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More general expression for the torsional warping of a thin-walled open-section beam

Wen Yi Lin[a,*], Kuo Mo Hsiao[b]

[a] *Department of Mechanical Engineering, De Lin Institute of Technology, 1 Alley 380, Ching Yun Road, Tucheng, Taiwan, ROC*

[b] *Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, Taiwan, ROC*

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Abstract

In this study, an analytical formulation for the torsional warping function of a thin-walled open-section beam is built based on the combination of the Vlasov assumption and the Kirchhoff assumption of plate/shell theory, essentially due to Goodier and Gjelsvik. Vlasov, Timoshenko and many authors (follow Vlasov) only consider the contour warping function as the real warping function. Goodier and Gjelsvik (follow Goodier) not only consider the contour warping but also the thickness warping. For some cross-sectional constants related to the warping function such as the torsional constant, the adoption of the warping function based on Vlasov's theory or Timoshenko's theory may cause them incorrectness. On the other hand, the torsional constant based on the Goodier's theory is consistent with the one based on the membrane analogy. Thus, Goodier's theory is a good approximation for the torsional warping of a thin-walled open-section beam. To the authors' limited knowledge, the analytical expression for the complete torsional warping of a thin-walled open-section beam has not been found in the literature. In this study, a more general expression for the torsional warping of a thin-walled open-section beam is presented. The position formulas to determine the twist center are also given.

1. Introduction

The torsional warping function for a thin-walled open-section beam may contain two parts: the contour warping function (the primary warping) and the thickness warping function (the secondary warping) [1–3]. Vlasov [4], Timoshenko and Gere [5], and many authors only consider the contour

* Corresponding author.
E-mail address: wylin@dlit.edu.tw (W.Y. Lin).

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warping function as the real warping function. Gjelsvik [3] followed Goodier [1] to have the warping function of a thin-walled open-section beam composed of the warping of the contour and the warping of the wall relative to the contour. Due to some thin sections where the contour warping is much larger than the thickness warping and the contribution of the thickness warping to the warping constant [2] may be small, the vast majority of researchers only consider the contour warping function as the warping function. However, this terminology can be misleading, since occasionally the thickness warping function is the dominant [2,3]. In this study, all the cross-sectional constants, related to the warping function, arising from a second-order linearization of the fully geometrically nonlinear beam theory [6–8] will show the differences between Vlasov's theory and the Goodier's theory. It can been seen that for some cross-sectional constants related to the warping function such as the warping constant, the adoption of the warping function based on Vlasov's theory or Timoshenko's theory or Goodier's theory might make only a few differences. On the other hand, for some cross-sectional constants related to the warping function such as the torsional constant, the adoption of the warping function based on Vlasov's theory or Timoshenko's theory may cause them incorrectness, rather than only a few difference. Moreover, the torsional constant based on the Goodier's theory is consistent with the one based on the membrane analogy [9]. Therefore, Goodier's theory is a good approximation for the warping of a thin-walled open-section beam and then the thickness warping function should be considered in theory. To the authors' limited knowledge, the analytical expression for the complete torsional warping of a thin-walled open-section beam has not been found in the literature. The examples of open thin sections in Ref. [3] only include bisymmetric and limited monosymmetric ones.

The object of this paper is to obtain a more general expression for the torsional warping function of a thin-walled open-section beam. In this study, an analytical formulation for the warping function is built based on the combination of the Vlasov assumption [4] and the Kirchhoff assumption of plate/shell theory [10], essentially due to Goodier [1] and Gjelsvik [3]. The position formulas to determine the twist center are also given. Note that the previous derivations of the position formulas whether for Vlasov [4] or for Gjelsvik [3] did not consider the thickness warping. The present derivation of the position formulas of the twist center will use the complete expression for the torsional warping obtained in this study.

Goodier's theory cannot be applied to the warping of a thick-walled open-section beam. The warping of a thick-walled open-section beam is worth investigating, but that is beyond the scope of this paper.

2. Analytical formulation for the torsional warping

Based on the Vlasov assumption [4] and the Kirchhoff assumption of plate/shell [10], the analytical formulation for the warping function of a thin-walled open-section beam subjected to a pure torque is built. First of all, the terminology used by Gjelsvik [3] is introduced as follows.

The surface midway through the walls of the beams is known as the middle surface. The intersection of the middle surface with the cross section is referred to as the contour of the cross section. Sharp corners and junctions in the contour are permissible and are collectively called junctions. The section of the contour lying between two junctions or between a junction and an end point is termed a branch of the contour. The corresponding part of the beam is called an element.

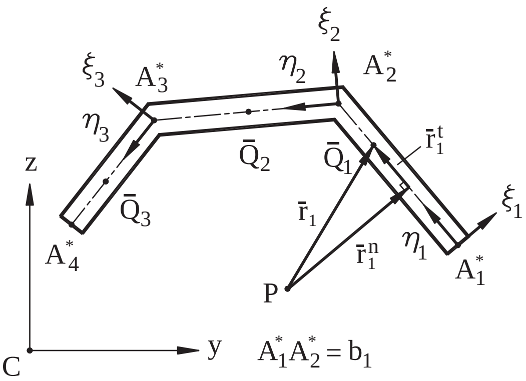

Fig. 1. Branches and local coordinate systems for the torsional warping.

The four assumptions are made: (1) small deformations, (2) the contour does not deform in its own plane, (3) the shear strain of the middle surface is zero in each element, and (4) each element behaves as a thin plate, that is, the Kirchhoff hypothesis is valid. Note that the Kirchhoff hypothesis contains the following: (a) an arbitrary lineal element extending through the plate thickness remains perpendicular to the mid surface of the plate (assumption (4a)), and (b) the lineal element is rigid (assumption (4b)), that is, the lineal element through the plate thickness does not elongate or contract and remains straight upon the application of load [10].

Consider the contour of any thin-walled open cross section as shown in Fig. 1. In Fig. 1, $Cxyz$ is the principal centroidal (right-handed Cartesian) coordinate system with its origin located at the centroid C of the cross section. Determine branches of the cross section and the corresponding elements. Define local coordinates of branches as shown in Fig. 1.

