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Analytic Functions and Integrable Hierarchies – Characterization of Tau Functions

LEE-PENG TEO
Department of Applied Mathematics, National Chiao Tung University, 1001, Ta-Hsueh Road, Hsinchu City, 30050, Taiwan, R.O.C. e-mail: lpteo@math.nctu.edu.tw

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Abstract. We prove the dispersionless Hirota equations for the dispersionless Toda, dispersionless coupled modified KP and dispersionless KP hierarchies using an idea from classical complex analysis. We also prove that the Hirota equations characterize the tau functions for each of these hierarchies. As a result, we establish the links between the hierarchies.

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1. Introduction

Dispersionless integrable hierarchies have been under active research in recent years (see, e.g., [13, 14, 19–22]). One of the reasons is due to its close relation with the other area of mathematics and physics, such as topological field theory, string theory, 2D gravity, matrix models and conformal maps (see, e.g., [1–4, 7, 10, 12, 15, 17, 18, 26]). The tau functions of the dispersionless integrable hierarchies play an important role in topological field theories ([8, 9]), for they give solutions to the so-called WDVV equation. Conversely, in [4], it was proved that the tau functions of the dispersionless KP (dKP) and dispersionless Toda (dToda) hierarchies satisfy the associativity equation. One of the main ingredients of the proof of [4] is the dispersionless Hirota equations satisfied by the dKP and dToda hierarchies. The dispersionless Hirota equation for the dKP hierarchy was first derived by Takasaki and Takebe [22] as the dispersionless limit of the differential Fay identity. Later, the Hirota equation was further studied by Carroll and Kodama [5]. In connection to conformal mappings which give rise to solutions of the dToda hierarchy, Wiegmann, Zabrodin et al. derive the Hirota equations for the dToda hierarchy [12, 15, 25]. In this paper, we point out that these Hirota equations are closely related to some concepts in classical complex analysis, namely Faber polynomials and Grunsky coefficients.

In Section 2, we review some concepts from classical complex analysis. We define some classes of formal power series which appear in dKP, dcmKP (dispersionless coupled modified KP, see [24]) and dToda hierarchies. We generalize the definition of the Grunsky coefficients and Faber polynomials to these classes of formal power series and review their properties. In Section 3, we review the dToda, dcmKP and dKP hierarchies and their tau functions. We derive the dispersionless Hirota equations by establishing the relation between the tau functions and the Grunsky coefficients. We also prove that the dispersionless Hirota equations for each of these hierarchies uniquely characterize the tau functions of their solutions. As a corollary, we show that some solutions of the dToda hierarchy will give rise to solutions of dcmKP hierarchy, which in turn will give rise to solutions of dKP hierarchy.

We omit some details in the exposition below. We refer to our preprint [23] on the web for further reference.

2. Algebraic Analysis

2.1. SPACES OF FORMAL POWER SERIES

We consider the following classes of formal power series:

$$\tilde{\Sigma} = \left\{ g(z) = bz + b_0 + \frac{b_1}{z} + \cdots = bz + \sum_{n=0}^{\infty} b_n z^{-n};\ b \neq 0 \right\},$$
$$\Sigma = \left\{ g \in \tilde{\Sigma}, b = 1 \right\}, \qquad \Sigma_0 = \left\{ g \in \Sigma, b_0 = 0 \right\}.$$

$\tilde{\Sigma}$ can be considered as the completion* of the space of analytic functions that fix the point ∞ and univalent in a small neighbourhood of ∞. Σ and Σ_0 are subspaces of $\tilde{\Sigma}$ consist of formal power series satisfying certain normalization conditions. By post-composing $\tilde{g} \in \tilde{\Sigma}$ with the linear map $z \mapsto (1/\tilde{b})z$, we get a function g in Σ. Further post-composition with the linear map $z \mapsto z - b_0$, we get a function g_0 in Σ_0. As their counterpart, we consider another three classes of formal power series:

$$\tilde{S} = \left\{ f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots = \sum_{n=1}^{\infty} a_n z^n;\ a_1 \neq 0 \right\},$$
$$S = \left\{ f \in \tilde{S}, a_1 = 1 \right\}, \qquad S_0 = \left\{ f \in S, a_2 = 0 \right\}.$$

$\tilde{S}$ can be considered as the completion of the space of analytic functions fixing the point 0 and univalent in a neighbourhood of 0. S and S_0 are subspaces of $\tilde{S}$ consist of formal power series subjecting to additional normalization conditions. By post-composing $\tilde{f} \in \tilde{S}$ with the linear map $z \mapsto (1/\tilde{a}_1)z$, we get a function $f \in S$. Further

*The completion is with respect to the filtration

$$\tilde{\Sigma}_N = \left\{ g(z) = bz + \sum_{n=N}^{\infty} b_n z^{-n}, b \neq 0 \right\}$$

on $\tilde{\Sigma}$.

post-composition with the Möbius transformation $z \mapsto z/(1 + a_2 z)$, we get a function $f_0 \in S_0$. Observe that the map

$$f \mapsto g(z) = \frac{1}{f\left(\frac{1}{z}\right)}$$

is a bijection between $\tilde{S}$ and $\tilde{\Sigma}$, S and Σ, S_0 and Σ_0, respectively.

2.2. GENERALIZED FABER POLYNOMIALS AND GRUNSKY COEFFICIENTS

We review and generalize some concepts from classical complex analysis. For details, see [11, 16].

One of the important problems in classical complex analysis is the determination of the upper bound satisfied by the coefficients a_n's in order that $f \in S$ is univalent on the unit disc. A lot of effort has been devoted to the proof of the famous Bieberbach conjecture (1916): If $f \in S$ is univalent on the unit disc, then $|a_n| \leqslant n$*. In the early attempts of the proof, one of the important tools is the Grunsky's inequality and its generalization. In 1939, Grunsky found a sequence of inequalities that should be satisfied by the so called Grunsky coefficients in order that $g \in \Sigma$ is univalent on $\{z \,|\, |z| > 1\}$. Surprisingly, we found this Grunsky coefficients appear everywhere in dispersionless limit of integrable hierarchies (especially in association with tau functions) without being realized its connection to complex analysis. This is the purpose of this paper to point out this connection and, thus, give simplified proofs of some of the facts related to dispersionless integrable hierarchies.

