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A survey on multi-loop networks

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Abstract

Bermond, Comellas and Hsu gave an excellent survey on multi-loop networks, directed and undirected, in 1995, but only one and half page is on loop networks other than the directed double-loop. Hwang recently gave a substantial survey, but only on the directed double-loop. This survey is a companion of the latter survey by focusing on the other loop networks.

1. Introduction

Multi-loop networks were first proposed by Wong and Coppersmith [28] for organizing multimodule memory services. Fiol et al. [17] slightly extended its definition in their study of the data alignment problem in SIMD processors. Nowadays, it is used for both local area computer networks [23, 25] and large area communication networks like SONET [13, 26].

A multi-loop network, denoted by $L(N; s_1, \ldots, s_l)$, can be represented by a digraph on N nodes, $0, 1, \ldots, N-1$ and lN links of l types: $i \to i+s_1, i \to i+s_2, \ldots, i \to i+s_l \pmod N$, $i = 0, 1, \ldots, N-1$. When l is specified, we can also call it an *l-loop network*. A symmetric $2l$-loop network, $L(N; s_1, \ldots, s_l, -s_1, \ldots, -s_l)$ can be represented by a graph on N vertices with edges of l types: $(i, i+s_1), (i, i+s_2), \ldots, (i, i+s_l) \pmod N$ for $i = 0, 1, \ldots, N-1$. Such a multi-loop network will be called an *undirected l-loop network*, and denoted by $L(N; \pm s_1, \ldots, \pm s_l)$.

The single-loop network (also called ring network) is mathematically trivial. The double-loop network is the most important and most-studied multi-loop network, but it

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has been recently surveyed [21]. So this survey will focus on l-loop networks, directed or undirected, for $l \geqslant 3$, and also on the undirected double-loop network (these have been briefly included in a survey by Bermond et al. [3]).

2. Multi-loop networks

We first discuss connectivity. For r a divisor of N and $S=\{s_1,\dots,s_l\}$ define $M(r,S) = \{i\colon\ i \equiv s_j \in S \pmod r \text{ for } 1 \leqslant i < r\}$. van Dorne [14] proved

Theorem 2.1. *The strong connectivity of* $L(N;S)$ *is* $\min_{r|N}\{(N/r)|M(r,S)|\colon\ |M(r,S)| < r-1\}$.

Divide the set $\{0,1,\dots,N-1\}$ into r residue classes modulo r. Then $M(r,S)$ represents the set of classes nodes in class 0 can go to via a s_j-step for some $s_j \in S$. Therefore, $M(r,S)$ is a cutset except when there is no other class besides class 0 and those in $M(r,S)$, which happens when $|M(r,S)| = r-1$. Since each class has N/r nodes, $(N/r)|M(r,S)|$ is the size of the cutset.

Corollary 2.2. $L(N;s_1,\dots,s_l)$ *is strongly connected if and only if* $\gcd(N,s_1,\dots,s_l)=1$.

Proof. $L(N;S)$ is strongly connected if and only if $|M(r,S)|>0$ for every $r|N$. □

In the following, we say $L(N;S)$ does not exist if it is not strongly connected.

Hamidoune [18] proved that the link-connectivity of a strongly connected vertex-transitive digraph is the vertex degree. Hence

Theorem 2.3. *The arc-connectivity of an l-loop is l if and only if it is strongly connected.*

Since the l-loops cannot be distinguished by the link-connectivity, we introduce a finer notion of link-connectivity. A digraph is *super*-λ if all its maximum link-cutsets are trivial (links adjacent to a node). Hamidoune and Tindell [19] characterized super-λ vertex-transitive digraphs from which the super-λ multi-loop can be characterized.

A *minimum distance diagram* (MDD) exhibits the shortest paths from node 0 to node i for all i (since the multi-loop network is vertex-transitive, there is no loss of generality in using node 0 as the source). Wong and Coppersmith gave a simple algorithm to construct the MDD. The $\mathbb{R}^l$ space is first divided into unit l-dimensional hypercubes. The MDD is constructed by labeling a connected set of N hypercubes by the set of residues $(\bmod N)$. The labeling is done step-wise. Let $l(x_1,\dots,x_l)$ denote the label assigned to the cube with coordinate $(x_1,\dots,x_l)$. At step 0, set $l(0,\dots,0)=0$, i.e., the origin has label 0. At step r we label cubes distance r away from the origin and set $l(x_1,\dots,x_l) \equiv \sum_{i=1}^{l} x_i s_i \pmod N$. The order of labeling follows the lexicographical order of $(x_l,x_{l-1},\dots,x_1)$, small first. For example for $l=3$ and $r=2$ the order is

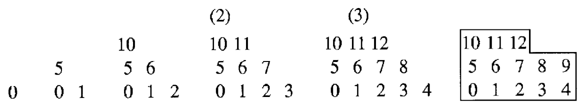

Fig. 1. Constructing the MDD for $L(13;1,5)$.

