z) = 0$ in $\mathbb{CP}^2$ is said to be *nonsingular* if at least one of the quantities $\partial p/\partial x$, $\partial p/\partial y$ and $\partial p/\partial z$ is nonzero at $[a, b, c]$.

We are now ready for the proof of our theorem via Bézout's theorem.

Proof. We only need to prove that $E \subseteq W(A)$. Let L be any of the common supporting lines $L_1, \dots, L_l$ of $W(A)$ and E, and let P be the tangent point of L with ∂E. We have several cases to consider.

(I) P is not in $W(A)$. Since L is tangent to $C(A)$, it corresponds, by duality, to an intersection point L^* of the dual curves $C(A)^*$ and ∂E^*. Hence $I_{L^*}(C(A)^*, \partial E^*) \geqslant 1$ by the definition of the intersection multiplicity.

(II) P is in $W(A)$ and lies on $C(A)$. In this case, the intersection point L^* of $C(A)^*$ and ∂E^* has a common tangent line P^*. Hence $I_{L^*}(C(A)^*, \partial E^*) \geqslant 2$ by the Proposition.

(III) P is in $W(A)$ but not on $C(A)$. This implies that L is tangent to $C(A)$ at two other points Q and R. We have two further cases to consider:

(1) Q and R are in two distinct components, say, C_1 and C_2 of $C(A)$. Then C_1^* and C_2^* are components of $C(A)^*$ and

$$I_{L^*}(C(A)^*, \partial E^*) \geqslant I_{L^*}(C_1^*, \partial E^*) + I_{L^*}(C_2^*, \partial E^*) \geqslant 1 + 1 = 2,$$

where the first inequality follows from one of the properties of the intersection multiplicity [6, Theorem 3.18 (v)].

(2) Q and R lie on the same component, say, C of $C(A)$. In this case, the point L^* on C^* has at least two tangent lines Q^* and R^* to $C(A)^*$. Thus L^* is a singular point of $C(A)^*$ and we deduce from Proposition that $I_{L^*}(C(A)^*, \partial E^*) \geqslant 2$.

In all the cases considered above, $I_{L^*}(C(A)^*, \partial E^*)$ is at least as large as the amount (1 or 2) contributed by the supporting line L to the sum $l + m$. Hence if $l + m \geqslant 2n + 1$, then $\sum_j I_{L_j^*}(C(A)^*, \partial E^*) \geqslant 2n + 1$. Since the degrees of $C(A)^*$ and ∂E^* are n and 2, respectively, Bézout's theorem implies that $C(A)^*$ and ∂E^* have common components. We infer from the irreducibility of the ellipse ∂E^* that ∂E^* is a component of $C(A)^*$. Hence ∂E is a component of $C(A)$ as required. □

Note added in proof

The Poncelet property conjecture [3, Conjecture 5.1], of which our Corollary is a special case, has since been shown to be false by Mirman in his upcoming paper "UB-matrices and conditions for Poncelet polygon to be closed".

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