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A new method for mixed H_2/H_∞ control with regional pole constraints

Jenq-Lang Wu[1,*,†] and Tsu-Tian Lee[2,‡,§]

[1] *Department of Electronic Engineering, Hwa Hsia College of Technology and Commerce, 111, Hwa-Shin Street, Chung-Ho 235, Taipei Country, Taiwan, R.O.C.*
[2] *Department of Electrical and Control Engineering, National Chiao Tung University, 111, Ta-Hseuh Road, Hsinchu 300, Taiwan, R.O.C.*

SUMMARY

In this paper, the problem of state feedback mixed H_2/H_∞ control with regional pole constraints is studied. The constraint region is represented by several algebraic inequalities. This constrained optimization problem cannot be solved via the LMI approach. Based on the barrier method, we instead solve an auxiliary minimization problem to get an approximate solution. We shall show that the obtained minimal solution of the auxiliary minimization problem can be arbitrarily close to the infimal solution of the original problem. An example is provided to illustrate the benefits of the approach. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: mixed H_2/H_∞ control; regional pole placement; barrier method; Lagrange multiplier method

1. INTRODUCTION

The mixed H_2/H_∞ control problem has attracted much attention in recent years (see References [1–7]). The mixed H_2/H_∞ control theory offers a way of combining the design criteria of quadratic performance and disturbance attenuation. But such a controller design method cannot guarantee that the closed-loop systems have good transient responses. The systems' transient responses are determined mainly by the locations of the systems' poles. In References [8–12,24], the optimal regional pole placement problem, which involves determining a feedback controller that minimizes a cost functional subject to the requirement that the closed-loop poles

*Correspondence to: Jenq-Lang Wu, Department of Electronic Engineering, Hwa Hsia College of Technology and Commerce, 111, Hwa-Shin Street, Chung-Ho 235, Taipei Country, Taiwan, R.O.C.
† E-mail: wujl@cc.hwh.edu.tw
‡ Fellow, IEEE.
§ E-mail: ttlee@cn.nctu.edu.tw

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lie within a specified region, has been studied. In Reference [13], Yedavalli and Liu studied the state feedback H_∞ control problem with single regional pole constraint via the Lagrange multiplier method. Recently, Wang and Bernstein [14] provided an approach to achieve mixed H_2/H_∞ control with an α-shifted pole constraint via output feedback. Bambang *et al.* [15] considered the mixed H_2/H_∞ control with pole placement in a circular region. In Reference [15], a generalized Riccati equation is employed in order to satisfy H_∞ -norm and poles' constraints as well as to obtain an upper bound on the H_2 cost to be minimized. These approaches have the disadvantage that they only minimize an 'upper bound' of the actual cost but not the actual cost. The obtained solution may be far from the actual solution. Yaz *et al.* [16] presented an approach to find the controller for a discrete system which simultaneously meets the following three criteria: pole placement in a specified disk, assignment of an upper bound to the H_2 cost and satisfaction of an H_∞ disturbance bound. More recently, Bambang *et al.* [17] provided a unified treatment for pole placement in the mixed H_2/H_∞ optimization problem. The cost function they minimized is also an upper bound of the actual cost. Moreover, the existence of the minimum point of the chosen auxiliary cost cannot be guaranteed. In fact, its infimal solution may lie on the boundary of the admissible solution set and may not be a stationary point. In this case, the infimal solution does not satisfy the obtained necessary conditions in Reference [17]. Chilali and Gahinet [18] used the linear matrix inequality (LMI) approach to solve the mixed H_2/H_∞ problem with regional pole constraint. The constraint region must be convex in LMI approach. Moreover, for tractability in the LMI framework, a single matrix that enforces several constraints must be found. This will lead to the result that the obtained solution may not be the global minimal solution, may not even be the local minimal solution, of the original constraint optimization problem. Furthermore, even in the case that the original constraint optimization problem is solvable, the LMI approach may fail and no solution may be found thus.

In this paper, we consider the regional pole constraints mixed H_2/H_∞ state feedback control problem. The constraint region is represented by several algebraic inequalities. It may be non-convex and may not be representable in the form of LMI region. In some special cases, it may contain several disjoint subregions. This problem is difficult to solve and its analytic solution has not been presented yet. Based on the barrier method [23], in this paper we instead solve an auxiliary minimization problem to get an approximate solution of the original optimization problem. The cost function of the auxiliary optimization problem is the sum of the cost function of the original problem and a weighted 'barrier function'. We shall show that if the weighting factor of the barrier function approaches zero, then the optimal solution of the auxiliary minimization problem will approach the infimal solution of the original optimization problem. The necessary conditions for local optimum of the auxiliary problem are derived. Furthermore, a solution algorithm is provided. The advantages of the presented approach are: (1) it is simple, (2) the existence of the minimum point of the auxiliary minimization problem is guaranteed, (3) the auxiliary minimization problem can be solved via some unconstrained search techniques and (4) the obtained solution can be arbitrarily close to the infimal solution of the original problem.

In what follows, $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ denote the real part and the imaginary part, respectively, of a complex number λ, $\sigma(\mathbf{M})$ is the spectrum of the matrix $\mathbf{M}$, $\mathbf{M}>0$ ($\mathbf{M}<0$) means that the matrix $\mathbf{M}$ is positive (negative) definite, $\|\mathrm{G}(s)\|_\infty$ ($\|\mathrm{G}(s)\|_2$) denotes infinity-norm (2-norm) of the transfer function G(s), $\otimes$ denotes Kronecker product, $\mathrm{vec}(\mathbf{M}) \equiv [\mathbf{m}_1^{\mathrm{T}}\mathbf{m}_2^{\mathrm{T}}\ldots\mathbf{m}_n^{\mathrm{T}}]^{\mathrm{T}}$, where $\mathbf{m}_i$ is the ith column of the matrix $\mathbf{M}$, and $\mathrm{vec}^{-1}(\cdot)$ is the inverse operator of $\mathrm{vec}(\cdot)$.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider a linear time-invariant continuous system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_1\mathbf{w}_1(t) + \mathbf{B}_2\mathbf{w}_2(t) + \mathbf{B}_3\mathbf{u}(t)\\ \mathbf{z}_1(t) &= \mathbf{C}_1\mathbf{x}(t) + \mathbf{D}_1\mathbf{u}(t)\\ \mathbf{z}_2(t) &= \mathbf{C}_2\mathbf{x}(t) + \mathbf{D}_2\mathbf{u}(t)\end{aligned} \tag{1}$$

where $\mathbf{w}_1 \in R^{n_{w1}}$ and $\mathbf{w}_2 \in R^{n_{w2}}$ denote the exogenous inputs, $\mathbf{x} \in R^n$ is the state, $\mathbf{u} \in R^m$ is the control, and $\mathbf{z}_1 \in R^{n_{z1}}$ and $\mathbf{z}_2 \in R^{n_{z2}}$ denote the controlled outputs. All matrices are assumed to be of appropriate dimensions. Define

$$\Omega_i \equiv \left\{ (a + \mathbf{i}b) \middle| f_i(a,b) = \sum_{f_i}\sum_{h_i} r_{f_i h_i} a^{f_i} b^{h_i} < 0 \right\} \quad i = 1, 2, \ldots, c$$

where $r_{f_i h_i}$ is real. Let $\Omega \equiv \bigcap_{i=1,\ldots,c} \Omega_i$ belong to the open left-half complex plane.