$(2,0,0),(1,1,0),(0,2,0),(1,0,1),(0,1,1),(0,0,2)$. The order is important since if two hypercubes have the same label, only the first one is labeled. Labeling stops when every label has appeared. Fig. 1 illustrates this algorithm for $L(13;1,5)$.

Hsu and Jia [20] extended a result of Wong and Coppersmith from $l=2$ to general l.

Theorem 2.4. *If $(x_1,\ldots,x_l)$ is not in the MDD, then nor is $(y_1,\ldots,y_l)$ where $y_i \geqslant x_i$ for $1 \leqslant i \leqslant l$. Furthermore, there exists a unique $(x_1,\ldots,x_l)$ not in the MDD such that the l hypercubes $(x_1-1,x_2,\ldots,x_l),(x_1,x_2-1,x_3,\ldots,x_l),\ldots,(x_1,\ldots,x_{l-1},x_l-1)$ are all in the MDD, and $\sum_{i=1}^{l} x_i s_i \equiv 0 \,(\mathrm{mod}\, N)$.*

Suppose that $\mathbb{R}^d$ is divided into unit hypercubes and a *shape* is a connected set of hypercubes. A shape is said to tessellate $\mathbb{R}^d$ if any number of it can be connected together with no internal gaps (rotation not allowed). Recently, Chen et al. [9] gave a sufficient condition for a shape to tessellate. The following result follows as a special case.

Theorem 2.5. *Every MDD tessellates $\mathbb{R}^d$.*

Wong and Coppersmith gave lower bounds and upper bounds for the diameter D and the mean distance of a multi-loop. The lower bounds are obtained by a packing argument, namely, for a given diameter, the number of nodes is at most the number of hypercubes within distance D from the origin. Since a maximum packing yields $\binom{l+r-1}{l-1}$ hypercubes whose distance to the origin is exactly r,

$$N \leqslant \sum_{r=0}^{D} \binom{l+r-1}{l-1} = \binom{l+D}{l} \leqslant [D+(l+1)/2]^l/l!.$$

On the other hand let M denote the mean distance. Then

$$NM \geqslant \sum_{r=0}^{D-1} r \binom{l+r-1}{l-1} = \frac{l(D-1)}{l+1}\binom{l+D-1}{l}$$

$$\geqslant \frac{l}{(l+1)!}\left[(l!D)^{1/l} - \frac{l+3}{2}\right]^{l+1}.$$

Thus we have

Theorem 2.6. $D(N;s_1,\ldots,s_l)\geqslant(l!N)^{1/l}-(l+1)/2\sim(l!N)^{1/l}$.

$$M(N;s_1,\ldots,s_l)\geqslant\frac{l}{(l+1)!N}\left[(l!N)^{1/l}-\frac{l+3}{2}\right]^{l+1}\sim\frac{l}{l+1}D(N;s_1,\ldots,s_l).$$

Wong and Coppersmith argued the upper bounds by giving a construction of an l-loop. Consider $L(u^l;1,u,\ldots,u^{l-1})$. It is easily seen that its MDD is a $u\times u\times\cdots\times u$ hypercube, hence

Theorem 2.7. $D(u^l;1,u,\ldots,u^{l-1})=l(u-1)\sim lN^{1/l}$;

$M(u^l;1,u,\ldots,u^{l-1})=l(u-1)/2.$

It should be noted that this upper bound is derived only for a special form of N. To apply it to all N, we say, if $(u-1)^l<N\leqslant u^l$, then its diameter is upper bounded by $l(u-1)$. Actually, we can obtain better bounds if there is some gap between N and u^l. Clearly, if $N\geqslant u^l-1$, then we can remove node u^l-1 (which lies opposite node 0 in the hypercube MDD) to reduce the diameter by 1. To reduce the diameter by 2, one cannot simply further remove the l nodes surrounding node u^l-1 since these nodes $(u^l-1)-1,(u^l-1)-u,\ldots,(u^l-1)-u^{l-1}$ are not the l consecutive nodes u^l-l-1, $u^l-l,\ldots,u^l-2$. The following result shows a way of doing it.