First, we introduce the Faber polynomials. For $g \in \tilde{\Sigma}$ that is analytic in a neighbourhood of ∞ and $w \in \mathbb{C}$, consider the function $\log(g(z) - w)/bz$. It defines an analytic function for large $|z|$ and vanishes at ∞. Hence it has an expansion at ∞ which can be written as

$$\log \frac{g(z) - w}{bz} = -\sum_{n=1}^{\infty} \frac{\Phi_n(w)}{n} z^{-n}. \tag{2.1}$$

Φ_n is called the nth Faber polynomial of g. Differentiate (2.1) with respect to z and define $\Phi_0(w) \equiv 1$, we have

$$\frac{g'(z)}{g(z) - w} = \sum_{n=0}^{\infty} \Phi_n(w) z^{-n-1}. \tag{2.2}$$

From this, we can deduce the recursion formula

$$\Phi_{n+1}(w) = \frac{w - b_0}{b} \Phi_n(w) - \frac{1}{b} \sum_{k=1}^{n-1} b_{n-k} \Phi_k(w) - (n+1) \frac{b_n}{b},$$

which implies that $\Phi_n(w)$ is a polynomial of degree n.

*This conjecture was completely proved by de Branges in 1984 [6].

Next we introduce the Grunsky coefficients. If $g \in \tilde{\Sigma}$ is univalent in a neighbourhood U of ∞, the function $\log (g(z) - g(\zeta))/(z-\zeta)$ is analytic in the neighbourhood $U \times U$ of (∞, ∞). Its expansion about (∞, ∞) has the form

$$\log \frac{g(z)-g(\zeta)}{z-\zeta} = \log b - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} z^{-m}\zeta^{-n}. \tag{2.3}$$

b_{mn}'s are known as Grunsky coefficients of g. They are symmetric, i.e. $b_{mn} = b_{nm}$. Putting $w = g(\zeta)$ into (2.1) and compare with (2.3), we have

$$\Phi_n(g(\zeta)) = \zeta^n + n\sum_{m=1}^{\infty} b_{nm}\zeta^{-m}. \tag{2.4}$$

There is a characterization of the Faber polynomials which plays an important role in our discussion later. Let $z = G(w) = w/b + \sum_{n=0}^{\infty} c_n w^{-n}$ be the inverse function of $w = g(z)$ in the neighbourhood U where g is univalent. From (2.2), we have

$$\Phi_n(w) = \operatorname*{Res}_{z=\infty} \frac{g'(z)z^n}{g(z)-w}\mathrm{d}z = \operatorname*{Res}_{\zeta=\infty} \frac{G(\zeta)^n}{\zeta - w}\mathrm{d}\zeta.$$

Using the expansion about $\zeta = \infty$,

$$G(\zeta)^n = \sum_{m=-\infty}^{n} c_{n,m}\zeta^m \quad \text{and} \quad \frac{1}{(\zeta - w)} = \sum_{k=0}^{\infty} w^k \zeta^{-k-1},$$

we obtain immediately $\Phi_n(w) = \sum_{m=0}^{n} c_{n,m} w^m$. Namely, $\Phi_n(w)$ is the polynomial part of $G(w)^n$, which we denote by $(G(w)^n)_{\geqslant 0}$, i.e.

$$\Phi_n(w) = (G(w)^n)_{\geqslant 0}. \tag{2.5}$$

In general, if $A = \sum_{n=-\infty}^{\infty} A_n w^n$ is a formal power series, and S a subset of integers, we define $(A)_S = \sum_{n\in S} A_n w^n$.

The Grunsky coefficients are generalized to a pair of functions f and g as follows. Let $f \in \tilde{S}$ be univalent in a neighbourhood V of 0 and $g \in \Sigma$ be univalent in a neighbourhood U of ∞. We say that (f, g) are disjoint relative to (U, V) if the sets $f(V)$ and $g(U)$ are disjoint. In this case, the functions

$$\log\frac{g(z)-g(\zeta)}{z-\zeta}, \qquad \log\frac{g(z)-f(\zeta)}{z}, \qquad \log\frac{f(z)-f(\zeta)}{z-\zeta}$$

are analytic in $U \times U$, $U \times V$ and $V \times V$ respectively. Hence, we can write down their series expansion about (∞, ∞), $(\infty, 0)$ and $(0, 0)$ respectively:

$$\log\frac{g(z)-g(\zeta)}{z-\zeta} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} z^{-m}\zeta^{-n}, \tag{2.6}$$

$$\log\frac{g(z)-f(\zeta)}{z} = -\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} b_{m,-n} z^{-m}\zeta^{n}, \tag{2.7}$$

$$\log\frac{f(z)-f(\zeta)}{z-\zeta}=-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}b_{-m,-n}z^m\zeta^n. \tag{2.8}$$

Obviously, when m, n are both positive or both negative, $b_{mn}=b_{nm}$. Hence, for $m\geqslant 0$, $n>0$, we define $b_{-m,n}=b_{n,-m}$. Letting $\zeta=0$ in (2.7) and (2.8), we obtain

$$\log\frac{g(z)}{z}=-\sum_{m=1}^{\infty}b_{m,0}z^{-m},\qquad \log\frac{f(z)}{z}=-\sum_{m=0}^{\infty}b_{-m,0}z^m. \tag{2.9}$$

In particular, $b_{00}=-\log a_1$. We define the generalized Faber polynomials $\Psi_n(w)$ for f by

$$\log\frac{w-f(z)}{w}=\log\frac{f(z)}{a_1 z}-\sum_{n=1}^{\infty}\frac{\Psi_n(w)}{n}z^n. \tag{2.10}$$

To see the relations of Ψ_n with the Grunsky coefficients, we define the function $g_f\in\tilde{\Sigma}$ by

$$g_f(z)=\frac{1}{f\left(\frac{1}{z}\right)}=\frac{z}{a_1}-\frac{a_2}{a_1^2}+\cdots.$$

Using the second equation in (2.9), equations (2.7), (2.8) and (2.10) can be rewritten in terms of g_f:

$$\log\left(1-\frac{1}{g(z)g_f(\zeta)}\right)=-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}b_{m,-n}z^{-m}\zeta^{-n}, \tag{2.11}$$

$$\log\frac{g_f(z)-g_f(\zeta)}{z-\zeta}=-\log a_1-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}b_{-m,-n}z^{-m}\zeta^{-n}, \tag{2.12}$$

$$\log\frac{g_f(z)-\frac{1}{w}}{z}=-\log a_1-\sum_{n=1}^{\infty}\frac{\Psi_n(w)}{n}z^{-n}. \tag{2.13}$$

Hence, $b_{-m,-n}$'s are Grunsky coefficients of g_f and $\Psi_n(w)$ is a polynomial of degree n in $1/w$. If we denote by $z=G_f(w)$ the inverse of $w=g_f(z)$, and by $z=F(w)$ the inverse of $w=f(z)$, equation (2.5) implies that

$$\Psi_n(w)=(G_f(w^{-1})^n)_{\leqslant 0}=(F(w)^{-n})_{\leqslant 0}. \tag{2.14}$$