The design objective is to determine the state feedback controller

$$\mathbf{u}(t) = \mathbf{F}\mathbf{x}(t) \tag{2}$$

to achieve the infimum of $\|T_{\mathbf{z}_2\mathbf{w}_2}(\mathbf{F})\|_2$, i.e.

$$\inf_{\mathbf{F}} J(\mathbf{F}) \equiv \|T_{\mathbf{z}_2\mathbf{w}_2}(\mathbf{F})\|_2$$

subject to the following constraints:

(1) $\|T_{\mathbf{z}_1\mathbf{w}_1}(\mathbf{F})\|_\infty < \gamma$
(2) $\sigma(\mathbf{A} + \mathbf{B}_3\mathbf{F}) \subset \Omega$.
This problem is referred as the $Q_{2\infty p}$ problem. □

Let $\Gamma_s \equiv \{\mathbf{F} \in R^{m\times n} | \mathbf{A} + \mathbf{B}_3\mathbf{F} \text{ is stable}\}$, $\Gamma_{\Omega_i} \equiv \{\mathbf{F} \in R^{m\times n} | \sigma(\mathbf{A}+\mathbf{B}_3\mathbf{F}) \subset \Omega_i\}$, $\Gamma_\Omega \equiv \bigcap_{i=1,2,\ldots,c} \Gamma_{\Omega_i}$, $\Gamma_\infty(\gamma) \equiv \left\{\mathbf{F} \in R^{m\times n} \middle| \|T_{\mathbf{z}_1\mathbf{w}_1}(\mathbf{F})\|_\infty < \gamma\right\}$, and $\Gamma \equiv \Gamma_\Omega \bigcap \Gamma_\infty(\gamma)$. Let $\mathbf{A}_C = \mathbf{A} + \mathbf{B}_3\mathbf{F}$, $\boldsymbol{C}_{1C} = \boldsymbol{C}_1 + \mathbf{D}_1\mathbf{F}$, and $\boldsymbol{C}_{2C} = \boldsymbol{C}_2 + \mathbf{D}_2\mathbf{F}$.

Assumption 1

Suppose all the eigenvalues of $\mathbf{A}$, which are outside Ω, are $\mathbf{B}_3$-controllable, i.e.

$$rank[\lambda\mathbf{I} - \mathbf{A}\ \mathbf{B}_3] = n \quad \text{for all } \lambda \in \sigma(\mathbf{A}) \text{ and } \lambda \notin \Omega$$

Assumption 1 guarantees that the set Γ_Ω is non-empty. From (1) and (2), it is known that the closed-loop system transfer function from $\mathbf{w}_1$ to $\mathbf{z}_1$ is given by

$$T_{\mathbf{z}_1\mathbf{w}_1}(\mathbf{F}) = (\mathbf{C}_1 + \mathbf{D}_1\mathbf{F})(s\mathbf{I} - \mathbf{A} - \mathbf{B}_3\mathbf{F})^{-1}\mathbf{B}_1$$

and the closed-loop system transfer function from $\mathbf{w}_2$ to $\mathbf{z}_2$ is given by

$$T_{\mathbf{z}_2\mathbf{w}_2}(\mathbf{F}) = (\mathbf{C}_2 + \mathbf{D}_2\mathbf{F})(s\mathbf{I} - \mathbf{A} - \mathbf{B}_3\mathbf{F})^{-1}\mathbf{B}_2$$

Let $\mathbf{L}_o$ be the positive semidefinite solution of the equation

$$\mathbf{A}_c^{\mathrm{T}}\mathbf{L}_o + \mathbf{L}_o\mathbf{A}_c + \mathbf{C}_{2c}^{\mathrm{T}}\mathbf{C}_{2c} = \mathbf{0} \tag{3}$$

It can be shown that

$$\|T_{\mathbf{z}_2\mathbf{w}_2}(\mathbf{F})\|_2 = \begin{cases} \sqrt{\mathrm{Tr}(\mathbf{B}_2^{\mathrm{T}}\mathbf{L}_o\mathbf{B}_2)}, & \text{if } \mathbf{F} \in \Gamma_s \\ \infty, & \text{otherwise} \end{cases} \tag{4}$$

Lemma 1

Suppose $\mathbf{A}_c$ is stable. Then the following statements are equivalent:

(1) $\|T_{\mathbf{z}_1\mathbf{w}_1}(\mathbf{F})\|_\infty < \gamma$.

(2) For any given matrix $\hat{\mathbf{Q}}_\infty = \hat{\mathbf{Q}}_\infty^{\mathrm{T}} > 0$, there exists a positive definite solution $\hat{\mathbf{P}}_\infty = \hat{\mathbf{P}}_\infty^{\mathrm{T}}$ to

$$(\mathbf{A}_c + \frac{1}{\gamma^2}\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty)^{\mathrm{T}}\hat{\mathbf{P}}_\infty + \hat{\mathbf{P}}_\infty(\mathbf{A}_c + \frac{1}{\gamma^2}\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty) + \mathbf{F}^{\mathrm{T}}\mathbf{F} + \hat{\mathbf{Q}}_\infty = 0 \tag{5}$$

where $\mathbf{P}_\infty = \mathbf{P}_\infty^{\mathrm{T}} \geq 0$ is the stabilizing solution of

$$\mathbf{A}_c^{\mathrm{T}}\mathbf{P}_\infty + \mathbf{P}_\infty\mathbf{A}_c + \mathbf{C}_{1c}^{\mathrm{T}}\mathbf{C}_{1c} + \frac{1}{\gamma^2}\mathbf{P}_\infty\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty = 0 \tag{6}$$

Proof

According to the *Bounded Real Lemma* (see Reference [19]), it is known that (6) has a stabilizing solution $\mathbf{P}_\infty = \mathbf{P}_\infty^{\mathrm{T}} \geq 0$. Since the matrix $\mathbf{F}^{\mathrm{T}}\mathbf{F} + \hat{\mathbf{Q}}_\infty$ is positive definite and the matrix $\mathbf{A}_c + 1/\gamma^2\,\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty$ is stable, from the Lyapunov stability theorem, the unique solution $\hat{\mathbf{P}}_\infty$ of (5) is positive definite. □

Moreover, we have the following result.