Theorem 2.8. $D(u^l-2l;1,u,\ldots,u^{l-2},u^{l-1}-1)=l(u-1)-2$.

Proof. We label the $x^l=0$ hyperplane according to $L(u^{l-1};1,u,\ldots,u^{l-2})$ except removing node $u^{l-1}-1$. Each of the $x^l=k$, $1\leqslant k\leqslant l-2$, hyperplane is labeled exactly like the $x^l=0$ hyperplane except with the set of nodes $\{k(u^{l-1}-1),k(u^{l-1}-1)+1,\ldots,k(u^{l-1}-1)+u^{l-1}-2\}$. The $x^l=l-1$ hyperplane is labeled similarly except removing nodes $u^l-2l,u^l-2l+l,\ldots,u^l-1$. □

Example. $D(21;1,3,8)=4$

$x_3=0$ plane			$x_3=1$ plane			$x_3=2$ plane		
6	7		14	15				
3	4	5	11	12	13	19	20	
0	1	2	8	9	10	16	17	18

Another way of generalization is to construct a hyper-rectangle with side lengths $u_1,u_2,\ldots,u_l$. It is easily verified that

$$N=\prod_{i=1}^{l}u_i,\quad s_1=1,\quad s_j=\prod_{i=1}^{j-1}u_i\quad\text{for } 2\leqslant j\leqslant l.$$

Theorem 2.9. $D(\prod_{i=1}^{l}u_i;1,u_1,u_1u_2,\ldots,\prod_{i=1}^{l-1}u_i)=\sum_{i=1}^{l}(u_i-1)$.

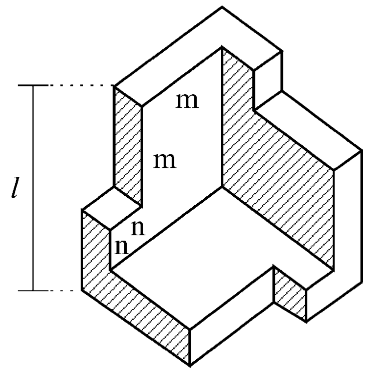

Fig. 2. A hyper-L shape.

For example, for $N=28$, Theorem 2.7 selects $u=3$ and obtains a bound of 6, while Theorem 2.9 selects $u_1=2$, $u_2=u_3=3$ and obtains a smaller bound of 5.

3. Triple-loop networks

For $l=3$, the lower bound of diameter from Theorem 2.6 is $D \geqslant (6N)^{1/3}$. Note that the above bound is obtained by packing a tetrahedron of cubes (x_1,x_2,x_3) satisfying $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$ and $x_1+x_2+x_3 \leqslant D$. But the cubes on the surface $x_1+x_2+x_3=D$ violates Theorem 2.1 since it neighbors many hypercubes (x_1,x_2,x_3) with (x_1-1,x_2,x_3), (x_1,x_2-1,x_3) and (x_1,x_2,x_3-1) all in the tetrahedron. By using Theorem 2.1 to tighten the packing, Hsu and Jia were able to obtain a better lower bound of D.

Theorem 3.1. $D \geqslant [(14-3\sqrt{3})N]^{1/3}-3$.

Hsu and Jia also gave an explicit construction of triple-loops with parameters $a=1$, $b=3D-8r+5$, $c=rb+D-3r+2$ and $N=rc+2D-6r+3$, where $r=\lfloor D/4 \rfloor$, and D is the diameter, which results in

Theorem 3.2. $D \leqslant (16N)^{1/3}$.

While the MDD of a triple-loop may not necessarily produce a regular shape, Aguilo et al. [1] took the approach of bypassing the triple-loop and going directly to the following highly regular *hyper-L shape*.

The hyper-L shape is symmetric with respect to the three dimensions and has three parameters l,m and n. It can be described as the l^3-cube with several parts removed: an $(l-m-n)^3$-cube at one corner and an $(l-m-n)\times m\times n$ hyper-rectangle from each of the three remaining hyper-rectangle pieces (see Fig. 2). We will denote such a hyper-L shape by $HL(l,m,n)$. It is easily verified that the hyper-L shape tessellates $\mathbb{R}^3$.