Now we derive the counterparts of (2.4). First, compare (2.13) to (2.1) and (2.4), we get

$$\Psi_n(f(\zeta))=\zeta^{-n}+n\sum_{m=1}^{\infty}b_{-n,-m}\zeta^m. \tag{2.15}$$

Next, we put $w=f(\zeta)$ into (2.1) and compare with (2.7). We obtain

$$\Phi_n(f(\zeta))=n\sum_{m=0}^{\infty}b_{n,-m}\zeta^m. \tag{2.16}$$

Finally, putting $w = g(\zeta)$ into (2.10), compare with (2.7) and using equations in (2.9), we have

$$\Psi_n(g(\zeta)) = -nb_{-n,0} + n\sum_{m=1}^{\infty} b_{m,-n}\zeta^{-m}. \tag{2.17}$$

For the convenience of next section, we gather again the formulas (2.9), and the formulas of the Faber polynomials in terms of the Grunsky coefficients (2.4), (2.15), (2.16), (2.17).

$$\begin{aligned}
&\log\frac{g(z)}{z} = -\sum_{m=1}^{\infty} b_{m,0}z^{-m}, \qquad \log\frac{f(z)}{z} = -\sum_{m=0}^{\infty} b_{-m,0}z^{m}\\
&\Phi_n(g(\zeta)) = \zeta^n + n\sum_{m=1}^{\infty} b_{nm}\zeta^{-m}, \qquad \Phi_n(f(\zeta)) = nb_{n,0} + n\sum_{m=1}^{\infty} b_{n,-m}\zeta^{m},\\
&\Psi_n(g(\zeta)) = -nb_{-n,0} + n\sum_{m=1}^{\infty} b_{m,-n}\zeta^{-m}, \qquad \Psi_n(f(\zeta)) = \zeta^{-n} + n\sum_{m=1}^{\infty} b_{-n,-m}\zeta^{m}.
\end{aligned} \tag{2.18}$$

The analysis above can be extended formally to the whole space $\tilde{\Sigma}$ and $\tilde{S}$. All the Taylor (Laurent) expansions are considered as formal power series expansions. All the identities hold formally.

3. Dispersionless Hierarchies and Tau Functions

We quickly review dispersionless Toda (dToda), dispersionless coupled modified KP (dcmKP) and dispersionless KP (dKP) hierarchies and their tau functions. For details, see [20–22, 24]. For each of these dispersionless hierarchies, we give a new derivation of the dispersionless Hirota equation satisfied by the tau function, using the algebraic analysis we discussed in the previous section. We also prove that the dispersionless Hirota equations uniquely characterize the tau functions.

3.1. DISPERSIONLESS TODA HIERARCHY

The fundamental quantities in dToda hierarchy are two formal power series in p:

$$\mathcal{L}(p) = p + \sum_{n=0}^{\infty} u_{n+1}(t)p^{-n}, \qquad \tilde{\mathcal{L}}^{-1}(p) = \tilde{u}_0(t)p^{-1} + \sum_{n=0}^{\infty} \tilde{u}_{n+1}(t)p^{n}.$$

Here $u_n(t)$ and $\tilde{u}_n(t)$ are functions of the independent variables $t_n, n \in \mathbb{Z}$, which we denote collectively by t. The Lax representation is*

$$\begin{aligned}
&\frac{\partial\mathcal{L}}{\partial t_n} = \{(\mathcal{L}^n)_{\geqslant 0}, \mathcal{L}\}_T, \qquad \frac{\partial\mathcal{L}}{\partial t_{-n}} = \{(\tilde{\mathcal{L}}^{-n})_{<0}, \mathcal{L}\}_T,\\
&\frac{\partial\tilde{\mathcal{L}}}{\partial t_n} = \{(\mathcal{L}^n)_{\geqslant 0}, \tilde{\mathcal{L}}\}_T, \qquad \frac{\partial\tilde{\mathcal{L}}}{\partial t_{-n}} = \{(\tilde{\mathcal{L}}^{-n})_{<0}, \tilde{\mathcal{L}}\}_T.
\end{aligned} \tag{3.1}$$

*Here it is understood that p is a variable and does not depend on t.

Here $\{\cdot,\cdot\}_T$ is the Poisson bracket for dToda hierarchy

$$\{f,g\}_T = p\frac{\partial f}{\partial p}\frac{\partial g}{\partial t_0} - p\frac{\partial f}{\partial t_0}\frac{\partial g}{\partial p}.$$

There exists a tau function τ_{dToda} which generates the coefficients of $\mathcal{L}$ and $\tilde{\mathcal{L}}$. More precisely, we have the following identities:

$$\log p = \log\mathcal{L} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_0\partial t_m}\mathcal{L}^{-m},$$

$$(\mathcal{L}^n)_{\geqslant 0} = \mathcal{L}^n - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_n\partial t_m}\mathcal{L}^{-m} = \frac{\partial^2\mathcal{F}}{\partial t_0\partial t_n} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_n}\tilde{\mathcal{L}}^m,$$

$$\log p = \log\tilde{\mathcal{L}} + \frac{\partial^2\mathcal{F}}{\partial t_0^2} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_0}\tilde{\mathcal{L}}^m, \tag{3.2}$$

$$(\tilde{\mathcal{L}}^{-n})_{<0} = -\sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_m\partial t_{-n}}\mathcal{L}^{-m}$$
$$= \tilde{\mathcal{L}}^{-n} + \frac{\partial^2\mathcal{F}}{\partial t_0\partial t_{-n}} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_{-n}}\tilde{\mathcal{L}}^m.$$

Here $\mathcal{F} = \log\tau_{\mathrm{dToda}}$ is called the free energy. Now we identify p with w, $\mathcal{L}$ with z, then the first equation defines a function $w = g(z) \in \Sigma$. $\mathcal{L}(p)$ corresponds to $z = G(w)$, the inverse of g. The third equation defines a function $w = f(z) \in \tilde{S}$ and $\tilde{\mathcal{L}}(p)$ corresponds to $z = F(w)$, the inverse of f. Under these identifications, we see that the Faber polynomials $\Phi_n(w)$'s are identified with $(\mathcal{L}^n(w))_{\geqslant 0}$, and $\Psi_n(w)$'s are identified with $(\tilde{\mathcal{L}}^{-n}(w))_{\leqslant 0}$. Now compare (3.2) with (2.18), we find that the Grunsky coefficients b_{nm} of the pair $(g = w(\mathcal{L}), f = w(\tilde{\mathcal{L}}))$ are related to the tau function or free energy by

$$b_{00} = -\frac{\partial^2\mathcal{F}}{\partial t_0^2}, \qquad b_{n,0} = \frac{1}{n}\frac{\partial^2\mathcal{F}}{\partial t_0\partial t_n}, \qquad b_{-n,0} = \frac{1}{n}\frac{\partial^2\mathcal{F}}{\partial t_0\partial t_{-n}}, \quad n \geqslant 1,$$

$$b_{m,n} = -\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_m\partial t_n}, \qquad b_{-m,-n} = -\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_{-n}}, \quad n,m \geqslant 1, \tag{3.3}$$

$$b_{-m,n} = b_{n,-m} = -\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_{-m} t_n}, \quad n,m \geqslant 1.$$