Lemma 2

$\mathrm{Tr}(\hat{\mathbf{P}}_\infty)$ approaches infinity as $\mathbf{F}$ approaches the boundary of $\Gamma_\infty(\gamma)$.

Proof

As $\mathbf{F}$ approaches the boundary of $\Gamma_\infty(\gamma)$, there exists at least one eigenvalue $\lambda \in \left(\mathbf{A} + \mathbf{B}_3\mathbf{F} + (1/\gamma^2)\,\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty\right)$ that will approach the imaginary axis. Suppose $\mathbf{v}$ is the normalized eigenvector corresponding to λ. Premultiplying and postmultiplying (5) by $\mathbf{v}^*$ and $\mathbf{v}$, respectively, and after some manipulations, we obtain

$$\mathbf{v}^*\hat{\mathbf{P}}_\infty\mathbf{v} = -\frac{\mathbf{v}^*(\mathbf{F}^{\mathrm{T}}\mathbf{F} + \hat{\mathbf{Q}}_\infty)\mathbf{v}}{2Re(\lambda)}$$

Since $\mathbf{v}^*(\mathbf{F}^{\mathrm{T}}\mathbf{F} + \hat{\mathbf{Q}}_\infty)\mathbf{v} > 0$ and $\mathrm{Re}(\lambda) \to 0^-$, it follows that $\mathbf{v}^*\hat{\mathbf{P}}_\infty\mathbf{v} \to \infty$. Note that $\hat{\mathbf{P}}_\infty$ is positive definite. This leads to the conclusion that $\mathrm{Tr}(\hat{\mathbf{P}}_\infty) \to \infty$. This completes the proof. □

Suppose $\lambda = a + \mathbf{i}b$. Then Ω_i can be represented equivalently by

$$\Omega_i = \{\lambda \in C | g_i(\lambda) < 0\}$$

where $g_i(\lambda) = f_i(a,b)|_{a=(\lambda+\bar{\lambda})/2, b=(\lambda-\bar{\lambda})/2\mathbf{i}} = \sum_{p_i}\sum_{q_i} c_{p_iq_i}\lambda^{p_i}\bar{\lambda}^{q_i}$. Note that $c_{p_iq_i} = \bar{c}_{q_ip_i}$ (see Reference [20]). Now define $\varphi_i(\alpha,\beta) = \sum_{p_i}\sum_{q_i} c_{p_iq_i}\alpha^{p_i}\beta^{q_i}$.

Lemma 3 (Bambang *et al.* [17])

Suppose that for all $\alpha_j \in \Omega_i$ ($j = 1,\ldots, n$), the following conditions are satisfied.

(1) $\varphi_i(\alpha_j, \bar{\alpha}_k) \neq 0$ for all $j,k = 1,2,\ldots,n$,
(2) The Hermitian matrix $\Psi_i \equiv \left[-\varphi_i^{-1}(\alpha_j,\bar{\alpha}_k)\right]_{j,k=1}^n$ is non-negative definite.

Then $\sigma(\mathbf{A}_c) \subset \Omega_i$ if, and only if, for any given positive definite Hermitian matrix $\hat{\mathbf{Q}}$, the unique solution $\hat{\mathbf{P}}$ of the following equation

$$\sum_{p_i}\sum_{q_i} c_{p_iq_i}\left(\mathbf{A}_c^{q_i}\right)^*\mathbf{P}\mathbf{A}_c^{p_i} + \mathbf{Q}_i = 0 \tag{9}$$

is positive definite. □

A class of regions that satisfies conditions (1) and (2) of Lemma 3 include, among the many possibile, hyperbolic, circular, elliptic, parabolic, etc. More examples and details can be found in References [17] and [21], and references therein. The regions discussed in the following are all assumed to belong to this class of regions.

Remark 1

In order to get a real solution $\mathbf{F}$, the region $\mathbf{\Omega}$ must be symmetric with respect to the real axis of the complex plane. However, each individual $\mathbf{\Omega}_i$ is not required to be symmetric with respect to the real axis. This class of regions may be non-convex, may even contain several disjoint subregions. It should be noted that the LMI regions discussed in the literature (see e.g. Reference [18]) are restricted to be convex.

Let $\mathbf{A}_c = \mathbf{A} + \mathbf{B}_3\mathbf{F}$ in (9) and suppose $\mathbf{P}_i$, $i = 1, \ldots, c$, are the positive definite solutions of (9). It is known that $\mathbf{P}_i$ is a function of $\mathbf{F}$. Then we have the following theorem.

Lemma 4

$\mathrm{Tr}(\mathbf{P}_i)$ approaches infinity as $\mathbf{F}$ approaches the boundary of Γ_{Ω_i},

Proof

As $\mathbf{F}$ approaches the boundary of Γ_{Ω_i}, then $\sum_{p_i}\sum_{q_i} c_{p_iq_i}\lambda^{p_i}\bar{\lambda}^{q_i} \to 0^-$ for some $\lambda \in \sigma(\mathbf{A}+\mathbf{BF})$. Let v be the normalized eigenvector corresponding to λ. Premultiplying and postmultiplying (9) by $\mathbf{v}^*$ and $\mathbf{v}$, respectively, and after some manipulations, we obtain

$$\mathbf{v}^*\mathbf{P}_i\mathbf{v} = -\frac{\mathbf{v}^*\mathbf{Q}_i\mathbf{v}}{g_i(\lambda)}$$

Since $\mathbf{v}^*\mathbf{Q}_i\mathbf{v} > 0$ and $\gamma_i(\lambda) \to 0^-$, it follows that $\mathbf{v}^*\mathbf{P}_i\mathbf{v} \to \infty$. Note that $\mathbf{P}_i$ is positive definite, this implies $Tr(\mathbf{P}_i)$ approaches infinity as $\mathbf{F}$ approaches the boundary of Γ_{Ω_i}. □

3. THE MAIN RESULTS

The $Q_{2\infty p}$ problem is a constrained optimization problem. The infimal solution of the $Q_{2\infty p}$ problem may lie on the boundary of Γ and may not be a stationary point. Thus far, no analytic solution to this problem has been derived. In this section, we instead solve an auxiliary problem, denoted by the problem $Q_{2\infty p\text{aux}}$ to obtain an approximate solution of the problem $Q_{2\infty p}$. The cost function of the problem $Q_{2\infty p\text{aux}}$ is defined as

$$J_{\text{aux}}(\gamma, \mathbf{F}) = \begin{cases} J(\mathbf{F}) + w \cdot J_b(\gamma, \mathbf{F}), & \text{if } \mathbf{F} \in \Gamma \\ \infty, & \text{otherwise} \end{cases} \tag{10}$$

where $w > 0$ is the weighting factor and $J_b(\mathbf{F})$ is defined by

$$J_b(\gamma, \mathbf{F}) = l_\infty \text{Tr}(\hat{\mathbf{P}}_\infty) + \sum_{i=1}^{c} w_i \text{Tr}(\mathbf{P}_i)$$

where $l_\infty > 0$ and $w_i > 0$, $i = 1, \ldots, c$, are the weighting factors, and $\hat{\mathbf{P}}_\infty$ and $\mathbf{P}_i, i = 1, \ldots, c$, are the solutions of matrix equations (5) and (9).

