PERGAMON

Microelectronics Reliability 43 (2003) 651–664

MICROELECTRONICS
RELIABILITY

www.elsevier.com/locate/microrel

A reliable procedure for testing linear regulators with one-sided specification limits based on multiple samples

W.L. Pearn [a,*], G.H. Lin [b]

[a] *Department of Industrial Engineering and Management, National Chiao Tung University, 1001 Ta Hsueh Road, Hsin Chu 30050, Taiwan*
[b] *Department of Communication Engineering, National Penghu Institute of Technology, Taiwan 880, ROC*

Received 30 July 2002; received in revised form 15 October 2002

Abstract

Process capability indices C_{PU} and C_{PL} are developed in the manufacturing industry to provide quantitative measures on process potential and performance, for processes with one-sided specification limits. Statistical properties of the estimators of C_{PU} and C_{PL} have been investigated extensively, but only restricted to cases with single samples. In this paper, we consider the estimation and capability testing of C_{PU} and C_{PL} based on multiple samples. We show that the proposed estimator of C_{PU} and C_{PL} are indeed the uniformly minimum variance unbiased estimators (UMVUEs). A simple procedure based on a hypothesis testing using the UMVUE is developed for the practitioners to use for judging whether a stable process meets the preset capability requirement.

---

1. Introduction

Process capability indices provide numerical measures on whether a manufacturing process meets the preset capability requirement. These have been the focus of recent research in quality assurance and process capability analysis. Three basic capability indices C_p, C_a, and C_{pk}, which establish the relationships between the actual process performance and the manufacturing specification tolerances, are defined as follows [1,2]:

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma},$$

$$C_a = 1 - \frac{|\mu - m|}{d},$$

$$C_{pk} = \min\left\{\frac{\text{USL} - \mu}{3\sigma}, \frac{\mu - \text{LSL}}{3\sigma}\right\},$$

where USL and LSL are the upper and lower specification limits preset by the process engineers or product designers, μ is the process mean, σ is the process standard deviation, $m = (\text{USL} + \text{LSL})/2$ is the mid-point between the upper and lower specification limits, and $d = (\text{USL} - \text{LSL})/2$ is half length of the specification interval.

While C_p, C_a, and C_{pk} are appropriate measures for processes with two-sided specifications (which require both upper and lower specification limits USL and LSL), C_{PL} and C_{PU} [1] have been designed particularly for processes with one-sided specification limits (which require only the upper or the lower specification limit). The quality and statistics literatures discuss the estimation of these indices based on a single sample, see [1,3–6]. In practice, however, process information is often derived from multiple samples rather then from one single sample. In this paper, we consider the estimation of the one-sided capability indices C_{PU} and C_{PL}

---

[*] Corresponding author. Tel.: +886-35-714261; fax: +886-35-722392.
E-mail address: roller@cc.nctu.edu.tw (W.L. Pearn).

0026-2714/02/$ - see front matter © 2002 Elsevier Science Ltd. All rights reserved.
doi:10.1016/S0026-2714(02)00323-2

Table 1
The five quality conditions

Quality condition	C_{I} values
Inadequate	$C_{\mathrm{I}} < 1.00$
Marginally capable	$1.00 \leqslant C_{\mathrm{I}} < 1.33$
Satisfactory	$1.33 \leqslant C_{\mathrm{I}} < 1.67$
Excellent	$1.67 \leqslant C_{\mathrm{I}} < 2.00$
Super	$2.00 \leqslant C_{\mathrm{I}}$

for multiple samples, and develop a simple but practical procedure to assist the practitioners to determine whether their processes meet the capability requirement.

In current practice, a process is called inadequate if $C_{\mathrm{I}} < 1.00$, where $C_{\mathrm{I}} = C_{\mathrm{PU}}$ or C_{PL}; it indicates that the process is not adequate with respect to the production tolerances (specifications), either the process variation σ^2 needs to be reduced or the process mean μ needs to be shifted closer to the target value T. A process is called marginally capable if $1.00 \leqslant C_{\mathrm{I}} < 1.33$; it indicates that caution needs to be taken regarding the process distribution and some process control is required. A process is called satisfactory if $1.33 \leqslant C_{\mathrm{I}} < 1.67$; it indicates that process quality is satisfactory, material substitution may be allowed, and no stringent quality control is required. A process is called excellent if $1.67 \leqslant C_{\mathrm{I}} < 2.00$; it indicates that process quality exceeds satisfactory. Finally, a process is called super if $C_{\mathrm{I}} \geqslant 2.00$. Table 1 summarizes the above five conditions and the corresponding C_{I} values.