From (3.2), we can express f and g in terms of the tau function or free energy:

$$g(z) = z\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_0\partial t_m}z^{-m}\right),$$
$$f(z) = z\exp\left(\frac{\partial^2\mathcal{F}}{\partial t_0^2} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_0}z^m\right).$$

As before, we define $g_f(z) = f(z^{-1})^{-1}$, then

$$g_f(z) = z\exp\left(-\frac{\partial^2\mathcal{F}}{\partial t_0^2} + \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_0}z^{-m}\right).$$

Hence rewriting the definition of the generalized Grunsky coefficients in terms of the tau function or free energy, we obtain the Hirota equation for dispersionless Toda hierarchy. Namely, from (2.6), (2.11), (2.12), we have

$$\begin{aligned}
&z_1\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_0\partial t_m}z_1^{-m}\right) - z_2\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_0\partial t_m}z_2^{-m}\right)\\
&\quad = (z_1 - z_2)\exp\left(\sum_{m,n=1}^{\infty}\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_m\partial t_n}z_1^{-m}z_2^{-n}\right),\\
&1 - \frac{1}{z_1z_2}\exp\left(\frac{\partial^2\mathcal{F}}{\partial t_0^2} + \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_0\partial t_m}z_1^{-m} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_0}z_2^{-m}\right)\\
&\quad = \exp\left(\sum_{m,n=1}^{\infty}\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_m\partial t_{-n}}z_1^{-m}z_2^{-n}\right),\\
&z_1\exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_0}z_1^{-m}\right) - z_2\exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_0}z_2^{-m}\right)\\
&\quad = (z_1 - z_2)\exp\left(\sum_{m,n=1}^{\infty}\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_{-n}}z_1^{-m}z_2^{-n}\right).
\end{aligned}\tag{3.4}$$

We should understand these identities as defining a sequence of relations satisfied by the second derivatives of $\mathcal{F}$ by comparing the coefficients of $z_1^{-m}z_2^{-n}$ on both sides.

Conversely, the tau function is uniquely characterized by these Hirota equations.

PROPOSITION 3.1. *If $\mathcal{F} = \log\tau$ is a function of t_n, $n \in \mathbb{Z}$ that satisfies the Hirota equations* (3.4)*, then τ is a tau function of a solution of the dToda hierarchy. More explicitly, if we define $\mathcal{L}(p)$ and $\tilde{\mathcal{L}}(p)$ by formally inverting the functions $\mathfrak{p}(\mathcal{L})$ and $\tilde{\mathfrak{p}}(\tilde{\mathcal{L}})$ defined by*

$$\begin{aligned}
\log\mathfrak{p}(\mathcal{L}) &= \log\mathcal{L} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_0\partial t_m}\mathcal{L}^{-m},\\
\log\tilde{\mathfrak{p}}(\tilde{\mathcal{L}}) &= \log\tilde{\mathcal{L}} + \frac{\partial^2\mathcal{F}}{\partial t_0^2} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_0}\tilde{\mathcal{L}}^{m},
\end{aligned}\tag{3.5}$$

then $(\mathcal{L}, \tilde{\mathcal{L}})$ satisfies the Lax equations (3.1) *for dToda hierarchy.*

Proof. In the first part of the proof, we trace the reasoning above backward. We define the function $g \in \Sigma$ and $f \in \tilde{S}$ by

$$\log\frac{g(z)}{z} = -\sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_0\partial t_m}z^{-m}, \qquad \log\frac{f(z)}{z} = \frac{\partial^2\mathcal{F}}{\partial t_0^2} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_0}z^{m}.$$

The generalized Grunsky coefficients of the pair (f, g) are defined by Equations (2.6), (2.7), (2.8). We also define $g_f(z) = 1/f(z^{-1})$, then

$$\log\frac{g_f(z)}{z} = -\frac{\partial^2\mathcal{F}}{\partial t_0^2} + \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_0}z^{-m}.$$

In terms of g and g_f, the Hirota equations (3.4) read as

$$\log\frac{g(z_1)-g(z_2)}{z_1-z_2} = \sum_{m,n=1}^{\infty}\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_m\partial t_n}z_1^{-m}z_2^{-n},$$

$$\log\left(1-\frac{1}{g(z_1)g_f(z_2)}\right) = \sum_{m,n=1}^{\infty}\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_m\partial t_{-n}}z_1^{-m}z_2^{-n},$$

$$\log\frac{g_f(z_1)-g_f(z_2)}{z_1-z_2} = -\frac{\partial^2\mathcal{F}}{\partial t_0^2} + \sum_{m,n=1}^{\infty}\frac{1}{mn}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_{-n}}z_1^{-m}z_2^{-n}.$$

Comparing with (2.6), (2.9), (2.11), (2.12), we find the Grunsky coefficients in terms of $\mathcal{F}$ are given by the equations in (3.3). Hence if we define $\mathcal{L}(p)$ to be the inverse function of $w = g(z)$, and $\tilde{\mathcal{L}}(p)$ to be the inverse function of $w = f(z)$ by replacing w with p, then the identities satisfied by the Faber polynomials of f and g (2.18) say that

$$(\mathcal{L}^n)_{\geqslant 0} = \mathcal{L}^n - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_n\partial t_m}\mathcal{L}^{-m} = \frac{\partial^2\mathcal{F}}{\partial t_0\partial t_n} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_n}\tilde{\mathcal{L}}^m,$$

$$(\tilde{\mathcal{L}}^{-n})_{<0} = -\sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_m\partial t_{-n}}\mathcal{L}^{-m} = \tilde{\mathcal{L}}^{-n} + \frac{\partial^2\mathcal{F}}{\partial t_0\partial t_{-n}} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_{-n}}\tilde{\mathcal{L}}^m.$$