The problem $Q_{2\infty p\text{aux}}$ is to find the feedback matrix $\mathbf{F}$ to minimize the cost function (10).

Lemma 5

$J_{\text{b}}(\mathbf{F})$ approaches infinity as $\mathbf{F}$ approaches the boundary of Γ.

Proof

This lemma can be proved directly from Lemmas 2 and 4. □

Now we shall show that the cost function $J_{\text{aux}}(\mathbf{F})$ has a minimum point in the interior of the set Γ if $\mathbf{D}_2$ has full column rank.

Lemma 6

Suppose $\mathbf{D}_2$ has full column rank. If the admissible set Γ is non-empty, then the cost function $J_{\text{aux}}(\mathbf{F})$ has a minimum point in the interior of the set Γ.

Proof

Define a level set

$$\Gamma_l(\mathbf{F}_0) \equiv \{\mathbf{F} \in \Gamma | J_{\text{aux}}(\mathbf{F}) \leq J_{\text{aux}}(\mathbf{F}_0), \text{ for } \mathbf{F}_0 \in \Gamma\}$$

The set $\Gamma_l(\mathbf{F}_0)$ is bounded since if $\mathbf{D}_2$ has full column rank, then $J(\mathbf{F}) \to \infty$ as $\|\mathbf{F}\| \to \infty$. Moreover, since $J_{\text{aux}}(\mathbf{F})$ is continuous in the set Γ and unbounded on the boundary of $\Gamma, \Gamma_l(\mathbf{F}_0)$ is closed. As a result, the level set $\Gamma_l(\mathbf{F}_0)$ is compact. From the Weiestrass theorem [10], there is a $\mathbf{F}_{\text{opt}} \in \Gamma_l(\mathbf{F}_0)$ such that

$$J_{\text{aux}}(\mathbf{F}_{\text{opt}}) \leq J_{\text{aux}}(\mathbf{F}), \text{ for all } \mathbf{F} \in \Gamma_l(\mathbf{F}_0)$$

This implies

$$J_{\text{aux}}(\mathbf{F}_{\text{opt}}) \leq J_{\text{aux}}(\mathbf{F}), \text{ for all } \mathbf{F} \in \Gamma_\Omega$$

and completes the proof. □

Since the minimum point of the auxiliary cost function $J_{\text{aux}}(\mathbf{F})$ lies in the interior of the admissible solution set, it must be a stationary point. The Lagrange multiplier method can be employed to derive the necessary conditions for local optimum of the $Q_{2\infty p\mathrm{aux}}$ problem to satisfy.

Theorem 1

The optimal solution F of the $Q_{2\infty p\mathrm{aux}}$ problem must satisfy

$$\begin{aligned} F_{\mathrm{grad}}(\gamma,\mathbf{F}) \equiv\; & 2\mathbf{D}_2^{\mathrm{T}}\mathbf{D}_2\mathbf{F}\mathbf{L}_2 + \mathbf{D}_1^{\mathrm{T}}\mathbf{D}_1\mathbf{F}(\mathbf{L}_\infty + \mathbf{L}_\infty^{\mathrm{T}}) + 2\mathbf{F}\hat{\mathbf{L}}_\infty + 2\sum_{i=1}^{c}\mathbf{F}\mathbf{S}_i \\ & + 2\mathbf{B}_3^{\mathrm{T}}\mathbf{L}_o\mathbf{L}_2 + 2\mathbf{D}_2^{\mathrm{T}}\mathbf{C}_2\mathbf{L}_2 + \mathbf{B}_3^{\mathrm{T}}\mathbf{P}_\infty\mathbf{L}_\infty^{\mathrm{T}} + \mathbf{B}_3^{\mathrm{T}}\mathbf{P}_\infty\mathbf{L}_\infty + \mathbf{D}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{L}_\infty^{\mathrm{T}} + \mathbf{D}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{L}_\infty \\ & + 2\mathbf{B}_3^{\mathrm{T}}\hat{\mathbf{P}}_\infty\hat{\mathbf{L}}_\infty + \sum_{i-1}^{c}\left[\sum_{p_i}\sum_{q_i} c_{p_iq_i}\left[\sum_{k=0}^{p_i-1}\mathbf{B}_3^{\mathrm{T}}(\mathbf{A}_c^*)^k\mathbf{P}_i\mathbf{A}_c^{q_i}\mathbf{S}_i(\mathbf{A}_c^*)^{p_i-1-k}\right.\right. \\ & \left.\left. + \sum_{k=0}^{q_i-1}\mathbf{B}_3^{\mathrm{T}}(\mathbf{A}_c^*)^k\mathbf{P}_i\mathbf{A}_c^{p_i}\mathbf{S}_i(\mathbf{A}_c^*)^{q_i-1-k}\right]\right] = 0 \end{aligned} \tag{11}$$

where $\mathbf{L}_o$, $\mathbf{L}_2$, $\mathbf{P}_\infty$, $\mathbf{L}_\infty$, $\hat{\mathbf{P}}_\infty\hat{\mathbf{L}}_\infty$, $\mathbf{P}_i, i = 1, \ldots, c$, and $\mathbf{S}_i$, $i = 1, \ldots, c$, satisfy the following matrix equations:

$$\mathbf{A}_c^{\mathrm{T}}\mathbf{L}_o + \mathbf{L}_o\mathbf{A}_c + \mathbf{C}_{2c}^{\mathrm{T}}\mathbf{C}_{2c} = 0 \tag{12}$$

$$\mathbf{A}_c\mathbf{L}_2 + \mathbf{L}_2\mathbf{A}_c^{\mathrm{T}} + \tfrac{1}{2}\big(Tr(\mathbf{B}_2^{\mathrm{T}}\mathbf{L}_o\mathbf{B}_2)\big)^{-1/2}\times\mathbf{B}_2\mathbf{B}_2^{\mathrm{T}} = 0 \tag{13}$$

$$\mathbf{A}_c\mathbf{P}_\infty + \mathbf{P}_\infty\mathbf{A}_c^{\mathrm{T}} + \mathbf{C}_{1c}^{\mathrm{T}}\mathbf{C}_{1c} + \frac{2}{\gamma^2}\mathbf{P}_\infty\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty = 0 \tag{14}$$