Assuming the existence of a triple-loop yielding $HL(l,m,n)$, then node 0 lies at the origin but also at $(l-m-n, l-m-n, l-m-n)$ by Theorem 2.1. By studying the distribution of nodes 0, we obtain the system of equations:

$$\begin{pmatrix} l & -m & -n \\ -n & l & -m \\ -m & -n & l \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \pmod{N},$$

or

$$\begin{pmatrix} l & -m & -n \\ -n & l & -m \\ -m & -n & l \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} N, \quad \text{for some integers } \alpha, \beta, \gamma \text{ not all 0.} \qquad (1)$$

It is easily verified that

$$N = l^3 - m^3 - n^3 - 3lmn, \quad \text{and}$$

$$D = 3l - 2n - m - 3 \quad (\text{assuming } m \geqslant n).$$

By setting $l=3w$, $m=n=w$, then $N=16w^3$ and $D=6w-3=6(N/16)^{1/13}-3$. Aguilo et al. thus claimed $(27N/2)^{1/3}-3$ to be an upper bound of D, which yields a better bound than Theorem 3.2. However, the validity of the new bound is based on the existence of a triple-loop yielding a hyper-L shape with $l=3w$, $m=n=w$. Chen et al. [8] provide a link between desirable hyper-L shapes and corresponding triple-loops by proving

Theorem 3.3. *A necessary and sufficient condition for* $L(N;a,b,c)$ *whose MDD is* $HL(l,m,n)$ *to exist is* $\gcd(l+m, l+n, m-n) = \gcd(l-m-n, l^2-mn) = 1$.

Corollary 3.4. *A necessary conditions for* $L(N;a,b,c)$ *to exist is* $m \neq n$.

Proof. Suppose $m=n$. Then

$$\gcd(l+m, l+n, n-n) = l+m > 1. \qquad \square$$

By Corollaries 3.4 and 2.2, we know that $L(N;a,b,c)$ for $l=3w$, $m=n=w$ does not exist. Nevertheless, Aguilo et al. gave three families of triple-loops:

(i) $L(16x^3+3x; 4x^2+x+1, 4x^2-x+1, 8x^2+1)$ with $l=3x$, $m=x+1$, $n=x-1$ and $D=6x-2$,

(ii) $L(16x^3+12x^2+3x; 4x^2+4x+1, 4x^2+3x+1, 8x^2+5x+1)$ with $l=3x+1$, $m=x+1$, $n=x$ and $D=6x-1$,

(iii) $L(16x^3+36x^2+27x+7; 4x^2+5x+1, 4x^2+4x+1, 8x^2+11x+4)$ with $l=3x+2$, $m=x+1$ and $D=6x+2$,

whose diameters approach the $(27N/2)^{1/3}$ bound. It is easily verified that these families satisfy the conditions of Theorem 3.3.

4. Undirected multi-loop networks

Most of the results in this section parallel those in Section 2.1 since an undirected l-loop can be treated as a directed $2l$-loop. Therefore by setting $S=\{\pm s_1,\ldots,\pm s_l\}$, Theorem 2.1 can be translated into

Theorem 4.1. *The connectivity of* $L(N;\pm s_1,\ldots,\pm s_l)$ *is* $\min_{r|N}\{(N/r)|M(r,S)|: |M(r,S)|<r-1\}$.

Corollary 4.2. $L(N;\pm s_1,\ldots,\pm s_l)$ *is connected if and only if* $\gcd(N,s_1,\ldots,s_l)=1$.

The edge-connectivity of an undirected multi-loop easily follows from a result of Mader [24] on connected vertex-transitive graphs.

Theorem 4.3. *The edge-connectivity of an undirected l-loop is $2l$ if and only if it is connected.*

Boesch and Tindell [6] characterized super-λ undirected l-loops.

Theorem 4.4. *The only connected undirected l-loops which are not super-λ are $L(N;s)$ for all s and $L(2l;2,4,6,\ldots,l-1,l)$ for l odd.*

As in the directed case, MDD can be constructed by filling in the nodes in the order of their distances to node 0 at the origin; and for nodes with the same distance r in the same hyperplane $(x_1,x_2,\ldots,x_j)$, $1\leqslant j\leqslant l$, in the order of $x_j=0,1,\ldots,r$. Fig. 3 illustrates this construction for $L(14;\pm 3,\pm 4)$.