Choose point A_1^* as the reference point of the axial displacement defined on the unwarping cross section. In this study, the superscript * denotes an end point or a junction. Assume y_p and z_p are the coordinates of the twist center P in the y- and z-axis, respectively. For the branch $A_1^*A_2^*$ and the corresponding element, the position vector of point $\bar{Q}_1$, an arbitrary point on the branch $A_1^*A_2^*$, relative to the twist point P in the undeformed configuration (see Fig. 1) may be expressed in the local coordinates $A_1^*\xi_1\eta_1$ as

$$\boldsymbol{r}_{\bar{Q}_1/P} = \bar{r}_1^n \boldsymbol{e}_1^n + \bar{r}_1^t \boldsymbol{e}_1^t, \tag{1}$$

where $\boldsymbol{e}_1^n$ and $\boldsymbol{e}_1^t$ represent the unit vectors of ξ_1 and η_1 axes, respectively. From assumptions (1) and (2), we may have the displacement of point $\bar{Q}_1$ due to the small counterclockwise rotation of the cross section as follows:

$$\theta_1 \boldsymbol{e}_1 \times \boldsymbol{r}_{\bar{Q}_1/P} = -\bar{r}_1^t \theta_1 \boldsymbol{e}_1^n + \bar{r}_1^n \theta_1 \boldsymbol{e}_1^t, \tag{2}$$

where θ_1 is the twist angle of the cross section and $\boldsymbol{e}_1$ represents the unit vector of x-axis. The individual component of the displacement of point $\bar{Q}_1$ in the directions of ξ_1- and η_1-axis may be expressed as

$$v_{\bar{Q}_1} = -\bar{r}_1^t \theta_1, \tag{3}$$

$$w_{\bar{Q}_1} = \bar{r}_1^n \theta_1. \tag{4}$$

From assumption (3), we have

$$\gamma_{x\eta_1} = \frac{\partial u_{\bar{Q}_1}}{\partial \eta_1} + \frac{\partial w_{\bar{Q}_1}}{\partial x} = 0, \tag{5}$$

where $\gamma_{x\eta_1}$ is the shear strain of the middle surface and $u_{\bar{Q}_1}$ is the displacement component of point $\bar{Q}_1$ in the x-axis. Using Eq. (4) in Eq. (5), we obtain

$$\frac{\partial u_{\bar{Q}_1}}{\partial \eta_1} = -\bar{r}_1^n \theta_1'. \tag{6}$$

In this study, the symbol $()'$ denotes the derivative with respect to x. From Eq. (6), we may get the warping displacement of point $\bar{Q}_1$ as follows:

$$u_{\bar{Q}_1} = \theta_1' \omega_{A_1^*}^c(\bar{Q}_1) + u_{A_1^*}, \tag{7}$$

$$\omega_{A_1^*}^c(\bar{Q}_1) = -\int_0^{\eta_1} \bar{r}_1^n \, \mathrm{d}\eta_1, \tag{8}$$

where $u_{A_1^*}$ is the warping displacement of the reference point A_1^*, and $\omega_{A_1^*}^c(\bar{Q}_1)$ is the contour warping function of point $\bar{Q}_1$ with the reference point A_1^*. The superscript c is for the contour warping. From Eq. (7), we have

$$u_{A_2^*} = \theta_1' \omega_{A_1^*}^c(A_2^*) + u_{A_1^*}, \tag{9}$$

$$\omega_{A_1^*}^c(A_2^*) = -\int_0^{b_1} \bar{r}_1^n \, \mathrm{d}\eta_1, \tag{10}$$

where b_1 is the length of the branch $A_1^* A_2^*$.

Following the above procedures for the branch $A_2^* A_3^*$ and the corresponding element, we may have, with the aid of Eq. (9),

$$u_{\bar{Q}_2} = -\theta_1' \int_0^{\eta_2} \bar{r}_2^n \, \mathrm{d}\eta_2 + u_{A_2^*} = \theta_1' \omega_{A_1^*}^c(\bar{Q}_2) + u_{A_1^*}, \tag{11}$$

$$\omega_{A_1^*}^c(\bar{Q}_2) = \omega_{A_1^*}^c(A_2^*) - \int_0^{\eta_2} \bar{r}_2^n \, \mathrm{d}\eta_2. \tag{12}$$

Similarly, for an arbitrary branch $A_n^* A_{n+1}^*$ and the corresponding element, the warping displacement of point $\bar{Q}_n$, an arbitrary point on the branch $A_n^* A_{n+1}^*$, may be given by

$$u_{\bar{Q}_n} = \theta_1' \omega_{A_1^*}^c(\bar{Q}_n) + u_{A_1^*}, \tag{13}$$

where

$$\omega^c_{A_1^*}(\bar{Q}_n) = \omega^c_{A_1^*}(A_n^*) - \int_0^{\eta_n} \bar{r}_n^n \, d\eta_n. \tag{14}$$

For the branch $A_1^* A_2^*$ and the corresponding element, the displacement vector of the point $Q_1(x, \xi_1, \eta_1)$ relative to the point $\bar{Q}_1(x, 0, \eta_1)$ may be obtained by means of assumptions (1) and (4b) as follows:

$$\boldsymbol{u}_{Q_1} - \boldsymbol{u}_{\bar{Q}_1} = (\theta_1 \boldsymbol{e}_1 + \psi \boldsymbol{e}_1^t) \times \xi_1 \boldsymbol{e}_1^n, \tag{15}$$

where ψ may be considered as the small counterclockwise rotation of rigid lineal element (lying on the unwarping cross section perpendicular to the middle surface) about the η-axis. From Eq. (15), we have

$$u_{Q_1} = u_{\bar{Q}_1} - \xi_1 \psi, \tag{16}$$

$$v_{Q_1} = v_{\bar{Q}_1}. \tag{17}$$

Assumption (4a) gives us the shear strain of the $x\xi_1$ plane with the rigid lineal element equal to zero, i.e.

$$\gamma_{x\xi_1} = \frac{\partial u_{Q_1}}{\partial \xi_1} + \frac{\partial v_{Q_1}}{\partial x} = 0. \tag{18}$$

By Eqs. (16)–(18), we may obtain ψ and then we have

$$u_{Q_1} = u_{\bar{Q}_1} - \xi_1 \frac{\partial v_{\bar{Q}_1}}{\partial x}. \tag{19}$$

Substituting Eqs. (3) and (7) into Eq. (19) yields

$$u_{Q_1} = \theta_1' \omega_{A_1^*}(Q_1) + u_{A_1^*}, \tag{20}$$

$$\omega_{A_1^*}(Q_1) = \omega^c_{A_1^*}(\bar{Q}_1) + \omega^t(Q_1), \tag{21}$$

$$\omega^t(Q_1) = \bar{r}_1^t \xi_1, \tag{22}$$

where $\omega^c_{A_1^*}(\bar{Q}_1)$ is given in Eq. (8) and $\omega^t(Q_1)$ is the thickness warping function of point Q_1 relative to the point $\bar{Q}_1$. The superscript t is for the thickness warping.