$$\mathbf{L}_\infty(\mathbf{A}_c + \frac{1}{\gamma^2}\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty) + (\mathbf{A}_c + \frac{1}{\gamma^2}\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty)^{\mathrm{T}}\mathbf{L}_\infty + \frac{1}{\gamma^2}\hat{\mathbf{L}}_\infty\hat{\mathbf{P}}_\infty\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}} = 0 \tag{15}$$

$$(\mathbf{A}_c + \frac{1}{\gamma^2}\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty)^{\mathrm{T}}\hat{\mathbf{P}}_\infty + \hat{\mathbf{P}}_\infty(\mathbf{A}_c + \frac{1}{\gamma^2}\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty) + \mathbf{F}^{\mathrm{T}}\mathbf{F} + \hat{\mathbf{Q}}_\infty = 0 \tag{16}$$

$$(\mathbf{A}_c + \frac{1}{\gamma^2}\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty)\hat{\mathbf{L}}_\infty + \hat{\mathbf{L}}_\infty(\mathbf{A}_c + \frac{1}{\gamma^2}\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty)^{\mathrm{T}} + w\cdot \boldsymbol{l}_\infty\cdot\mathbf{I} = 0 \tag{17}$$

$$\sum_{p_i}\sum_{q_i} c_{p_iq_i}(\mathbf{A}_c^*)^{q_i}\mathbf{P}_i\mathbf{A}_c^{p_i} + \mathbf{F}^{\mathrm{T}}\mathbf{F} + \mathbf{Q}_i = 0 \quad i = 1, 2, ..., c \tag{18}$$

$$\sum_{p_i}\sum_{q_i} c_{p_iq_i}\mathbf{A}_c^{p_i}\mathbf{S}_i(\mathbf{A}_c^*)^{q_i} + w\cdot w_i\cdot\mathbf{I} = 0 \quad i = 1, 2, ..., c \tag{19}$$

Proof

Define the Hamiltonian H_{am} by

$$H_{am} = \left(Tr\left(\mathbf{B}_2^{\mathrm{T}}\mathbf{L}_o\mathbf{B}_2\right)\right)^{1/2} + w \cdot \left(l_\infty Tr\left(\hat{\mathbf{P}}_\infty\right) + Tr\left(\sum_{i=1}^{c} w_i \mathbf{P}_i\right)\right)$$
$$+ Tr\left(\mathbf{L}_2\left(\mathbf{A}_c^{\mathrm{T}}\mathbf{L}_o + \mathbf{L}_o\mathbf{A}_c + \mathbf{C}_{2c}^{\mathrm{T}}\mathbf{C}_{2c}\right)\right)$$
$$+ Tr\left(\mathbf{L}_\infty\left(\mathbf{A}_c\mathbf{P}_\infty + \mathbf{P}_\infty\mathbf{A}_c^{\mathrm{T}} + \mathbf{C}_{1c}^{\mathrm{T}}\mathbf{C}_{1c} + \frac{1}{\gamma^2}\mathbf{P}_\infty\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty\right)\right)$$
$$+ Tr\left(\hat{\mathbf{L}}_\infty\left((\mathbf{A}_c + \frac{1}{\gamma^2}\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty)^{\mathrm{T}}\hat{\mathbf{P}}_\infty + \hat{\mathbf{P}}_\infty(\mathbf{A}_c + \frac{1}{\gamma^2}\mathbf{B}_1\mathbf{B}_1^{\mathrm{T}}\mathbf{P}_\infty) + \mathbf{F}^{\mathrm{T}}\mathbf{F} + \hat{\mathbf{Q}}_\infty\right)\right)$$
$$+ Tr\left(\sum_{i=1}^{c}\mathbf{S}_i\left(\sum_{p_i=0}^{n_{pi}}\sum_{q_i=0}^{n_{qi}} c_{p_i q_i}(\mathbf{A}_c^*)^{q_i}\mathbf{P}_i\mathbf{A}_c^{p_i} + \mathbf{F}^{\mathrm{T}}\mathbf{F} + \mathbf{Q}_i\right)\right)$$

where $\mathbf{L}_2, \mathbf{L}_\infty, \hat{\mathbf{L}}_\infty, \mathbf{S}_i (i = 1, 2, ..., c)$ are the Lagrange multipliers. Then the necessary conditions for local optimum are $\partial H_{am}/\partial \mathbf{L}_2 = 0$, $\partial H_{am}/\partial \mathbf{L}_o = 0$, $\partial H_{am}/\partial \mathbf{L}_\infty = 0$, $\partial H_{am}/\partial \mathbf{P}_\infty = 0$, $\partial H_{am}/\partial \hat{\mathbf{L}}_\infty = 0$, $\partial H_{am}/\partial \hat{\mathbf{P}}_\infty = 0$, $\partial H_{am}/\partial \mathbf{S}_i = 0$, $\partial H_{am}/\partial \mathbf{P}_i = 0$, and $\partial H_{am}/\partial \mathbf{F} = 0$. After some algebraic manipulations, results of (11)–(19) can be derived. □

For a fixed weighting factor w, suppose the optimal solution of problem $Q_{2\infty p\mathrm{aux}}$ is $\mathbf{F}_{\mathrm{opt}}(w)$. Suppose the infimal solution of the $Q_{2\infty p}$ problem is $\mathbf{F}_*$. Then we have the following result.

Proposition 1

$$\lim_{w\to 0^+} J(\mathbf{F}_{\mathrm{opt}}(w)) \to J(\mathbf{F}_*).$$

Proof

For any $\varepsilon > 0$, define the set Γ_ε by

$$\Gamma_\varepsilon \equiv \left\{\mathbf{F} \in \Gamma \middle| J(\mathbf{F}) - J(\mathbf{F}_*) < \tfrac{1}{2}\varepsilon\right\}$$

This set is non-empty since $J(\mathbf{F})$ is continuous in Γ. Set $w_\varepsilon = \varepsilon/2J_b(\mathbf{F}^a)$, for some $\mathbf{F}^{\mathrm{a}} \in \Gamma_\varepsilon$. Note that $J(\mathbf{F})$ is continuous over the set Γ, and $J(\mathbf{F}) > 0$ is bounded for any $\mathbf{F} \in \Gamma$, so $w_\varepsilon > 0$. If choose w such that $0 < w < w_\varepsilon$, then $w \cdot J_b(\mathbf{F}^a) < \frac{1}{2}\varepsilon$. Moreover,

$$J_{\mathrm{aux}}(\mathbf{F}^a) - J(\mathbf{F}_*) = J(\mathbf{F}^a) + w \cdot J_b(\mathbf{F}^a) - J(\mathbf{F}_*) < \varepsilon$$

That is, for any $\varepsilon > 0$, we can find w satisfying $0 < w < w_\varepsilon$ such that

$$\min_{\mathbf{F}\in\Gamma} J_{\mathrm{aux}}(\mathbf{F}) - J(\mathbf{F}_*) \le J_{\mathrm{aux}}(\mathbf{F}^a) - J(\mathbf{F}_*) < \varepsilon$$

It should be noted that $J_{\mathrm{aux}}(\mathbf{F}_{\mathrm{opt}}(w))$ is decreasing as w is decreasing. So

$$\lim_{w\to 0^+} J_{\mathrm{aux}}(\mathbf{F}_{\mathrm{opt}}(w)) \to J(\mathbf{F}_*)$$

Since $J_{aux}(\mathbf{F}_{opt}(w)) > J(\mathbf{F}_{opt}(w)) \geq J(\mathbf{F}_*)$, for all $w > 0$, it can be concluded that

$$\lim_{w\to 0^+} J(\mathbf{F}_{opt}(w)) \to J(\mathbf{F}_*)$$

This completes the proof. □

This proposition shows that the minimum solution of the problem $Q_{2\infty p\text{aux}}$ will converge to the infimal solution of the $Q_{2\infty p}$ problem if $w \to 0^+$.