Only the first part of Theorem 2.4 holds for the undirected case. Hence the MDD has less structure as a shape.

Theorem 4.5. *If* $(x_1,\ldots,x_l)$ *is not in the MDD, then nor is* $(y_1,\ldots,y_l)$, *where* $y_ix_i\geqslant 0$, $|y_i|\geqslant|x_i|$, $1\leqslant i\leqslant l$.

Wong and Coppersmith also gave a packing upper bound. The number of hypercubes r-distance away from the origin with exactly i coordinates equal to zero is 1 for

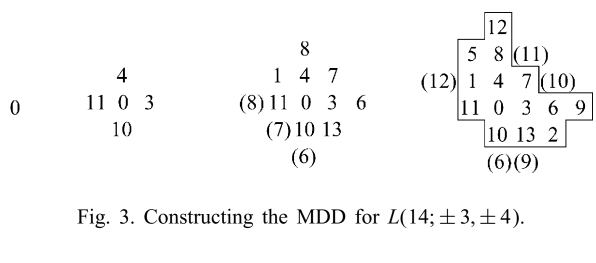

Fig. 3. Constructing the MDD for $L(14;\pm 3,\pm 4)$.

$r=0$ and

$$\binom{n-1}{l-i-1} \quad \text{for } r \geqslant 1,$$

and there are $\binom{l}{i}$ ways of selecting the i coordinates. By noting that each nonzero x_i can be set either positive or negative, we have

$$N \leqslant 1+\sum_{r=1}^{D}\sum_{i=0}^{l-1}\binom{l}{i}\binom{r-1}{l-i-1}2^{l-i} = 1+\sum_{i=0}^{l-1}\binom{l}{i}\binom{D}{l-i}2^{l-i}$$
$$=\sum_{i=0}^{l}\binom{l}{i}\binom{D}{l-i}2^{l-i} \leqslant 2^l\sum_{i=0}^{l}\binom{l}{i}\binom{D}{l-i}$$
$$=\binom{l+D}{l}2^l \leqslant (2D+l+1)^l/l!. \tag{1}$$

On the other hand

$$NM \geqslant \sum_{r=1}^{D-1} r\sum_{i=0}^{l-1}\binom{l}{i}\binom{r-l}{l-i-1}2^{l-i}$$
$$=\sum_{i=0}^{l-1} r\binom{r-1}{l-1}2^l$$
$$=\frac{l}{l+1}D\binom{D-1}{l}2^l$$
$$\geqslant l2^l\binom{D}{l+1} \geqslant \frac{l}{2(l+1)!}[(l!N)^{1/l}-(3l+1)]^{l+1}. \tag{2}$$

Thus we have

Theorem 4.6. $D(N;\pm s_1,\ldots,\pm s_l) \geqslant \frac{1}{2}(l!N)^{1/l}-(2l+1)/2 \sim \frac{1}{2}(l!N)^{1/l}$,

$$M(N;\pm s_1,\ldots,\pm s_l) \geqslant \frac{l}{2(l+1)!N}[(l!N)^{1/l}-(3l+1)]^{l+1} \sim \frac{l}{2(l+1)}(l!N)^{1/l}.$$

Wong and Coppersmith again used the l-dimensional hypercube with side u (see Theorem 2.5) to give upper bounds. By placing the center of the hypercube at the origin, we obtain

Theorem 4.7.

$$D(u^l;\pm 1,\pm u,\ldots,\pm u^{l-1}) = l(u-1)/2 \sim (l/2)N^{1/l}.$$
$$M(u^l;\pm 1,\pm u,\ldots,\pm u^{l-1}) = l(u^2-1)/4u.$$

Erdös and Hsu [15] gave a better bound than Theorem 4.7 provided some condition (not easily checked or met) is satisfied.

Theorem 4.8. *Suppose that there exists an optimal* $L(N; \pm 1, \pm s'_2, \dots, \pm s'_{l-1})$, $l \geqslant 4$, *i.e., its diameter is the lower bound* $[(l-1)!N]^{1/(l-1)} - l/2$. *Then there exists an* $L(N; \pm 1, \pm s_2, \dots, \pm s_l)$ *whose diameter is upper bounded by* $\min_c \{[(l-1)!c]^{1/(l-1)} + 1/c\} N^{1/l} - l/2 - 1$.

These values serve as upper bounds for $(u-1)^l < N \leqslant u^l$.