Following the above procedures for the general branch $A_n^* A_{n+1}^*$ and the corresponding element, we may obtain the warping displacement of point $Q_n(x, \xi_n, \eta_n)$ as follows:

$$u_{Q_n} = \theta_1' \omega_{A_1^*}(Q_n) + u_{A_1^*}, \tag{23}$$

$$\omega_{A_1^*}(Q_n) = \omega^c_{A_1^*}(\bar{Q}_n) + \omega^t(Q_n), \tag{24}$$

$$\omega^t(Q_n) = \bar{r}_n^t \xi_n, \tag{25}$$

where $\omega^c_{A_1^*}(\bar{Q}_n)$ is given in Eq. (14).

The form of the warping displacement may be essentially similar to the one according to the Saint-Venant torsion theory because of no consideration of the warping shear strain. Besides, because the warping should be independent of the choice of the reference point of the axial displacement, the warping displacement of an arbitrary point $Q_n(x, \xi_n, \eta_n)$ may be expressed as follows:

$$u_{Q_n} = \theta_1' \omega(Q_n), \tag{26}$$

$$\omega(Q_n) = \omega^c(\bar{Q}_n) + \omega^t(Q_n), \tag{27}$$

$$\omega^c(\bar{Q}_n) = \omega^c_{A_1^*}(\bar{Q}_n) - S, \tag{28}$$

where S is a constant to be determined.

Using Eq. (26) and the assumptions of small deformations, we have

$$\varepsilon_x = \theta_1'' \omega, \tag{29}$$

$$\sigma_x = E\theta_1'' \omega, \tag{30}$$

where E is Young's modulus. Due to the beam only subjected to a torque, we may have

$$\int \sigma_x \, \mathrm{d}A = 0, \tag{31}$$

$$\int \sigma_x y \, \mathrm{d}A = 0, \tag{32}$$

$$\int \sigma_x z \, \mathrm{d}A = 0, \tag{33}$$

where A is the cross-sectional area. Substituting Eq. (30) into Eq. (31), we have, in light of Eqs. (25) and (27),

$$\int \omega \, \mathrm{d}A = \int \omega^c \, \mathrm{d}A = 0. \tag{34}$$

Substituting Eq. (28) into Eq. (34) yields

$$S = \frac{\int \omega^c_{A_1^*} \, \mathrm{d}A}{A}. \tag{35}$$

The resulting warping displacements should be identical whether the reference point of the axial displacement locating at any end point or at any junction, because they all satisfy Eq. (34).

When a second-order linearization of the fully geometrically nonlinear beam theory is adopted for the nonlinear analysis of thin-walled open-section beams, there arise the following cross-sectional constants related to the warping function [6–8]:

$$I_\omega = \int \omega^2 \, \mathrm{d}A, \quad \alpha_\omega = \int \omega^3 \, \mathrm{d}A, \quad \alpha_{y\omega} = \int z^2 \omega \, \mathrm{d}A, \quad \alpha_{z\omega} = \int y^2 \omega \, \mathrm{d}A,$$

$$\alpha_{\omega y} = \int \omega^2 z \, dA, \quad \alpha_{\omega z} = \int \omega^2 y \, dA, \quad \alpha_{\omega yz} = \int \omega yz \, dA,$$

$$J = \int \{[-(z - z_p) + \omega_{,y}]^2 + [(y - y_p) + \omega_{,z}]^2\} \, dA,$$

$$J_y = \int [(y - y_p)(z\omega_{,z} - \omega) - z(z - z_p)\omega_{,y} + z(\omega_{,y}^2 + \omega_{,z}^2) + \omega\omega_{,z}] \, dA,$$

$$J_z = \int [(z - z_p)(y\omega_{,y} - \omega) - y(y - y_p)\omega_{,z} - y(\omega_{,y}^2 + \omega_{,z}^2) + \omega\omega_{y}] \, dA,$$

$$J_\omega = \int [(y - y_p)\omega\omega_{,z} - (z - z_p)\omega\omega_{,y} + \omega(\omega_{,y}^2 + \omega_{,z}^2)] \, dA, \tag{36}$$

where the symbols $()_{,y}$ and $()_{,z}$ denote $\partial()/\partial y$ and $\partial()/\partial z$, respectively, and J and I_ω are the so-called the torsional constant and the warping constant [2], respectively. For bisymmetric cross sections, the values of y_p, z_p, α_ω, $\alpha_{y\omega}$, $\alpha_{z\omega}$, $\alpha_{\omega y}$, $\alpha_{\omega z}$, J_y, J_z and J_ω are zero. For monosymmetric cross sections with $z_p = 0$, the values of α_ω, $\alpha_{y\omega}$, $\alpha_{z\omega}$, $\alpha_{\omega y}$, J_y, and J_ω are zero [6,7]. The associated details for these cross-sectional constants can be found in Refs. [6–8]. All the cross-sectional constants will show the differences between the present theory and Vlasov's theory in this study.

Substituting Eq. (30) into Eqs. (32) and (33), we have

$$\int y\omega \, dA = 0, \tag{37}$$

$$\int z\omega \, dA = 0. \tag{38}$$

With the aid of Eqs. (37) and (38), we may determine the position of the center of twist. According to the Betti reciprocal theorem, we may verify that the twist center coincides with the shear center [11].