Now from (3.5) and (3.6), we have

$$\begin{aligned}
\frac{1}{\mathfrak{p}(\mathcal{L})}\frac{\partial\mathfrak{p}(\mathcal{L})}{\partial t_n}\Big|_{\mathcal{L}\text{ fixed}} &= \frac{\partial(\mathcal{L}^n)_{\geqslant 0}}{\partial t_0}\Big|_{\mathcal{L}\text{ fixed}},\\
\frac{1}{\mathfrak{p}(\mathcal{L})}\frac{\partial\mathfrak{p}(\mathcal{L})}{\partial t_{-n}}\Big|_{\mathcal{L}\text{ fixed}} &= \frac{\partial(\tilde{\mathcal{L}}^{-n})_{<0}}{\partial t_0}\Big|_{\mathcal{L}\text{ fixed}},\\
\frac{1}{\tilde{\mathfrak{p}}(\tilde{\mathcal{L}})}\frac{\partial\tilde{\mathfrak{p}}(\tilde{\mathcal{L}})}{\partial t_n}\Big|_{\tilde{\mathcal{L}}\text{ fixed}} &= \frac{\partial(\mathcal{L}^n)_{\geqslant 0}}{\partial t_0}\Big|_{\tilde{\mathcal{L}}\text{ fixed}},\\
\frac{1}{\tilde{\mathfrak{p}}(\tilde{\mathcal{L}})}\frac{\partial\tilde{\mathfrak{p}}(\tilde{\mathcal{L}})}{\partial t_{-n}}\Big|_{\tilde{\mathcal{L}}\text{ fixed}} &= \frac{\partial(\tilde{\mathcal{L}}^{-n})_{<0}}{\partial t_0}\Big|_{\tilde{\mathcal{L}}\text{ fixed}}.
\end{aligned}\tag{3.7}$$

On the other hand, since $\mathfrak{p}\circ\mathcal{L}$ is the identity function in p, by chain rule, we have

$$\frac{\partial\mathfrak{p}(\mathcal{L})}{\partial\mathcal{L}}\frac{\partial\mathcal{L}}{\partial t}\Big|_{p\text{ fixed}} + \frac{\partial\mathfrak{p}(\mathcal{L})}{\partial t}\Big|_{\mathcal{L}\text{ fixed}} = 0,\tag{3.8}$$

and similarly for $\tilde{\mathcal{L}}$. Here t is any of the independent variables. Hence

$$\frac{\partial \mathcal{L}}{\partial t_n} = -\frac{\partial \mathcal{L}}{\partial p}\frac{\partial \mathfrak{p}(\mathcal{L})}{\partial t_n}\Big|_{\mathcal{L}\,\text{fixed}} = -p\frac{\partial \mathcal{L}}{\partial p}\frac{\partial (\mathcal{L}^n)_{\geqslant 0}}{\partial t_0}\Big|_{\mathcal{L}\,\text{fixed}}$$
$$= -p\frac{\partial \mathcal{L}}{\partial p}\left(\frac{\partial (\mathcal{L}^n)_{\geqslant 0}}{\partial t_0} - \frac{\partial (\mathcal{L}^n)_{\geqslant 0}}{\partial \mathcal{L}}\frac{\partial \mathcal{L}}{\partial t_0}\right) = \{(\mathcal{L}^n)_{\geqslant 0}, \mathcal{L}\}_T,$$

which is the first one of the Lax equations (3.1). The other equations are derived in the same way from (3.7). □

There are two variants of the dToda hierarchy that we would like to discuss here. Let $(\mathcal{L}, \tilde{\mathcal{L}})$ be a solution of the dToda hierarchy and $\phi = \partial\mathcal{F}/\partial t_0$. We make a Miura-type transformation (Lemma 2.1.3 in [22]) and define $\mathcal{L}' = \mathrm{e}^{-\mathrm{ad}\phi}\mathcal{L}$, $\tilde{\mathcal{L}}' = \mathrm{e}^{-\mathrm{ad}\phi}\tilde{\mathcal{L}}$. They are of the form

$$\mathcal{L}'(p) = \tilde{u}_0 p + \sum_{n=0}^{\infty} u'_{n+1}(t)p^{-n} = \tilde{u}_0 p + \sum_{n=0}^{\infty} u_{n+1}(t)(\tilde{u}_0 p)^{-n},$$

$$\tilde{\mathcal{L}}'^{-1}(p) = p^{-1} + \sum_{n=0}^{\infty} \tilde{u}'_{n+1}(t)p^{n} = p^{-1} + \sum_{n=0}^{\infty} \tilde{u}_{n+1}(t)(\tilde{u}_0 p)^{n},$$

and satisfy the Lax equations

$$\frac{\partial \mathcal{L}'}{\partial t_n} = \{((\mathcal{L}')^n)_{>0}, \mathcal{L}'\}_T, \qquad \frac{\partial \mathcal{L}'}{\partial t_{-n}} = \{((\tilde{\mathcal{L}}')^{-n}))_{\leqslant 0}, \mathcal{L}'\}_T,$$

$$\frac{\partial \tilde{\mathcal{L}}'}{\partial t_n} = \{((\mathcal{L}')^n)_{>0}, \tilde{\mathcal{L}}'\}_T, \qquad \frac{\partial \tilde{\mathcal{L}}'}{\partial t_{-n}} = \{((\tilde{\mathcal{L}}')^{-n})_{\leqslant 0}, \tilde{\mathcal{L}}'\}_T.$$

From the point of view of conformal maps, this transformation amounts to the pre-composition of $z = G(w)$ and $z = F(w)$ with the linear map $w \mapsto \tilde{u}_0 w$ ($\tilde{u}_0 = a_1$, the leading coefficient of $f(z)$). Hence for the inverse function, we have

$$g'(z) = \frac{g(z)}{\tilde{u}_0}, \qquad f'(z) = \frac{f(z)}{\tilde{u}_0}.$$

From these and the definition of the Grunsky coefficients, it is quite obvious that the Grunsky coefficients b_{mn}, $mn \neq 0$ of (f, g) and (f', g') are the same. Hence, the Hirota equations for $(\mathcal{L}', \tilde{\mathcal{L}}')$ assume the same form (3.4). However, now $\mathcal{L}'$, $\tilde{\mathcal{L}}'$ are defined by inverting the functions

$$\mathfrak{p}(\mathcal{L}') = g'(\mathcal{L}') = \mathcal{L}' \exp\left(-\frac{\partial^2\mathcal{F}}{\partial t_0^2} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_0 \partial t_m}(\mathcal{L}')^{-m}\right),$$
$$\tilde{\mathfrak{p}}(\tilde{\mathcal{L}}') = f'(\tilde{\mathcal{L}}') = \tilde{\mathcal{L}}' \exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\frac{\partial^2\mathcal{F}}{\partial t_{-m}\partial t_0}(\tilde{\mathcal{L}}')^{m}\right).$$