2. Estimating C_{PL} and C_{PU} based on multiple samples

To estimate the indices C_{PL} and C_{PU} in the presence of single samples, Chou and Owen [3] considered the following natural estimators of C_{PU} and C_{PL}:

$$\widehat{C}_{\mathrm{PU}} = \frac{\mathrm{USL} - \overline{X}}{3S},$$

$$\widehat{C}_{\mathrm{PL}} = \frac{\overline{X} - \mathrm{LSL}}{3S},$$

where n is the sample size and $\overline{X} = \sum_{i=1}^{n} X_i/n$, $S = [(n-1)^{-1}\sum_{i=1}^{n}(X_i - \overline{X})^2]^{1/2}$ are conventional estimators of μ and σ, which may be obtained from a process that is demonstrably stable (in-control). Chou and Owen [3] showed that under the normality assumption, the estimators $3\sqrt{n}\widehat{C}_{\mathrm{PU}}$ and $3\sqrt{n}\widehat{C}_{\mathrm{PL}}$ are distributed as $t_{n-1}(\delta_{\mathrm{L}})$ and $t_{n-1}(\delta_{\mathrm{U}})$, a non-central t distribution with $n-1$ degrees of freedom and non-central parameters $\delta_{\mathrm{U}} = 3\sqrt{n}C_{\mathrm{PU}}$ and $\delta_{\mathrm{L}} = 3\sqrt{n}C_{\mathrm{PL}}$. Pearn and Chen [4] obtained the UMVUEs of C_{PL} and C_{PU}. Based on the UMVUEs, Pearn and Chen [4] developed a simple but practical procedure based on single samples. Applying the proposed procedure, the practitioners can determine whether their processes meet the preset capability requirement.

A common practice of the process capability estimation in the microelectronics manufacturing industry is to first implement a daily-based data collection program for monitoring/controlling the process stability, then to analyze the past "in control" data. For multiple samples of m groups each of size n taken for preparing the $(\overline{X}, S)$ control charts, Kirmani et al. [7] considered the following natural estimators of C_{PU} and C_{PL}. Let $\overline{X}_i = \sum_{j=1}^{n} X_{ij}/n$ and $S_i = [(n-1)^{-1}\sum_{j=1}^{n}(X_{ij} - \overline{X}_i)^2]^{1/2}$ be the ith sample mean and the sample standard deviation respectively. Then $\overline{\overline{X}} = \sum_{i=1}^{m}\overline{X}_i/m$ and $S_{\mathrm{p}}^2 = \sum_{i=1}^{m} S_i^2/m$ are the unbiased estimators of μ and σ^2, respectively and the estimators of C_{PU} and C_{PL} are

$$\widetilde{C}^*_{\mathrm{PU}} = \frac{b_{m(n-1)}\left(\mathrm{USL} - \overline{\overline{X}}\right)}{3S_{\mathrm{p}}},$$

$$\widetilde{C}^*_{\mathrm{PL}} = \frac{b_{m(n-1)}\left(\overline{\overline{X}} - \mathrm{LSL}\right)}{3S_{\mathrm{p}}},$$

where b_g is the well-known correction factor defined as $b_g = (2/g)^{1/2}\Gamma(g/2)/\{\Gamma[(g-1)/2]\}$. Throughout this paper, we assume that the sample of the characteristic investigated, $(X_{i1}, X_{i2}, \ldots, X_{in})$, are chosen randomly from a stable process which follows a normal distribution $N(\mu, \sigma^2)$ for $i = 1, 2, \ldots, m$. In the following, we show that the proposed estimators $\widetilde{C}^*_{\mathrm{PU}}$ and $\widetilde{C}^*_{\mathrm{PL}}$ are the uniformly minimum variance unbiased estimators (UMVUEs) of C_{PU} and C_{PL} for multiple samples. The probability density function of the UMVUE of C_{PU} or C_{PL} may be obtained as

$$f(x) = k\left\{\sum_{j=0}^{\infty}\left(\frac{g^3 + 9mnx^2}{g^2}\right)^{-((j+g+1)/2)} \times \frac{(\sqrt{2}\delta g)^j}{j!}\frac{\Gamma[(g+j+1)/2]}{\Gamma[(g+1)/2]}\right\},$$

where $g = m(n-1)$, $\delta = 3\sqrt{mn}C_{\mathrm{I}}$, $k = 3\sqrt{mn/\pi}g^{(g/2)-1} \times \exp(-\delta^2/2)\{\Gamma[(g+1)/2]/\Gamma(g/2)\}$.