3. Position formulas to determine the twist center

In practice, we may determine the position of the center of twist as follows. If we do not know the position of the twist center P, we may take an arbitrary point $\tilde{P}$ as a fictitious twist center, so that the fictitious warping function $\tilde{\omega}$ arising from the fictitious twist center is as simple as possible. Then the position of the twist center P may be obtained with the aid of the fictitious twist center $\tilde{P}$ and the fictitious warping function $\tilde{\omega}$.

The line connecting point $\bar{Q}$ and point Q is perpendicular to the branch. Point $\bar{Q}$ in the coordinate systems $x\xi\eta$ and xyz may be expressed as $\bar{Q}(x, 0, \eta)$ and $\bar{Q}(x, \bar{y}, \bar{z})$, respectively, where

$$\bar{y} = \bar{y}(\eta),$$

$$\bar{z} = \bar{z}(\eta). \tag{39}$$

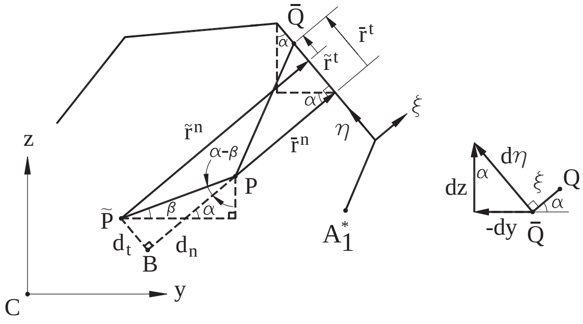

Fig. 2. Fictitious twist center for the position formulas of real twist center.

An arbitrary Point Q in the coordinate systems $x\xi\eta$ and xyz, as shown in Fig. 2, may be expressed as $Q(x,\xi,\eta)$ and $Q(x,y,z)$, respectively, where

$$y = \bar{y} + \xi\cos\alpha,$$

$$z = \bar{z} + \xi\sin\alpha \tag{40}$$

with

$$\sin\alpha = -\frac{d\bar{y}}{d\eta},$$

$$\cos\alpha = \frac{d\bar{z}}{d\eta}. \tag{41}$$

From Fig. 2, we have

$$\sin\beta = \frac{z_p - z_{\tilde{p}}}{P\tilde{P}},$$

$$\cos\beta = \frac{y_p - y_{\tilde{p}}}{P\tilde{P}}. \tag{42}$$

Let $PB = d_n, \tilde{P}B = d_t$. Using the geometrical relations (see Fig. 2) and Eq. (42), we have

$$d_n = P\tilde{P}\cos(\alpha-\beta) = P\tilde{P}(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

$$= (y_p - y_{\tilde{p}})\cos\alpha + (z_p - z_{\tilde{p}})\sin\alpha,$$

$$d_t = P\tilde{P}\sin(\alpha-\beta) = P\tilde{P}(\sin\alpha\cos\beta - \sin\beta\cos\alpha)$$

$$= (y_p - y_{\tilde{p}})\sin\alpha - (z_p - z_{\tilde{p}})\cos\alpha. \tag{43}$$

From Fig. 2, we have

$$\tilde{r}^n = \bar{r}^n + d_n,$$

$$\tilde{r}^t = \bar{r}^t - d_t. \tag{44}$$

The warping function ω may be expressed by

$$\begin{aligned}\omega(Q_n) &= \omega^c(\bar{Q}_n) + \omega^t(Q_n) = \omega^c_{A_1^*}(\bar{Q}_n) - S + \omega^t(Q_n)\\ &= \omega^c_{A_1^*}(A_n^*) - \int_0^{\eta_n} \bar{r}_n^n \, \mathrm{d}\eta_n - S + \bar{r}_n^t \xi_n\\ &= C_0 - \int_0^{\eta_n} \bar{r}_n^n \, \mathrm{d}\eta_n + \bar{r}_n^t \xi_n,\end{aligned} \tag{45}$$

where $C_0 = \omega^c_{A_1^*}(A_n^*) - S$ is a constant.

Alternatively,

$$\omega = C_0 - \int_0^{\eta} \bar{r}^n \, \mathrm{d}\eta + \bar{r}^t \xi. \tag{46}$$

The fictitious warping function $\tilde{\omega}$ may be expressed by

$$\begin{aligned}\tilde{\omega} &= \tilde{C}_0 - \int_0^{\eta} \tilde{r}^n \, \mathrm{d}\eta + \tilde{r}^t \xi\\ &= \tilde{C}_0 - \int_0^{\eta} (\bar{r}^n + d_n) \, \mathrm{d}\eta + (\bar{r}^t - d_t)\xi\\ &= C_1 + \omega - \int_0^{\eta} [(y_p - y_{\tilde{p}})\cos\alpha + (z_p - z_{\tilde{p}})\sin\alpha] \, \mathrm{d}\eta - [(y_p - y_{\tilde{p}})\sin\alpha - (z_p - z_{\tilde{p}})\cos\alpha]\xi,\\ &= C_1 + \omega - [(y_p - y_{\tilde{p}})(\bar{z} - \bar{z}_0) - (z_p - z_{\tilde{p}})(\bar{y} - \bar{y}_0)] - [(y_p - y_{\tilde{p}})\sin\alpha - (z_p - z_{\tilde{p}})\cos\alpha]\xi\\ &= C_1 + \omega - (y_p - y_{\tilde{p}})(\bar{z} + \xi\sin\alpha - \bar{z}_0) + (z_p - z_{\tilde{p}})(\bar{y} + \xi\cos\alpha - \bar{y}_0)\\ &= C_1 + \omega - (y_p - y_{\tilde{p}})(z - \bar{z}_0) + (z_p - z_{\tilde{p}})(y - \bar{y}_0),\end{aligned} \tag{47}$$

where

$$C_1 = \tilde{C}_0 - C_0, \quad \bar{y}_0 = \bar{y}(0), \quad \bar{z}_0 = \bar{z}(0). \tag{48}$$

The process of derivation of Eq. (47) has used Eqs. (39)–(41), (43), (44) and (46).