Another variant of the dToda hierarchy is the one favored by Wiegmann and Zabrodin in association with conformal maps [12, 15, 25]. Now the Miura type transformation is defined as $\mathcal{L}' = e^{-\mathrm{ad}\frac{\phi}{2}}\mathcal{L}, \tilde{\mathcal{L}}' = e^{-\mathrm{ad}\frac{\phi}{2}}\tilde{\mathcal{L}}$. Then they are of the form

$$\mathcal{L}'(p) = rp + \sum_{n=0}^{\infty} u'_{n+1}(t)p^{-n} = rp + \sum_{n=0}^{\infty} u_{n+1}(t)(rp)^{-n}$$

$$\tilde{\mathcal{L}}'^{-1}(p) = rp^{-1} + \sum_{n=0}^{\infty} \tilde{u}'_{n+1}(t)p^{n} = rp^{-1} + \sum_{n=0}^{\infty} \tilde{u}_{n+1}(t)(rp)^{n},$$

where r is a square root of $\tilde{u}_0$. The Lax equations become*

$$\frac{\partial \mathcal{L}'}{\partial t_n} = \{\mathcal{H}_n, \mathcal{L}'\}_T, \qquad \frac{\partial \mathcal{L}'}{\partial t_{-n}} = \{\tilde{\mathcal{H}}_n, \mathcal{L}'\}_T,$$

$$\frac{\partial \tilde{\mathcal{L}}'}{\partial t_n} = \{\mathcal{H}_n, \tilde{\mathcal{L}}'\}_T, \qquad \frac{\partial \tilde{\mathcal{L}}'}{\partial t_{-n}} = \{\tilde{\mathcal{H}}_n, \tilde{\mathcal{L}}'\}_T$$

$$\mathcal{H}_n = ((\mathcal{L}')^n)_{>0} + \frac{1}{2}((\mathcal{L}')^n)_0, \qquad \tilde{\mathcal{H}}_n = ((\tilde{\mathcal{L}}')^{-n})_{<0} + \frac{1}{2}((\tilde{\mathcal{L}}')^{-n})_0.$$

This version has the advantage that the roles of $\mathcal{L}$ and $\mathcal{L}'$ are symmetric. The same discussion above shows that the tau function and dispersionless Hirota equations assume the same form but now, $\mathcal{L}'$, $\tilde{\mathcal{L}}'$ are defined by inverting the functions

$$\mathfrak{p}(\mathcal{L}') = g'(\mathcal{L}') = \mathcal{L}' \exp\left(-\frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial t_0^2} - \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial^2 \mathcal{F}}{\partial t_0 \partial t_m}(\mathcal{L}')^{-m}\right),$$

$$\tilde{\mathfrak{p}}(\tilde{\mathcal{L}}') = f'(\tilde{\mathcal{L}}') = \tilde{\mathcal{L}}' \exp\left(\frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial t_0^2} - \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial^2 \mathcal{F}}{\partial t_{-m} \partial t_0}(\tilde{\mathcal{L}}')^{m}\right).$$

We can also view the dispersionless Hirota equations as a consequence of the definition of the Grunsky coefficients for the pair $(g, g_{\tilde{f}})$. From this point of view and our discussion above, we readily see that if $(\mathcal{L}, \tilde{\mathcal{L}})$ is a solution to the first version of the dToda hierarchy, then the pair $(\mathcal{L}', \tilde{\mathcal{L}}')$, where $\mathcal{L}'(p) = \tilde{\mathcal{L}}(1/p)^{-1}, \tilde{\mathcal{L}}'(p) = \mathcal{L}(1/p)^{-1}$ is a solution to the second version of the dToda hierarchy, if we redefine the independent variables as $t'_n = t_{-n}$ and $t'_{-n} = t_n$.

3.2. DISPERSIONLESS (COUPLED) MODIFIED KP HIERARCHY

We define the dispersionless coupled modified KP hierarchy (demKP) in [24]. Here we are only interested in a special case. (In the notation in [24], it is the case where $\mathcal{P} = k$.)

The fundamental quantity is a formal power series

$$\mathcal{L} = k + \sum_{n=0}^{\infty} u_{n+1}(t)k^{-n},$$

*Notice that the $\bar{t}_n$ in [25] is the complex conjugate of t_n. Here it amounts to $-t_{-n}$.

with coefficients depending on the parameters $t = (x, t_0, t_1, t_2 \ldots)$. The Lax representation of the dcmKP in our special case here is

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{(\mathcal{L}^n)_{>0}, \mathcal{L}\}, \quad n \geqslant 1; \qquad \frac{\partial \mathcal{L}}{\partial t_0} = \frac{1}{k}\frac{\partial \mathcal{L}}{\partial x}. \tag{3.9}$$

Here $\{\cdot,\cdot\}$ is the Poisson bracket defined as

$$\{f, g\} = \frac{\partial f}{\partial k}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial k}.$$

All the dependence on t_1 and x appears in the form $t_1 + x$. There exists a tau function τ which satisfies the following identities:

$$\begin{aligned} \log k &= \log \mathcal{L} - \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_0}\mathcal{L}^{-m}, \\ (\mathcal{L}^n)_{\geqslant 0} &= \mathcal{L}^n - \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial^2 \mathcal{F}}{\partial t_n \partial t_m}\mathcal{L}^{-m}, \qquad \frac{\partial^2 \mathcal{F}}{\partial t_0 \partial t_n} = (\mathcal{L}^n)_0. \end{aligned} \tag{3.10}$$

Here $\mathcal{F} = \log \tau$. Now we identify k with w and $\mathcal{L}$ with z. The first equation in (3.10) defines w as a function of z, which we denote by $g(z)$:

$$g(z) = z \exp\left(- \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_0} z^{-m} \right). \tag{3.11}$$

Obviously, $g \in \Sigma$. The Faber polynomials Φ_n for g are then identified with $(\mathcal{L}^n)_{\geqslant 0}$. Comparing (2.4) to (3.10), we find that the Grunsky coefficients of $g(z)$ are related to the tau function or the free energy by

$$b_{mn} = -\frac{1}{mn}\frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n}. \tag{3.12}$$

Together with (3.11), we can rewrite the definition of the Grunsky coefficients (2.3) as

$$\begin{aligned} &z_1 \exp\left(- \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial^2 \mathcal{F}}{\partial t_0 \partial t_m} z_1^{-m} \right) - z_2 \exp\left(- \sum_{m=1}^{\infty} \frac{1}{m}\frac{\partial^2 \mathcal{F}}{\partial t_0 \partial t_m} z_2^{-m} \right) \\ &\quad = (z_1 - z_2) \exp\left(\sum_{m,n=1}^{\infty} \frac{1}{mn}\frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} z_1^{-m} z_2^{-n} \right), \end{aligned} \tag{3.13}$$

which is the dispersionless Hirota equation for this special case of dcmKP hierarchy.