The coupling equations (13)–(21) are not easy to solve. Based on steepest descent method, solution algorithms are provided in Appendix A. It should be noted that the problem $Q_{2\infty p\text{aux}}$ may have several local optimal solutions, and the solution obtained via the proposed algorithms may not be the global optimal solution. However, in the following example we will see that the proposed approach is useful.

4. AN ILLUSTRATIVE EXAMPLE

Consider the following system:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 1/2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{w}_1(t) + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \mathbf{w}_2(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 3/5 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{z}_1(t) = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 0 & 1/3 & 1/3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{z}_2(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t)$$

Suppose the constraint region Ω (shown in Figure 1) is defined by

$$\Omega \equiv \left\{ (x + \mathbf{i}y) \middle| x > -50, x^2 + y^2 > 9,\ 10x + y^2 < 0 \right\}$$

The objective is to find $\mathbf{u}(t) = \mathbf{F}\mathbf{x}(t)$ to achieve the infimun of $\|T_{\mathbf{z}_2\mathbf{w}_2}(\mathbf{F})\|_2$ under the constraints (1) $\|T_{\mathbf{z}_1\mathbf{w}_1}(\mathbf{F})\|_\infty < 1$, and (2) $\sigma(\mathbf{A} + \mathbf{B}_3\mathbf{F}) \subset \Omega$.

It should be noted that the region Ω is not convex. It is not a LMI region.

For comparison, we consider seven cases. In case 1, we consider the H_2 optimal control problem without constraints (1) and (2). In case 2, we consider the H_2 optimal control problem under constraint (1) and without constraint (2). In case 3, we consider the H_2 optimal control problem under constraint (2) and without constraint (1). In cases 4–7, we consider the H_2 optimal control problem under both constraints (1) and (2) with different weighting factors. Let the matrix $\mathbf{Q}_i$ ($i = 1$, 2, and 3) in Equation (10) and the matrix $\hat{\mathbf{Q}}_\infty$ in Equation (5) be identity matrices. Without loss of generality, let the weighting factor $w = 1$ in cases 2–7. By applying the algorithms shown in Appendix A, the final results are shown in Table I, and the locations of the resultant closed-loop poles are shown in Figure 2.

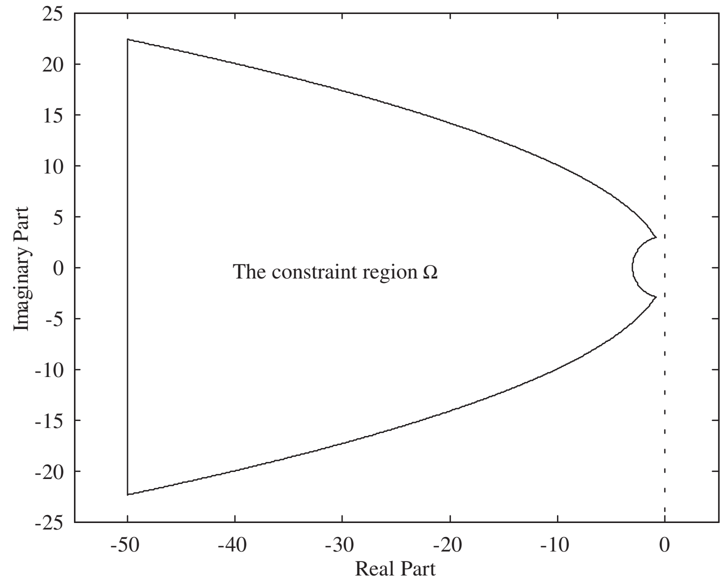

Figure 1. The constraint region Ω.

For case 1, we can see that the resultant $\|T_{\mathbf{z}_2\mathbf{w}_2}(\mathbf{F}_{\rm opt})\|_2 = 3.9572$ is the smallest value among all the seven cases. However, some of the resultant closed-loop poles are outside the region Ω and the resultant $\|T_{\mathbf{z}_1\mathbf{w}_1}(\mathbf{F}_{\rm opt})\|_\infty$ is larger than 1 since we did not put these constraints into the design procedure. In case 2, we only put the H_∞-norm constraint into the controller design procedure. Letting $l_\infty = 1$. It should be noted that the constraint of $\|T_{\mathbf{z}_1\mathbf{w}_1}(\mathbf{F}_{\rm opt})\|_\infty < 1$ is satisfied. However, some of the closed-loop poles are outside the region Ω. In case 3, we only put the poles' constraint into the design procedure. It can be verified that all the closed-loop poles lie in the desired region Ω. However, the resultant $\|T_{\mathbf{z}_1\mathbf{w}_1}(\mathbf{F}_{\rm opt})\|_\infty$ is larger than 1. In cases 4–7, both the pole constraint and the H_∞ -norm constraint are considered. From Table I, we can see that for these four cases, all the pole constraint and H_∞ -norm constraint are satisfied. In general, if both the weighting factors l_∞ and w_i are decreasing, the resultant value of $\|T_{\mathbf{z}_2\mathbf{w}_2}(\mathbf{F}_{\rm opt})\|_2$ will decrease. Note that the weighting factors in the Case 7 are nearly zero, we can expect that the infimum of $\|T_{\mathbf{z}_2\mathbf{w}_2}(\mathbf{F}_{\rm opt})\|_2$, under the constraints that $\|T_{\mathbf{z}_1\mathbf{w}_1}(\mathbf{F}_{\rm opt})\|_\infty < 1$ and all the closed-loop poles lying in the region Ω, is about 6.1988.