Thus, we have

$$\begin{aligned}\int y\tilde{\omega} \, \mathrm{d}A &= C_1 \int y \, \mathrm{d}A + \int y\omega \, \mathrm{d}A - (y_p - y_{\tilde{p}}) \int yz \, \mathrm{d}A + (y_p - y_{\tilde{p}})\bar{z}_0 \int y \, \mathrm{d}A\\ &\quad + (z_p - z_{\tilde{p}}) \int y^2 \, \mathrm{d}A - (z_p - z_{\tilde{p}})\bar{y}_0 \int y \, \mathrm{d}A,\\ \int z\tilde{\omega} \, \mathrm{d}A &= C_1 \int z \, \mathrm{d}A + \int z\omega \, \mathrm{d}A - (y_p - y_{\tilde{p}}) \int z^2 \, \mathrm{d}A + (y_p - y_{\tilde{p}})\bar{z}_0 \int z \, \mathrm{d}A\\ &\quad + (z_p - z_{\tilde{p}}) \int yz \, \mathrm{d}A - (z_p - z_{\tilde{p}})\bar{y}_0 \int z \, \mathrm{d}A.\end{aligned} \tag{49}$$

Rearranging the above equation, we yield the position formulas of the twist center P, with the aid of Eqs. (37) and (38) and the principal centroidal coordinates $Cxyz$

$$y_p = y_{\tilde{p}} - \frac{I_{\tilde{\omega}y}}{I_y},$$

$$z_p = z_{\tilde{p}} + \frac{I_{\tilde{\omega}z}}{I_z}, \tag{50}$$

where

$$I_{\tilde{\omega}y} = \int \tilde{\omega} z \, \mathrm{d}A, \quad I_{\tilde{\omega}z} = \int \tilde{\omega} y \, \mathrm{d}A. \tag{51}$$

4. Illustrative examples

The application of the analytical expression for the warping function is straightforward. Thus, only one example will show the detailed process, and the other examples will directly give the results.

4.1. Warping function of a monosymmetric channel section

Choose the point A_1^* as the reference point and determine $\bar{r}^n$ and $\bar{r}^t$ for each branch as shown in Fig. 3. top flange (branch 1): $-t_f/2 \leqslant \xi_1 \leqslant t_f/2$, $0 \leqslant \eta_1 \leqslant b$.

From Fig. 3, we have

$$\bar{r}_1^n = h/2,$$

$$\bar{r}_1^t = -(e + b - \eta_1),$$

then

$$\omega_{A_1^*}^c(\eta_1) = -\int_0^{\eta_1} \bar{r}_1^n \, \mathrm{d}\eta_1 = (-h/2)\eta_1,$$

$$\omega_{A_1^*}^t(\xi_1, \eta_1) = \bar{r}_1^t \xi_1 = \xi_1(\eta_1 - e - b).$$

Thus,

$$\omega_{A_1^*}^c(A_2^*) = -bh/2.$$

Web (branch 2): $-t_w/2 \leqslant \xi_2 \leqslant t_w/2$, $0 \leqslant \eta_2 \leqslant h$.

From Fig. 3, we have

$$\bar{r}_2^n = -e,$$

$$\bar{r}_2^t = \begin{cases} -(h/2 - \eta_2), & 0 \leqslant \eta_2 \leqslant h/2, \\ \eta_2 - h/2, & h/2 \leqslant \eta_2 \leqslant h, \end{cases}$$

$$= \eta_2 - h/2, \quad 0 \leqslant \eta_2 \leqslant h,$$

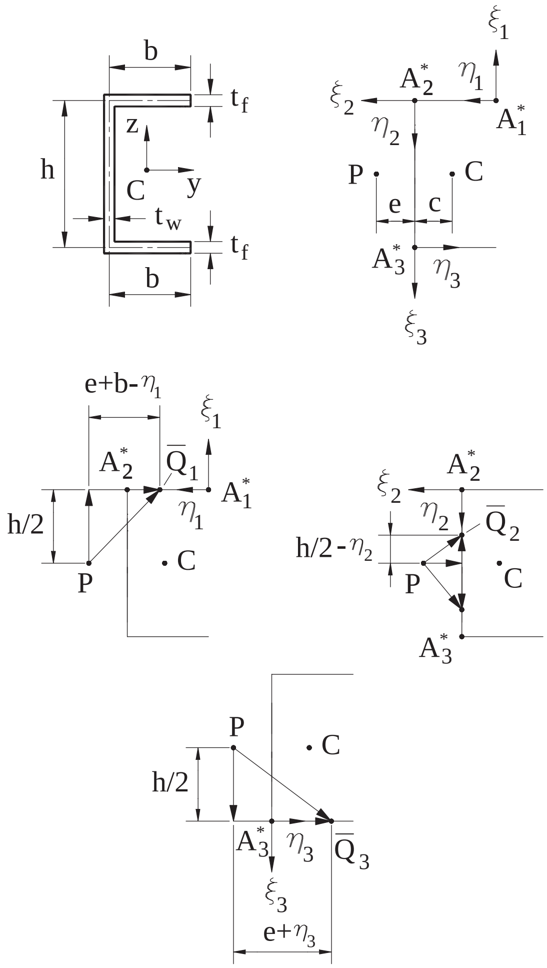

Fig. 3. Geometry of a monosymmetric channel section.

then

$$\omega^c_{A_1^*}(\eta_2) = \omega^c_{A_1^*}(A_2^*) - \int_0^{\eta_2} \bar{r}_2^n \, \mathrm{d}\eta_2 = (-bh/2) + e\eta_2,$$

$$\omega^t_{A_1^*}(\xi_2, \eta_2) = \bar{r}_2^t \xi_2 = \xi_2(\eta_2 - h/2).$$

Thus,

$$\omega^c_{A_1^*}(A_3^*) = (-bh/2) + eh.$$

Bottom flange (branch 3): $-t_f/2 \leqslant \xi_3 \leqslant t_f/2$, $0 \leqslant \eta_3 \leqslant b$.