Since the MDD of an undirected multi-loop has irregular shape, Zerovink and Pisanski [30] brought in some structure by introducing a different kind of diagram which we will call *base diagram* (BD). For a given l-loop, let Z denote the set of coordinates of the 0-nodes. A 0-*vector* is a vector from a 0-node to another 0-node. Let $b_1, \dots, b_l$ be a set of l independent 0-vectors. Then $\{b_1, \dots, b_l\}$ is a base of Z. A BD $[b_1, \dots, b_l)$ is defined as

$$[b_1, \dots, b_l) = \{x\colon\ x = \alpha_1 b_1 + \cdots + \alpha_l b_l, \text{ where } 0 \leqslant \alpha_i < 1\}.$$

Note that $[b_1, \dots, b_l)$ contains half of the boundary of the parallelepiped. Similarly, we can define $(b_1, \dots, b_l]$, $[b_1, \dots, b_l]$ and $(b_1, \dots, b_l)$ by specifying whether each of the inequalities $0 \leqslant \alpha_i$ and $\alpha_i \leqslant 1$ is strict (with a different effect on the boundary). They proved

Theorem 4.9. $[b_1, \dots, b_l)$ *contains every residue modulo* N *exactly once.*

Chen and Jia [11] gave a different construction to obtain a better bound. For $l \geqslant 3$, $n = \lfloor (D - l + 3)/l \rfloor$ and choose $s_i = (4n)^{i-1}$ for $1 \leqslant i \leqslant l$. They proved that for

$$N \leqslant 2n \sum_{i=0}^{l-1} (4n)^i = \left(\frac{1}{2}\right) \left(\frac{4}{l}\right)^l D^l + \mathrm{O}(D^{l-1}),$$

every residue $x \pmod N$ can be represented as

$$x = \sum_{i=1}^{l} x_i s_i, \quad \text{with } -2n < x_i \leqslant 2n \text{ for } 1 \leqslant i \leqslant l-1 \quad \text{and} \quad 0 \leqslant x_l \leqslant 2n,$$

furthermore,

$$\sum_{i=1}^{l} |x_i| \leqslant D.$$

Thus for such N,

Theorem 4.10. $D \leqslant (l/4)(2N)^{1/l}$ *for* $l \geqslant 3$.

5. Undirected double-loop networks

The connectivity of undirected double-loops can be determined without going through the computation of Theorem 4.1.

Theorem 5.1. *$L(N;\pm s_1,\pm s_2)$ is 4-connected for $N\geqslant 6$.*

Proof. Watkins [27] proved that a connected vertex-transitive graph is maximally connected if it does not contain K_4. It is easily verified that $L(N;\pm s_1,\pm s_2)$ does not contain K_4 for $N\geqslant 6$. □

Set $l=2$ in Eq. (1), we have

$$N \leqslant \sum_{i=0}^{2}\binom{2}{i}\binom{D}{2-i}2^{2-i} = 2D(D-1)+4D+1 = 2D^2+2D+1, \text{ or}$$

$$D \geqslant \left\lceil \frac{-1+\sqrt{2N-1}}{2} \right\rceil \triangleq D^*.$$

Boesch and Wong [7] proved that this lower bound of D can always be achieved by choosing $s_1=D^*$ and $s_2=D^*+1$.

Theorem 5.2. *$L(N;\pm D^*,\pm(D^*+1))$ achieves the minimum diameter D^* for all N.*

Bermond et al. [5] tightened the bound Eq. (2) of mean distance (their definition of mean distance is not counting self-distance) by taking account of the unfilled hypercubes in packing.

$$(N-1)M \geqslant \sum_{r=1}^{D-1} r \sum_{i=0}^{l-1}\binom{l}{i}\binom{r-1}{l-i-1}2^{l-i} + D(N-2D^2+2D-1). \tag{3}$$

For $l=2$, (3) becomes

$$(N-1)M \geqslant (N-1)D - \frac{2D}{3}(D^2-1), \quad \text{or}$$

$$M \geqslant D\left[1-\frac{2(D^2-1)}{3(N-1)}\right] \triangleq M^* \sim \sqrt{2N}/3.$$

They proved.