5. CONCLUSIONS

A new method for the regional pole constraint mixed H_2/H_∞ state feedback control problem is provided. The considered constrained region, which is represented by several inequalities, may be non-convex. In some special case, it may contain several disjoint subregions. The solution of the considered problem is approximately obtained via solving an auxiliarily optimization problem. We have proved that the obtained solution can be arbitrarily close to the infimal solution of the original constrained optimization problem. Moreover, solution algorithms are provided. Furthermore, an illustrative example is included to demonstrate the presented

Table I. The results ($w=1$)

	$\mathbf{F}_{opt}$	$\sigma(\mathbf{A}_c)$	$\|\mathbf{T}_{\mathbf{z}_2\mathbf{w}_2}\|_2$	$\|\mathbf{T}_{\mathbf{z}_1\mathbf{w}_1}\|_\infty$
Case 1	$\begin{bmatrix} 1.7705 & 0.2080 & -6.1090 \\ -0.4566 & 2.1091 & -3.4309 \end{bmatrix}$	$-0.7+\mathbf{i}\times 0.3317$, $-0.7-\mathbf{i}\times 0.3317$, -1	3.9572	5.4430
Case 2 $l_\infty=1$	$\begin{bmatrix} 5.8505 & 6.1651 & -17.2029 \\ 1.0439 & 7.5974 & -12.8610 \end{bmatrix}$	-0.5931, -2.3139, -8.8035	6.5900	0.7030
Case 3 $w_i=1$ $(i=1,2,3)$	$\begin{bmatrix} 8.7555 & 11.5623 & -19.1110 \\ -1.1783 & 14.2953 & -15.4660 \end{bmatrix}$	$-2.4409+\mathbf{i}\times 3.5040$, $-2.4409-\mathbf{i}\times 3.5040$, -7.1248	10.3340	1.3307
Case 4 $l_\infty=1000$, $w_i=1$ $(i=1,2,3)$	$\begin{bmatrix} 5.4808 & 11.7021 & -19.9334 \\ -5.3472 & 18.5253 & -19.1625 \end{bmatrix}$	$-3.0456+\mathbf{i}\times 0.7725$, $-3.0456-\mathbf{i}\times 0.7725$, -12.9847	7.9628	0.8508
Case 5 $l_\infty=1$, $w_i=1000$	$\begin{bmatrix} 7.7090 & 22.3760 & -30.6759 \\ -3.2914 & 23.3656 & -24.7824 \end{bmatrix}$	$-3.2894+\mathbf{i}\times 2.4050$, $-3.2894-\mathbf{i}\times 2.4050$, -13.8820	10.4491	0.9987
Case 6 $l_\infty=10^{-3}$, $w_i=10^{-3}$	$\begin{bmatrix} -1.6795 & 15.2044 & -17.0812 \\ -12.3278 & 21.2104 & -7.8760 \end{bmatrix}$	$-2.8048+\mathbf{i}\times 2.0075$, $-2.8048-\mathbf{i}\times 2.0075$, -11.4169	7.5411	0.9450
Case 7 $l_\infty=10^{-5}$, $w_i=10^{-5}$	$\begin{bmatrix} 1.4664 & 5.1435 & -9.4430 \\ -5.7785 & 10.6113 & -7.8760 \end{bmatrix}$	$-2.7411+\mathbf{i}\times 1.5956$, $-2.7411-\mathbf{i}\times 1.5956$, -3.3215	6.1988	0.9959

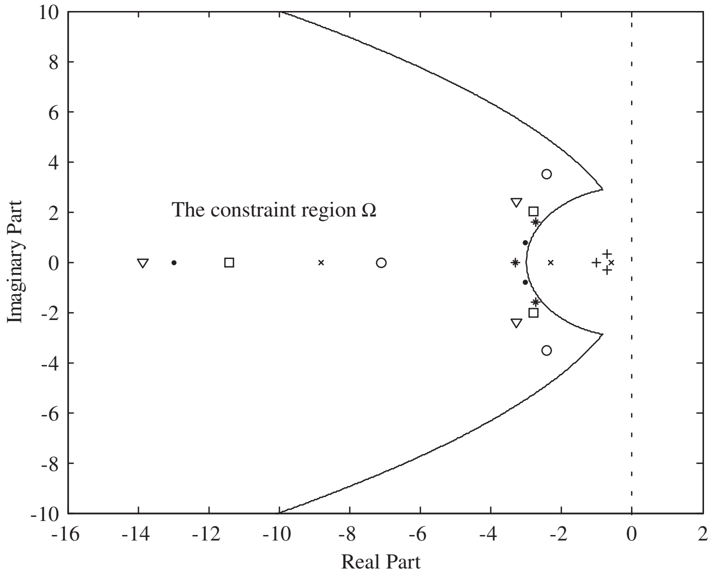

Figure 2. The locations of the resultant closed-loop poles of the example (+: case 1, ×: case 2, ○: case 3, ●: case 4, ▽: case 5, □: case 6, ✻: case 7).

approach. Although we only consider the state feedback case, in fact, the output feedback case can be solved without any difficult via the same procedure.

APPENDIX A

Here we provide a solution algorithm for the $Q_{2\infty paux}$ problem. Note that the $F_{\text{grad}}(\gamma,\mathbf{F})$ defined in (13) is the gradient of $J_{\text{aux}}(\mathbf{F})$ with respect to $\mathbf{F}$ under the constraints of (4)–(6) and (9) (see chapter 1 of [22]). If $\mathbf{F}_{\text{opt}}$ is a minimal solution, then $F_{\text{grad}}(\gamma,\mathbf{F}_{\text{opt}})=0$. Based on the steepest descent method, a solution algorithm is proposed below. Suppose $\varepsilon>0$ is a specified small number. The algorithm will stop if $\|F_{\text{grad}}(\mathbf{F})\|<\varepsilon$.

Algorithm 1

Find the optimal solution $\mathbf{F}_{\text{opt}}$ of the auxiliary minimization problem.

(1) Choose a $\mathbf{F}(0)\in\Gamma$. Set $k=0$.
(2) Solving (12)–(19), by substituting $\mathbf{F}$ by $\mathbf{F}(k)$, yield $\mathbf{L}_o(k)$, $\mathbf{L}_2(k)$, $\mathbf{P}_\infty(k)$, $\mathbf{L}_\infty(k)$, $\hat{\mathbf{P}}_\infty(k)$, $\hat{\mathbf{L}}_\infty(k)$, $\mathbf{P}_i(k)$, $\mathbf{S}_i(k)$.
(3) Find $F_{\text{grad}}(\gamma,\mathbf{F}(k))$.
(4) If $\|F_{\text{grad}}(\gamma,\mathbf{F}(k))\|<\varepsilon$, then $\mathbf{F}_{\text{opt}}=\mathbf{F}(k)$, end.
else find $\delta(k)>0$ via line search techniques such that $\mathbf{F}(k+1)=\mathbf{F}(k)-\delta(k)\cdot F_{\text{grad}}(\gamma,\mathbf{F}(k))$ shall minimize $J(\mathbf{F}(k+1))$. Let $k=k+1$, go to (2).

The step (1) in the Algorithm 1 is not a trivial task. In the following, we will provide an algorithm to find a $\mathbf{F}\in\Gamma$.