Conversely, we can characterize the tau function for dcmKP hierarchy as:

PROPOSITION 3.2. *If $\mathcal{F} = \log \tau$ is a function of t_n, $n \geqslant 0$ that satisfies the Hirota equation* (3.13) *and $\partial^3 \log \tau / \partial t_0^2 \partial t_1 = 0$, then τ is a tau function of a solution of the dcmKP hierarchy. More explicitly, if we define $\mathcal{L}(k)$ by formally inverting the function* $\mathsf{k}(\mathcal{L})$ *defined by*

$$\log \mathsf{k}(\mathcal{L}) = \log \mathcal{L} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \mathcal{F}}{\partial t_0 \partial t_m} \mathcal{L}^{-m}, \tag{3.14}$$

and replacing t_1 by $t_1 + x$, then $\mathcal{L}$ satisfies the Lax equations (3.9) *for dcmKP.*

Proof. Identifying $\mathcal{L}$ with z, we define the function $g(z) \in \Sigma$ by

$$\log g(z) = \log z - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \mathcal{F}}{\partial t_0 \partial t_m} z^{-m}.$$

The dispersionless Hirota equation (3.13) then read as

$$\log \frac{g(z_1) - g(z_2)}{z_1 - z_2} = \sum_{m,n=1}^{\infty} \frac{1}{mn} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} z_1^{-m} z_2^{-n}.$$

Comparing with the definition of the Grunsky coefficients of g given by equation (2.3), we find that the relation between the Grunsky coefficients and free energy is given by (3.12). Let $z = G(w)$ be the inverse of $g(z)$. Then $G(w)$ is the function $\mathcal{L}(k)$ defined by (3.14) if we identify k with w. The Faber polynomials of $g(z)$ is then identified with $(\mathcal{L}(w))_{\geqslant 0}$. We can then rewrite the identity satisfied by the Faber polynomials (2.4) in terms of the free energy by

$$(\mathcal{L}^n)_{\geqslant 0} = \mathcal{L}^n - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} \mathcal{L}^{-m}.$$

From (3.14), we also have

$$\frac{\partial^2 \mathcal{F}}{\partial t_n \partial t_0} = \operatorname{Res} \mathcal{L}^n d \log k = (\mathcal{L}^n)_0.$$

Hence

$$(\mathcal{L}^n)_{>0} = \mathcal{L}^n - \frac{\partial^2 \mathcal{F}}{\partial t_n \partial t_0} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} \mathcal{L}^{-m}. \tag{3.15}$$

From (3.14) and the $n = 1$ case in (3.15), we have

$$\frac{1}{\mathsf{k}} \frac{\partial \mathsf{k}}{\partial t_1}\Big|_{\mathcal{L} \text{ fixed}} = -\sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^3 \mathcal{F}}{\partial t_m \partial t_0 \partial t_1} = \frac{\partial \mathsf{k}}{\partial t_0}\Big|_{\mathcal{L} \text{ fixed}}.$$

Identity (3.8) (with p replaced by k) then gives the second equation in the Lax equations (3.9). From (3.15) again, we have

$$\frac{\partial \mathsf{k}}{\partial t_n}\Big|_{\mathcal{L} \text{ fixed}} = \frac{\partial (\mathcal{L}^n)_{>0}}{\partial t_1}\Big|_{\mathcal{L} \text{ fixed}},$$

which by (3.8) is equivalent to the first equation of the Lax equations (3.9). □

Compare the dispersionless Hirota equations for dToda and dcmKP hierarchies, we immediately have

COROLLARY 3.3. *If* $(\mathcal{L}, \tilde{\mathcal{L}})$ *is a solution to the dToda hierarchy* (3.1), *and* $\partial(\mathcal{L})_0/\partial t_0 = 0$, *then* $\mathcal{L}$ *is a solution to the dcmKP hierarchy* (3.9), *when we replace* t_1 *by* $t_1 + x$ *and regard the* t_{-n}*'s,* $n \geqslant 1$ *as parameters. The tau function for the dToda hierarchy is the tau function for the corresponding dcmKP hierarchy.*

Proof. In [24], we proved this proposition by comparing the Lax equations. Here we just notice that the first equation in the Hirota equations for dToda (3.4) is identical with the Hirota equation for dcmKP (3.13). The result follows from the proposition above. □

3.3 DISPERSIONLESS KP HIERARCHY

This is the most well known case. The dispersionless Hirota equation for dKP hierarchy was first derived as the quasi-classical limit of the differential Fay identity by Takasaki and Takebe in [22], see also the work of Carroll and Kodama [5]. Here we derive the Hirota equation along the same line as we do for dToda and dcmKP hierarchies.

The fundamental quantity in dKP hierarchy is a formal power series

$$\mathcal{L} = k + \sum_{n=1}^{\infty} u_{n+1}(t) k^{-n}$$

with coefficients depending on the independent variables $t = (x, t_1, t_2, \ldots)$. The Lax equation is

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{(\mathcal{L}^n)_{\geqslant 0}, \mathcal{L}\}. \tag{3.16}$$

Here the Poisson bracket is the same as in the dcmKP hierarchy. The dependence on t_1 and x appears in the combination $t_1 + x$.

There exists a tau function τ which satisfies the following identities:

$$(\mathcal{L}^n)_{\geqslant 0} = \mathcal{L}^n - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} \mathcal{L}^{-m}. \tag{3.17}$$

Here $\mathcal{F} = \log \tau$. Now we identify k with w and $\mathcal{L}$ with z, the function $z = G(w)$ is defined to be $\mathcal{L}(k)$, and the function $w = g(z) \in \Sigma_0$ the inverse of $G(w)$. Then the Faber polynomials $\Phi_n(w)$ of g are identified with $(\mathcal{L}^n(w))_{\geqslant 0}$. Define the Grunsky coefficients of g by Equation (2.3), then comparing (2.4) with (3.17), we find the relation between the Grunsky coefficients and the free energy is given by

$$b_{mn} = -\frac{1}{mn} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n}. \tag{3.18}$$

From the $n = 1$ case of Equation (3.17) and the fact that $(\mathcal{L})_{\geqslant 0} = k$, we have

$$g(z) = z - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_1} z^{-m}.$$

Using this, and (3.18), we can rewrite the definition of the Grunsky coefficients (2.3) in terms of the free energy:

$$1 - \frac{1}{z_1 - z_2} \sum_{m=1}^{\infty} \frac{z_1^{-m} - z_2^{-m}}{m} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_1} = \exp\left(\sum_{m,n=1}^{\infty} \frac{1}{mn} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} z_1^{-m} z_2^{-n} \right), \tag{3.19}$$

which is the dispersionless Hirota equation for dKP.