Theorem 1. *The estimators* $(3\sqrt{mn}/b_{m(n-1)})\widetilde{C}^*_{\mathrm{PU}}$ *and* $(3\sqrt{mn}/b_{m(n-1)})\widetilde{C}^*_{\mathrm{PL}}$ *are distributed as the non-central t distribution with* $m(n-1)$ *degrees of freedom and non-central parameters* $\delta_{\mathrm{U}} = 3\sqrt{mn}C_{\mathrm{PU}}$ *and* $\delta_{\mathrm{L}} = 3\sqrt{mn}C_{\mathrm{PL}}$, *respectively.*

Proof. We note that $W = m(n-1)S_{\mathrm{p}}^2/\sigma^2$ is distributed as $\chi^2(m(n-1))$, a chi-square distribution with $m(n-1)$ degrees of freedom. On the other hand, $Z_{\mathrm{L}} =$

$\sqrt{mn}[(\overline{\overline{X}} - \text{LSL})/\sigma]$ and $Z_{\text{U}} = \sqrt{mn}[(\text{USL} - \overline{\overline{X}})/\sigma]$ are distributed as a normal distribution $N(\delta_{\text{L}}, 1)$ and $N(\delta_{\text{U}}, 1)$ respectively. Both Z_{L} and Z_{U} are independent statistics of W under the normality assumption. Therefore $(3\sqrt{mn}/b_{m(n-1)})\widetilde{C}^*_{\text{PL}} = Z_{\text{L}}/\sqrt{W/[m(n-1)]}$ and $(3\sqrt{mn}/b_{m(n-1)})\widetilde{C}^*_{\text{PU}} = Z_{\text{U}}/\sqrt{W/[m(n-1)]}$ are distributed as $t(m(n-1), \delta_{\text{L}})$ and $t(m(n-1), \delta_{\text{U}})$, a non-central t distribution with $m(n-1)$ degrees of freedom and non-central parameters $\delta_{\text{L}} = 3\sqrt{mn}C_{\text{PL}}$ and $\delta_{\text{U}} = 3\sqrt{mn}C_{\text{PU}}$, respectively. □

Theorem 2. *Both $\widetilde{C}^*_{\text{PU}}$ and $\widetilde{C}^*_{\text{PL}}$ are the UMVUEs of C_{PU} and C_{PL}, respectively.*

Proof. We note that $E\{b_g[t(g,\delta)]\} = \delta$. The joint probability density function of m samples of size n is given in the following, which is in the exponential family so that the statistic $\Delta = \left(\sum_{i=1}^m \sum_{j=1}^n x_{ij}^2, \sum_{i=1}^m \sum_{j=1}^n x_{ij}\right)$ is sufficient for (μ, σ^2).

$$(2\pi)^{-mn/2}\sigma^{-mn}\exp[-mn\mu^2/(2\sigma^2)]\exp\left\{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}^2\right) \times [-1/(2\sigma^2)] + \left(\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}\right)(\mu/\sigma^2)\right\}.$$

The components of $(-1/(2\sigma^2), \mu/\sigma^2)$ are functionally independent. Therefore, the statistic Δ is a complete sufficient statistic. Also, $(\overline{\overline{X}}, S_{\text{p}}^2)$ is an invertible function of the statistic Δ, and is consequently a complete sufficient statistic. Since $\widetilde{C}^*_{\text{PU}}$ and $\widetilde{C}^*_{\text{PL}}$ depends on $(\overline{\overline{X}}, S_{\text{p}}^2)$ alone, it follows that $\widetilde{C}^*_{\text{PU}}$ and $\widetilde{C}^*_{\text{PL}}$ are the UMVUEs of C_{PU} and C_{PL} based on multiple samples, respectively.