Algorithm 2

Find $\mathbf{F}$ such that $\mathbf{F} \in \Gamma$.

(1) Choose any $\mathbf{F}(0) \in \Gamma_\Omega$.
If $\|(\mathbf{C}_1 + \mathbf{D}_1\mathbf{F}(0))(s\mathbf{I} - \mathbf{A} - \mathbf{B}_3\mathbf{F}(0))^{-1}\mathbf{B}_1\|_\infty < \gamma$, then let $\mathbf{F} = \mathbf{F}(0)$, end.
else choose $\hat{\gamma}(0) > \|(\mathbf{C}_1 + \mathbf{D}_1\mathbf{F}(0))(s\mathbf{I} - \mathbf{A} - \mathbf{B}_3\mathbf{F}(0))^{-1}\mathbf{B}_1\|_\infty$ and set $k = 1$, go to (2).

(2) Solving (12)–(19), by substituting F by $\mathbf{F}(k)$ and γ by $\hat{\gamma}(k)$, yield $\mathbf{L}_o(k)$, $\mathbf{L}_2(k)$, $\mathbf{P}_\infty(k)$, $\mathbf{L}_\infty(k)$, $\hat{\mathbf{P}}_\infty(k)$, $\hat{\mathbf{L}}_\infty(k)$, $\mathbf{P}_i(k)$, $\mathbf{S}_i(k)$.

(3) Find $F_{\text{grad}}(\hat{\gamma}(k), \mathbf{F}(k))$.

(4) Find $\delta(k) > 0$, via line search technique, such that $\mathbf{F}(k+1) = \mathbf{F}(k) - \delta(k) \cdot F_{\text{grad}}(\gamma, \mathbf{F}(k))$ shall minimize $J_{\text{aux}}(\hat{\gamma}(k), \mathbf{F}(k+1))$.

(5) Let $\tilde{\gamma} = (k)\|(\mathbf{C}_1 + \mathbf{D}_1\mathbf{F}(k+1))(s\mathbf{I} - \mathbf{A} - \mathbf{B}_3\mathbf{F}(k+1))^{-1}\mathbf{B}_1\|_\infty$.
If $\tilde{\gamma}(k) < \gamma$, then $\mathbf{F} = \mathbf{F}(k+1) \in \Gamma$, end
else set $\hat{\gamma}(k+1) = \hat{\gamma}(k) - \eta(\hat{\gamma}(k) - \tilde{\gamma}(k))$, for $0 < \eta < 1$, and let $k = k+1$, go to (2).

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REFERENCES

1. Bernstein DS, Haddad WM. LQG control with an H_∞ performance bound: a riccati equation approach, *IEEE Transactions on Automatic Control* 1989; **34**: 293–305.
2. Doyle JC, Zhou K, Glover K, Bodenheimer B. Mixed H_2 and H_∞ performance objectives II: optimal control. *IEEE Transactions on Automatic Control* 1994; **39**: 1575–1587.
3. Khargonekar PP, Rotea MA. Mixed H_2/H_∞ Control: a convex optimization approach. *IEEE Transactions on Automatic Control* 1991; **36**: 824–837.
4. Limebeer DJN, Anderson BDO, Hendel B. A nash game approach to mixed H_2/H_∞ Control. *IEEE Transactions on Automatic Control* 1994; **39**: 69–82.
5. Rotea MA, Khargonekar PP. H_2 -optimal control with an H_∞ -constraint: the state feedback case. *Automatica* 1991; **27**: 307–316.
6. Saberi A, Chen BM, Sannuti P, Ly UL. Simultaneous H_2/H_∞ optimal control: the state feedback case. *Automatica*, 1993; **29**: 1611–1614.
7. Zhou K, Glover K, Bodenheimer B, Doyle JC. Mixed H_2 and H_∞ performance objectives I: robust performance analysis. *IEEE Transactions on Automatic Control* 1994; **39**: 1564–1574.
8. Gutman S, Jury EI. A general theory for mixed root-clustering in subregions of the complex plane, *IEEE Transactions on Automatic Control* 1981; **26**: 853–863.
9. Haddad WM, Bernstein DS. Controller Design with Regional Pole Constraints. *IEEE Transactions on Automatic Control* 1992; **37**: 54–69.
10. Juang JC, Lee TT. On optimal pole assignment in a specified region. *International Journal of Control* 1984; **40**: 65–79.
11. Kawaski N, Shimemura E. pole placement in a specified region based on a linear quadratic regulator. *International Journal of Control* 1988; **48**: 225–240.
12. Kim JS, Lee CW. Optimal pole assignment into specified region and its application to rotating mechanical systems. *Optimal Control Applications & Methods* 1990; **11**: 197–210.
13. Yedavalli RK, Liu Y. H_∞ control with regional stability constraints. *Automatica* 1995; **31**: 611–615.
14. Wang YW, Bernstein DS. Optimal control synthesis with an α-shifted pole constraint. *Proceedings of the 31th Conference on Decision and Control*, Tucson, Arizona, 1992, 3711–3716.
15. Bambang RT, Shimemura E, Uchida K. Mixed H_2/H_∞ Control with Pole Placement: State Feedback case. Proceedings of 1993 American Control Conference San Francisco, CA, 1993; 2777–2779.
16. Yaz E, Grigoriadis K, Skelton RE. Discrete-time H_2/H_∞ norm-bounding pole placement control and estimation. *Proceedings of the 31th Conference on Decision and Control*, Tucson, Arizona, 1992; 487–492.
17. Bambang RT, Shimemura E, Uchida K. Mixed H_2/H_∞ Control with pole placement in a class of regions. *Optimal Control Applications & Methods* 1994; **15**: 151–173.

18. Cholali M, Gahinet P. H_∞ design with pole placement constraints: an LMI approach. *IEEE Transactions on Automatic Control* 1996; **41**: 358–367.
19. Petersen IR, Anderson BDO, Jonckheere EA. A first principles solution to the non-singular H_∞ control problem. *International Journal of Robust and Nonlinear Control* 1991; **1**: 171–185.
20. Furuta K, Kim SB. Pole assignment in a specified disk. *IEEE Transactions on Automatic Control* 1987; **32**: 423–427.
21. Gutman S. Root Clustering in parameter Space. *Lecture Notes in Control and Information Science*, vol. 141. Springer: Berlin, 1990.
22. Bryson AE, Ho, YC. *Applied Optimal Control*. Hemisphere Publishing Corporation: Washington, DC, 1975.
23. Luenberger DG. *Linear and Nonlinear programming* (2nd edn). Addison-Wesley: Reading, 1989.
24. Shieh LS, Dib HM, Mcinnis BC. Linear quadratic regulators with eigenvalue placement in a vertical strip. *IEEE Transactions on Automatic Control* 1986; **31**: 241–243.