From Fig. 3, we have

$$\bar{r}_3^n = h/2,$$

$$\bar{r}_3^t = \eta_3 + e,$$

then

$$\omega^c_{A_1^*}(\eta_3) = \omega^c_{A_1^*}(A_3^*) - \int_0^{\eta_3} \bar{r}_3^n \, \mathrm{d}\eta_3 = (-bh/2) + eh - (h/2)\eta_3,$$

$$\omega^t_{A_1^*}(\xi_3, \eta_3) = \bar{r}_3^t \xi_3 = \xi_3(\eta_3 + e).$$

From Eq. (35), we have

$$S = -h(b-e)/2.$$

Therefore, the warping function of a monosymmetric channel section shown in Fig. 3 is given as follows:

Top flange: $-t_f/2 \leqslant \xi_1 \leqslant t_f/2$, $0 \leqslant \eta_1 \leqslant b$.
Warping function:

$$\omega(\xi_1, \eta_1) = \xi_1\eta_1 - (b+e)\xi_1 - (h/2)\eta_1 + h(b-e)/2.$$

Web: $-t_w/2 \leqslant \xi_2 \leqslant t_w/2$, $0 \leqslant \eta_2 \leqslant h$.
Warping function:

$$\omega(\xi_2, \eta_2) = \xi_2\eta_2 - (h/2)\xi_2 + e\eta_2 - he/2.$$

Bottom flange: $-t_f/2 \leqslant \xi_3 \leqslant t_f/2$, $0 \leqslant \eta_3 \leqslant b$.
Warping function:

$$\omega(\xi_3, \eta_3) = \xi_3\eta_3 + e\xi_3 - (h/2)\eta_3 + he/2.$$

The warping constant and the torsional constant according to the present theory are

$$I_\omega = \frac{h^2 t_f[(b-e)^3 + e^3]}{6} + \frac{h^3 e^2 t_w}{12} + \frac{t_f^3[(b+e)^3 - e^3]}{18} + \frac{h^3 t_w^3}{144},$$

$$J = \tfrac{2}{3}\, b t_f^3 + \tfrac{1}{3}\, h t_w^3.$$

Note that the torsional constant based on the present theory is consistent with the one based on the membrane analogy. The warping constant and the torsional constant according to Vlasov's theory are

$$I_\omega = \frac{h^2 t_f[(b-e)^3 + e^3]}{6} + \frac{h^3 e^2 t_w}{12},$$

$$J = \tfrac{2}{3}\, b^3 t_f + \tfrac{1}{6}\, b t_f^3 + \tfrac{1}{12}\,(h^3 t_w + h t_w^3) + 2bet_f(b+e).$$

For a monosymmetric channel section with $b = 6$ mm, $h = 12$ mm, and $t_f = t_w = 0.8$ mm, the results for the cross-sectional constants related to the warping function based on Vlasov's theory (or Timoshenko's theory) and the present (Goodier's) theory are tabulated in Table 1. As can been seen from Table 1, the values of some cross-sectional constants I_ω, α_ω, $\alpha_{y\omega}$, $\alpha_{z\omega}$, $\alpha_{\omega y}$, $\alpha_{\omega z}$, $\alpha_{\omega yz}$, J_y and J_ω almost make no difference between Vlasov's theory and the present theory. However, the torsional constant obtained by the present theory is 4.096 mm^4 consistent with the one based on

Table 1
Cross-sectional constants based on Vlasov's theory and Goodier's theory for a monosymmetric channel section

Cross-sectional constants	Vlasov's theory	Present (Goodier's theory)
I_ω (mm^6)	1.814×10^3	1.836×10^3
$\alpha_{y\omega}$ (mm^6)	0	0
$\alpha_{z\omega}$ (mm^6)	0	0
$\alpha_{\omega yz}$ (mm^6)	1.814×10^3	1.817×10^3
$\alpha_{\omega y}$ (mm^7)	0	0
$\alpha_{\omega z}$ (mm^7)	2.539×10^3	2.563×10^3
α_ω (mm^8)	0	0
J (mm^4)	409.6	4.096
J_y (mm^5)	0	0
J_z (mm^5)	260.4	7.668
J_ω (mm^6)	0	0

Monosymmetric channel section with $b = 6$ mm, $h = 12$ mm, and $t_f = t_w = 0.8$ mm

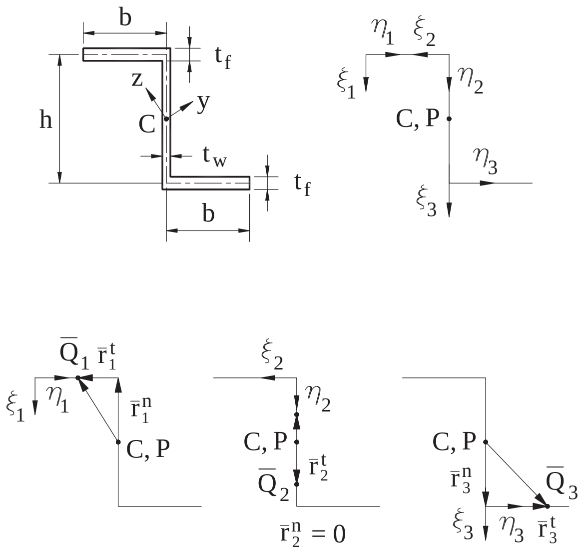

Fig. 4. Geometry of an unsymmetric Z-section.

the membrane analogy, as against 409.6 mm^4 according to Vlasov's theory. Moreover, the difference of the constant J_z between Vlasov's theory and the present theory is remarkable.

4.2. Warping function of an unsymmetric Z-section

The warping function of an unsymmetric Z-section shown in Fig. 4 is given as follows:
Top flange: $-t_f/2 \leqslant \xi_1 \leqslant t_f/2$, $0 \leqslant \eta_1 \leqslant b$.

Warping function:

$$\omega(\xi_1,\eta_1)=\xi_1\eta_1-b\xi_1+(h/2)\eta_1-S,$$

where

$$S=\frac{bh(bt_f+ht_w)}{2(2bt_f+ht_w)}.$$

Web: $-t_w/2\leqslant\xi_2\leqslant t_w/2,\ 0\leqslant\eta_2\leqslant h$.
Warping function:

$$\omega(\xi_2,\eta_2)=\xi_2\eta_2-(h/2)\xi_2+bh/2-S.$$

Bottom flange: $-t_f/2\leqslant\xi_3\leqslant t_f/2,\ 0\leqslant\eta_3\leqslant b$.
Warping function:

$$\omega(\xi_3,\eta_3)=\xi_3\eta_3-(h/2)\eta_3+bh/2-S.$$

The warping constant and the torsional constant according to the present theory are

$$I_\omega=\frac{b^3h^2t_f(bt_f+2ht_w)}{12(2bt_f+ht_w)}+\frac{b^3t_f^3}{18}+\frac{h^3t_w^3}{144},$$