Theorem 5.3. *$L(N;\pm D^*,\pm(D^*+1))$ achieves M^* for $2(D^*)^2-1\leqslant N\leqslant 2(D^*)^2+2(D^*)+1$; $L(N;\pm(D^*-1),\pm D^*)$ achieves M^* for $2(D^*)^2-2(D^*)+1\leqslant N\leqslant 2(D^*)^2+1$.*

We observe

Corollary 5.4. *The undirected double-loops in Theorem 5.3 achieve both minimum diameter and minimum average distance in their respective range. Furthermore, since*

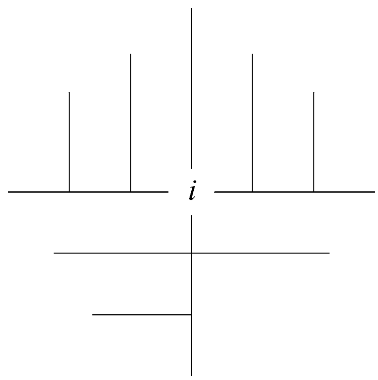

Fig. 4. The Fàbrega and Zaragoza's tree.

$2(D^*-1)^2+2(D^*-1)+1=2(D^*)^2-2D^*+1$, *for every N, there exists an undirected double-loop achieving both minimum diameter and minimum mean distance.*

Bermond et al. also proved

Theorem 5.5. *The diameter of* $L(N;\pm D^*,\pm(D^*+1))$ *after one fault* (*node or edge*) *is at most* D^*+1.

Theorem 4.3 says that an undirected double-loop has edge-connectivity 4. Bermond et al. [4] obtained a stronger result.

Theorem 5.6. *An undirected double-loop has two link-disjoint hamiltonian cycles.*

In view of Corollary 5.4, Theorems 5.5 and 5.6, it seems that we do have an optimal undirected double-loop in either $L(N;\pm D^*,\pm(D^*+1))$ or $L(N;\pm(D^*-1),\pm D^*)$ for all N. Therefore we fail to understand the existence of a large body of literature studying the diameters of $L(N;\pm 1,\pm s)$ which are often suboptimal.

For a given routing ρ and a set F of faulty elements (nodes or edges), a *surviving route graph* is a graph whose nodes are the nodes of the undirected double-loop and an edge $[i,j]$ exists if and only if $\rho(i,j)$ contains no element of F. So an edge in a surviving route graph represents a surviving route. The diameter of a surviving route graph, representing the maximum number of surviving routes a path has to bypass through, is a common measure for fault tolerance.

Fàbrega and Zaragozà [16] considered the surviving route graphs for optimal undirected double-loops, i.e., (s_1,s_2) is either $(D,D+1)$ or $(D-1,D)$. They defined a routing ρ by the rule: for any node i draw a shortest spanning tree t_i in the MDD centered at i, and route (i,j) coincides with the path on t_i. The t_i chosen by them takes a-steps before b-steps if j is in the upper half (including the horizontal line) and b-steps first if otherwise (see Fig. 4).

Clearly, if F consists of nodes in the boundary of t_i and edges not in t_i, then all nodes not in G are adjacent to i in the surviving route graph. By symmetry, a node j can be put at any place of an MDD, including the boundary. This makes it clear that for any single node-fault, the surviving graph has diameter 2. Fàbrega and Zaragoza in fact proved.

Theorem 5.7. *Suppose* $N=2D^2+2D+1$. *Then the diameter of the surviving route graph is* 2 *for any* F *consisting of two nodes*; *the diameter is* 3 *if* F *consists of three nodes.*

There are not many results for undirected double-loops with general (s_1,s_2).

Zerovink and Pisanski were able to obtain results beyond Theorem 4.9 for $l=2$. They proved the following two results.

Theorem 5.8. *Suppose* b_1 *and* b_2 *are the two shortest* (*in* L_1-*metric*) *independent* 0-*vectors*, *then for every node in* $[b_1,b_2)$, *the closest* 0-*node is always one in the four corners of* $[b_1,b_2]$.