Conversely, we can characterize the tau function for the dKP hierarchy as follows:

PROPOSITION 3.4. *If $\mathcal{F} = \log\tau$ is a function of t_n, $n \geqslant 1$ that satisfies the Hirota equation* (3.19)*, then τ is a tau function of a solution of the dKP hierarchy. More explicitly, if we define $\mathcal{L}(k)$ by formally inverting the function $\mathsf{k}(\mathcal{L})$ defined by*

$$\mathsf{k}(\mathcal{L}) = \mathcal{L} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_m} \mathcal{L}^{-m}, \tag{3.20}$$

and replacing t_1 by $t_1 + x$, then $\mathcal{L}$ satisfies the Lax equations (3.16) *for dKP.*

Proof. We define the function $g \in \Sigma_0$ by identifying $\mathcal{L}$ with z in (3.20), the dispersionless Hirota equation (3.19) says that

$$\log \frac{g(z_1) - g(z_2)}{z_1 - z_2} = \sum_{m,n=1}^{\infty} \frac{1}{mn} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} z_1^{-m} z_2^{-n}. \tag{3.21}$$

Compare with the definition of the Grunsky coeffcients b_{mn} (2.3) of g, we find that the b_{mn} can be expressed in terms of the free energy by (3.18). Now define $z = G(w)$ to be the formal inverse of $w = g(z)$. In other words, $G(w)$ is $\mathcal{L}(k)$ if we identify k with w. Then the Faber polynomials $\Phi_n(w)$ of g are identified with $(\mathcal{L}^n(k))_{\geqslant 0}$. Hence, the identities satisfied by the Faber polynomials (2.4) can be rewritten as

$$(\mathcal{L}^n)_{\geqslant 0} = \mathcal{L}^n - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} \mathcal{L}^{-m}. \tag{3.22}$$

The same argument as in Proposition 3.2 gives the Lax equation (3.16) of dKP. □

From this characterization of the tau functions, we can also see that a solution of the dcmKP hierarchy will give rise to a solution of the dKP hierarchy.

COROLLARY 3.5. *If*

$$\mathcal{L} = k + \sum_{n=0}^{\infty} u_{n+1} k^{-n}$$

is a solution of the dcmKP hierarchy, then the Miura transform of $\mathcal{L}$,

$$\mathcal{L}' = k + \sum_{n=1}^{\infty} u'_{n+1} k^{-n} = k + \sum_{n=1}^{\infty} u_{n+1} (k - u_1)^{-n} \tag{3.23}$$

is a solution of the dKP hierarchy, where we regard t_0 as a constant. Moreover, $\mathcal{L}$ and $\mathcal{L}'$ have the same tau function.

Proof. Since $\mathcal{L} = k + \sum_{n=0}^{\infty} u_{n+1}k^{-n}$ is a solution of the dcmKP hierarchy, the free energy of $\mathcal{F}$ satisfies the dispersionless Hirota equation (3.13). Taking the limit $z_2 \to \infty$ in (3.13), we obtain the relation

$$z \exp\left(-\sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \mathcal{F}}{\partial t_0 \partial t_m} z^{-m}\right) = z - \frac{\partial^2 \mathcal{F}}{\partial t_0 \partial t_1} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_1} z^{-m}.$$

Substituting this into the Hirota equation (3.13) again, we obtain

$$z_1 - z_2 - \sum_{m=1}^{\infty} \frac{z_1^m - z_2^m}{m} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_1} = (z_1 - z_2) \exp\left(\sum_{m,n=1}^{\infty} \frac{1}{mn} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} z_1^{-m} z_2^{-n}\right),$$

which is equivalent to the dispersionless Hirota equation for dKP (3.19). Hence, from Proposition 3.4 above, the function $\mathcal{L}'$ defined by inverting

$$\mathsf{k}(\mathcal{L}') = \mathcal{L}' - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_1} (\mathcal{L}')^{-m}$$

is a solution of the dKP hierarchy. Comparing with (3.14) and using the fact that $\partial^2 \mathcal{F}/\partial t_0 \partial t_1 = u_1$ give (3.23). □

Remark 3.6. The proof of this corollary also shows that the dispersionless Hirota equation for dcmKP implies the dispersionless Hirota equation for dKP.

4. Concluding Remarks

We rederive the dispersionless Hirota equations for dToda, dcmKP and dKP hierarchies and prove that they uniquely characterize the tau functions associated to a solution of the hierarchies. This might be helpful in classifying the solutions of the hierarchies. The transformation that relate the three versions of the dToda hierarchies and the Miura map that transform a solution of the dcmKP hierarchy to a solution of the dKP hierarchy are just the linear maps that relate the three spaces of formal power series we discuss in Section 2.

Given any formal power series $f \in \tilde{S}$, $g \in \Sigma$, if we define $\mathcal{F}_{mn}$'s as

$$\mathcal{F}_{m,n} = -|mn| b_{m,n}, \qquad m \neq 0,\ n \neq 0,$$

$$\mathcal{F}_{m,0} = \mathcal{F}_{0,m} = |m| b_{m,0}, \qquad m \neq 0, \qquad \mathcal{F}_{0,0} = -b_{0,0},$$

where $b_{m,n}$'s are the Grunsky coefficients associated to the pair (f, g), then $\mathcal{F}_{m,n}$'s satisfy the dispersionless Hirota equations (3.4), (3.13) and (3.19) if we replace $\partial^2 \mathcal{F}/\partial t_m \partial t_n$ by $\mathcal{F}_{m,n}$. In [3], Sorin and Bonora proved that the Neumann coefficients that appear in string field theory satisfy the dispersionless Hirota equations. By definition, the Neumann coefficients coincide with the Grunsky coefficients $b_{m,n}$ defined above. This explain their results.

However, it is still an open question to find a function $\mathcal{F}$ such that

$$\frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} = \mathcal{F}_{m,n}.$$

In [25], Wiegmann and Zabrodin provided a solution to this problem (see also [12, 15, 26]) when G, the inverse function of g is an analytic function that maps the outer disc $\{|z| > 1\}$ to the exterior of an analytic curve, and $f(z) = \tilde{g}(z^{-1})^{-1}$. It will be interesting to solve the general problem.

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