$$J=\tfrac{2}{3}\,bt_f^3+\tfrac{1}{3}\,ht_w^3.$$

Note that the torsional constant based on the present theory is consistent with the one based on the membrane analogy. The warping constant and the torsional constant according to Vlasov's theory are

$$I_\omega=\frac{b^3h^2t_f(bt_f+2ht_w)}{12(2bt_f+ht_w)},$$

$$J=\tfrac{2}{3}\,b^3t_f+\tfrac{1}{6}\,bt_f^3+\tfrac{1}{12}\,(h^3t_w+ht_w^3).$$

For an unsymmetric Z-section with $b=85.73$ mm, $h=142.88$ mm, and $t_f=t_w=12.7$ mm, the results for the cross-sectional constants related to the warping function based on Vlasov's theory (or Timoshenko's theory) and the present (Goodier's) theory are tabulated in Table 2. As can been seen from Table 2, the values of some cross-sectional constants I_ω, α_ω, $\alpha_{y\omega}$, $\alpha_{z\omega}$, $\alpha_{\omega y}$, $\alpha_{\omega z}$, $\alpha_{\omega yz}$, and J_z almost make no difference between Vlasov's theory and the present theory. However, the torsional constant obtained by the present theory is 2.146×10^5 mm^4 consistent with the one based on the membrane analogy, as against 8.475×10^6 mm^4 according to Vlasov's theory. Moreover, the differences of the constants J_y and J_ω between Vlasov's theory and the present theory are remarkable.

4.3. Warping function of an unsymmetric angle section

The warping function of an unsymmetric angle section shown in Fig. 5 is given as follows:
Horizontal leg (flange): $-t_f/2\leqslant\xi_1\leqslant t_f/2,\ 0\leqslant\eta_1\leqslant b$.

Table 2
Cross-sectional constants based on Vlasov's theory and Goodier's theory for an unsymmetric Z-section

Cross-sectional constants	Vlasov's theory	Present (Goodier's theory)
I_ω (mm^6)	1.609×10^{10}	1.620×10^{10}
$\alpha_{y\omega}$ (mm^6)	-2.491×10^{10}	-2.474×10^{10}
$\alpha_{z\omega}$ (mm^6)	-9.973×10^{8}	-9.824×10^{8}
$\alpha_{\omega yz}$ (mm^6)	1.107×10^{10}	1.113×10^{10}
$\alpha_{\omega y}$ (mm^7)	0	0
$\alpha_{\omega z}$ (mm^7)	0	0
α_{ω} (mm^8)	-2.584×10^{13}	-2.626×10^{13}
J (mm^4)	8.475×10^{6}	2.146×10^{5}
J_y (mm^5)	4.302×10^{7}	0
J_z (mm^5)	0	0
J_ω (mm^6)	0	-1.793×10^{8}

Unsymmetric Z-section with $b = 85.73$ mm, $h = 142.88$ mm, and $t_f = t_w = 12.7$ mm

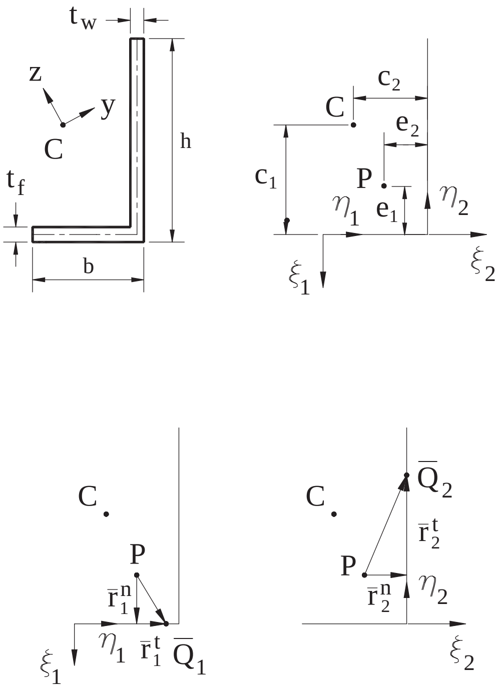

Fig. 5. Geometry of an unsymmetric angle section.

Warping function:

$$\omega(\xi_1, \eta_1) = \xi_1\eta_1 + (e_2 - b)\xi_1 - e_1\eta_1 - S,$$

Table 3
Cross-sectional constants based on Vlasov's theory and Goodier's theory for an unsymmetric angle section

Cross-sectional constants	Vlasov's theory	Present (Goodier's theory)
I_ω (mm^6)	0	3.742×10^6
$\alpha_{y\omega}$ (mm^6)	0	2.780×10^5
$\alpha_{z\omega}$ (mm^6)	0	1.347×10^6
$\alpha_{\omega yz}$ (mm^6)	0	-2.520×10^6
$\alpha_{\omega y}$ (mm^7)	0	2.682×10^7
$\alpha_{\omega z}$ (mm^7)	0	7.112×10^7
α_{ω} (mm^8)	0	1.772×10^7
J (mm^4)	0	1.103×10^4
J_y (mm^5)	0	-9.665×10^4
J_z (mm^5)	0	8.658×10^4
J_ω (mm^6)	0	3.762×10^4

Unsymmetric angle section with $b_1 = 47.75$ mm, $h = 72.75$ mm, and $t_f = t_w = 6.5$ mm

where

$$S = -\frac{b^2 t_f e_1 + h^2 t_w e_2 + 2bht_w e_1}{2(bt_f + ht_w)}.$$

Vertical leg (web): $-t_w/2 \leqslant \xi_2 \leqslant t_w/2,\ 0 \leqslant \eta_2 \leqslant h.$
Warping function:

$$\omega(\xi_2, \eta_2) = \xi_2\eta_2 - e_1\xi_2 - e_2\eta_2 - be_1 - S.$$

For an unsymmetric angle section with $b = 47.75$ mm, $h = 72.75$ mm, and $t_f = t_w = 6.5$ mm, the results for the cross-sectional constants related to the warping function based on Vlasov's theory and the present (Goodier's) theory are tabulated in Table 3. The warping constant obtained by the present theory is 3.742×10^6 mm^6. The torsional constant obtained by the present theory is 1.103×10^4 mm^4 consistent with the value based on the membrane analogy. According to Vlasov's theory or Timoshenko's theory, the angle section twists without warping.