Clearly, the center of $[b_1,b_2]$ must be the point farthest away from the closest corner, the diameter of $L(N;\,\pm s_1,\,\pm s_2)$ can be obtained by computing only the distances of the four nodes surrounding the center to their respective closest corners. Let d_1 and d_2 be the two diagonals of $[b_1,b_2]$. Then

$$\|d_1\| = \min\{\|b_1+b_2\|,\|b_1-b_2\|\},$$

$$\|d_2\| = \max\{\|b_1+b_2\|,\|b_1-b_2\|\},$$

Theorem 5.9.

$$D(N;\pm\, s_1,\pm\, s_2) = \begin{cases} \lfloor \|d_1\|/2 \rfloor - 1 & \text{if } \|d_1\| = \|d_2\| > \max\{\|b_1\|,\|b_2\|\} \text{ and} \\ & \text{both } \|b_1\| \text{ and } \|b_2\| \text{ are odd}, \\ \lfloor \|d_1\|/2 \rfloor & \text{otherwise.} \end{cases}$$

Corollary 5.10. *Any two shortest* 0-*vectors* $\{b_1,b_2\}$ *yield the same diameters.*

6. Chordal ring networks

Arden and Lee [2] first proposed the *chordal ring network* on an even number N of nodes with two types of edges:

ring-edges: $(i,i+1)$ for $i=0,1,\ldots,N-1$, and

chords: $(i,i+h)$, where h is odd, for $i=1,3,5,\ldots,N-1$ (i odd).

Note that there are only $N/2$ chords, where a loop typically has N edges. Therefore a chordal ring can be viewed as an undirected one-and-half loop.

Arden and Lee considered an optimal mixture of the following three kinds of moves:
(i) at an even node: a 1-step followed by an h-step,
(ii) at an even node: a (-1)-step followed by an h-step,
(iii) 1-step or (-1)-step.
For a given diameter D, they found the maximum N using the above three moves and mistakenly claimed it to be an upper bound of N for the chordal ring. Yebra et al. [29] considered two additional kinds of moves:
(iv) at an odd node: a 1-step followed by a $(-h)$-step,
(v) at an odd node: a (-1)-step followed by a $(-h)$-step,
and obtained better results. They proved

Theorem 6.1. (i) $N \leqslant (3D^2+1)/2$ *for* D *odd*; *achievable by setting*

$$h = 3k.$$

(ii) $N \leqslant (3/2)D^2 - D$ *for* D *even*; *achievable by setting* $h = 3k+1$.

Corollary 6.2. $D \sim (2N/3)^{1/2}$.

For comparison, the diameter of an optimal undirected double-loop is about $(1/\sqrt{2})N^{1/2} \cong (0.7)N^{1/2} < (0.82)N^{1/2} \cong (2N/3)^{1/2}$.

Hwang and Wright [22] considered *directed chordal ring network* which replaces ring-edges with links, namely, it has two types of links:

$$\text{ring-link}: \quad i \to i+1 \text{ for } i = 0,1,\ldots,N-1 \ (N \text{ even}), \text{ and}$$
$$\text{chords}: \quad i \to i+h \text{ for odd } i.$$

Similarly, a directed chordal ring can be viewed as a one-and-half loop.

Hwang and Wright observed that by combining nodes i and ith into one super-node, then a directed chordal ring with parameters (N,h) is reduced to a double-loop $L(N/2;1,(h+1)/2)$. They used this equivalence to compute the reliability of a directed chordal ring when each node, each ring-link and each chord fails with the probabilities p, p_1, p_h, respectively. Chen et al. [10] observed that the diameter of a directed chordal ring can also be computed by using the $O(\log N)$ algorithm developed by Cheng and Hwang [12] for the double-loop. They further extended the directed chordal ring to allow the replacement of $i \to i+1$ link by $i \to i+s$ for some odd s.

Chen et al. also proposed the *mixed chordal ring*, denoted by $M(N;s,h)$, with

$$\text{ring-link}: \ i \to i+s \text{ for } i = 0,1,\ldots,N-1 \ (N \text{ even}), \text{ and}$$
$$\text{chords}: \quad (i, \pm h) \text{ for odd } i.$$

By treating an edge (i,j) as the set of two links $i \to j$ and $j \to i$, then a mixed chordal ring is a regular digraph with degree 2, hence comparable in hardware with a double-loop. Chen et al. gave the surprising result that a mixed chordal ring can achieve shorter diameter than a double-loop.

Theorem 6.3. *$D(N;s,h) \leqslant \sqrt{2N}+3$, where s and h are computed by the following rules: set N' to be the smallest even integer $\geqslant \lceil \sqrt{N/2} \rceil$. Then $s=N'+1$ and $h=N'-1$.*

They also proved

Theorem 6.4. *$M(N;s,h)$ is hamiltonian if and only if $L(N/2;(s-h)/2,(s+h)/2)$ does. Furthermore, it has two link disjoint hamiltonian circuits if and only if* $\gcd(N, s-h)=\gcd(N,s+h)=2$.

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