4.4. Warping function of an unsymmetrical channel section

The warping function of an unsymmetrical channel section shown in Fig. 6 is given as follows:
Top flange: $-t_1/2 \leqslant \xi_1 \leqslant t_1/2,\ 0 \leqslant \eta_1 \leqslant b_1.$
Warping function:

$$\omega(\xi_1, \eta_1) = \xi_1\eta_1 - (b_1 + e_1)\xi_1 + (c_2 + e_2)\eta_1 - S,$$

where

$$S = \frac{(c_2 + e_2)(\frac{1}{2}\, b_1^2 t_1 - \frac{1}{2}\, b_2^2 t_2 + b_1 h t_w + b_1 b_2 t_2) + \frac{1}{2}\, b_2^2 h t_2 - \frac{1}{2}\, h^2 t_w e_1 - e_1 h b_2 t_2}{b_1 t_1 + h t_w + b_2 t_2}.$$

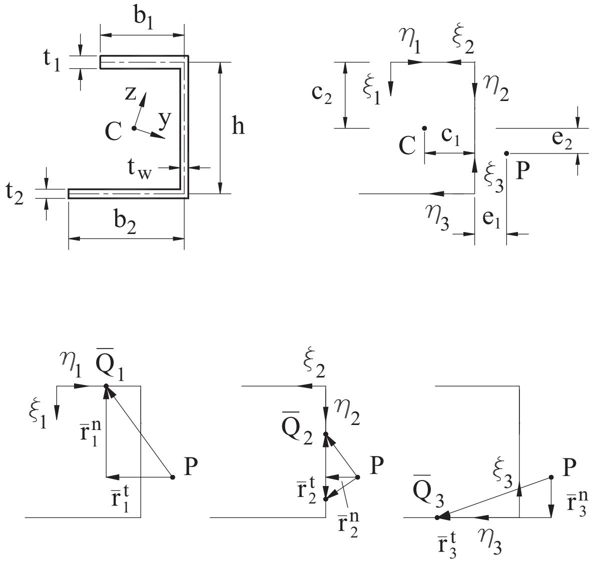

Fig. 6. Geometry of an unsymmetric channel section.

Web: $-t_w/2 \leqslant \xi_2 \leqslant t_w/2,\ 0 \leqslant \eta_2 \leqslant h$.
Warping function:

$$\omega(\xi_2, \eta_2) = \xi_2\eta_2 - (c_2 + e_2)\xi_2 - e_1\eta_2 + (c_2 + e_2)b_1 - S.$$

Bottom flange: $-t_2/2 \leqslant \xi_3 \leqslant t_2/2,\ 0 \leqslant \eta_3 \leqslant b_2$.
Warping function:

$$\omega(\xi_3, \eta_3) = \xi_3\eta_3 + e_1\xi_3 + (h - c_2 - e_2)\eta_3 + b_1(c_2 + e_2) - e_1 h - S.$$

For an unsymmetric channel section with $b_1 = 2.0$ mm, $h = 10.0$ mm, $b_2 = 4.0$ mm, and $t_1 = t_2 = t_w = 0.5$ mm, the results for the cross-sectional constants related to the warping function based on Vlasov's theory and the present (Goodier's) theory are tabulated in Table 4. The warping constants I_ω are 71.033 mm^6 and 73.348 mm^6 according to Vlasov's theory and the present theory, respectively. The difference between them is about 3%. The constants $\alpha_{z\omega}$ are -1.813 mm^6 and -2.175 mm^6 according to Vlasov's theory and the present theory, respectively. The difference between them reaches about 20%. Moreover, the torsional constant J obtained by the present theory is 0.667 mm^4 as against 312.989 mm^4 according to Vlasov's theory. Obviously, the former is consistent with the value based on the membrane analogy and the latter is not correct. The differences of the constants J, J_y, J_z, and J_ω between Vlasov's theory and the present theory are remarkable.

5. Conclusions

The torsional warping function for a thin-walled open-section beam may contain two parts: the contour warping function and the thickness warping function. For some cross-sectional constants

Table 4
Cross-sectional constants based on Vlasov's theory and Goodier's theory for an unsymmetric channel section

Cross-sectional constants	Vlasov's theory	Present (Goodier's theory)
I_ω (mm^6)	71.03	73.35
$\alpha_{y\omega}$ (mm^6)	−28.81	−32.64
$\alpha_{z\omega}$ (mm^6)	−1.813	−2.175
$\alpha_{\omega yz}$ (mm^6)	78.16	78.01
$\alpha_{\omega y}$ (mm^7)	221.0	237.3
$\alpha_{\omega z}$ (mm^7)	−70.86	−69.77
α_ω (mm^8)	−306.0	−304.5
J (mm^4)	313.0	0.6667
J_y (mm^5)	−736.4	−0.8242
J_z (mm^5)	121.7	−0.5288
J_ω (mm^6)	122.4	0.2706

Unsymmetric channel section with $b_1 = 2.0$ mm, $h = 10.0$ mm, $b_2 = 4.0$ mm, and $t_1 = t_2 = t_w = 0.5$ mm

such as the warping constant I_ω, α_ω, $\alpha_{y\omega}$, $\alpha_{z\omega}$, $\alpha_{\omega y}$, $\alpha_{\omega z}$, and $\alpha_{\omega yz}$, the adoption of the warping function based on Vlasov's theory or Timoshenko's theory or Goodier's theory might make only a few difference. On the other hand, for some cross-sectional constants such as the torsional constant J, J_y, J_z and J_ω, the adoption of the warping function based on Vlasov's theory or Timoshenko's theory might cause them incorrectness. Thus, the thickness warping should be considered, especially in the nonlinear analysis of thin-walled open-section beams. The torsional constant based on Goodier's theory is consistent with the one based on the membrane analogy. Thus, Goodier's theory is a good approximation for the warping of a thin-walled open-section beam. In this study, a more general analytical expression for the torsional warping function of a thin-walled beam with generic open sections is built based on the Vlasov assumption and the Kirchhoff assumption of plate/shell theory. A practical procedure to determine the twist center is also given

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