The first two moments of $t(g,\delta)$ are δ/b_g and $[g/(g-2)]\delta$. Therefore, the first two moments and the mean squared error (MSE) of C_{I} based on multiple samples can be calculated as $E(\widetilde{C}^*_{\text{I}}) = C_{\text{I}}$, $E(\widetilde{C}^*_{\text{I}})^2 = b_g^2[g/(g-2)][C_{\text{I}}^2 + (9mn)^{-1}]$, and MSE $(\widetilde{C}^*_{\text{I}}) = E(\widetilde{C}^*_{\text{I}})^2 - C_{\text{I}}^2$, where $g = m(n-1)$. The square root of the relative MSE may be used as a direct measurement, which presents the expected relative error of the estimation from the true C_{I}. For cases with unequal sample sizes, the estimators are

$$\widetilde{C}^*_{\text{PU}} = \frac{b_{\sum(n_i-1)}\left(\text{USL} - \overline{\overline{X}}\right)}{3S_{\text{p}}},$$

$$\widetilde{C}^*_{\text{PL}} = \frac{b_{\sum(n_i-1)}\left(\overline{\overline{X}} - \text{LSL}\right)}{3S_{\text{p}}},$$

$$\left(3\sqrt{\sum n_i}/b_{\sum(n_i-1)}\right)\widetilde{C}^*_{\text{PL}} = Z_{\text{L}}\Big/\sqrt{W/\left[\sum(n_i-1)\right]},$$

and

$$\left(3\sqrt{\sum n_i}/b_{\sum(n_i-1)}\right)\widetilde{C}^*_{\text{PU}} = Z_{\text{U}}\Big/\sqrt{W/\left[\sum(n_i-1)\right]}$$

are distributed as $t(\sum(n_i-1), \delta_{\text{L}})$ and $t(\sum(n_i-1), \delta_{\text{U}})$, a non-central t distribution with $\sum(n_i-1)$ degrees of freedom and non-central parameters $\delta_{\text{L}} = 3(\sum n_i)C_{\text{PL}}$ and $\delta_{\text{U}} = 3(\sum n_i)C_{\text{PU}}$, respectively. □

3. Testing C_{PL} and C_{PU} based on multiple samples

To test whether a given process meets the preset capability requirement and runs under the desired quality condition, we can consider the following hypotheses:

$H_0: \ C_{\text{I}} \leqslant C,$
$H_1: \ C_{\text{I}} > C,$

where $C_{\text{I}} = C_{\text{PU}}$ or C_{PL}. A process fails to meet the capability (quality) requirement if $C_{\text{I}} \leqslant C$ (a preset known constant), and meets the capability requirement if $C_{\text{I}} > C$. Some C values commonly used in the capability testing for most industry applications are displayed in Table 1. We define the test $\psi_{C_0}(x) = 1$, if $\widetilde{C}^*_{\text{I}} \geqslant c_0$ and $\psi_{C_0}(x) = 0$, if $\widetilde{C}^*_{\text{I}} < c_0$. A critical value c_0 with confidence level α satisfies $\alpha(c_0) = \alpha$, which implies that

$$c_0 = \frac{b_g t_{1-\alpha}(g,\delta)}{3\sqrt{mn}},$$

where $g = m(n-1)$, $b_g = (2/g)^{1/2}\Gamma(g/2)/\{\Gamma[(g-1)/2]\}$, and $t_{1-\alpha}(g,\delta)$ denotes the $(1-\alpha)$th percentile of a non-central t distribution with $g = m(n-1)$ degrees of freedom and non-central parameter $\delta = 3\sqrt{mn}C_{\text{I}}$. Table 2 (Panels A–D) display the critical values for commonly used capability requirements, $C_{\text{I}} = 1.00$, 1.33, 1.67 and 2.00 based on multiple samples with $n = 3(1)6$, $m = 5(1)40$, and $\alpha = 0.05$, 0.025, 0.01. The power function may be expressed as

$$\beta\{C_{\text{I}}, \psi_{C_0}(\underset{\sim}{x})\} = E_{C_{\text{I}}}\{\psi_{C_0}(\underset{\sim}{x})\} = P_{C_{\text{I}}}\{\widetilde{C}^*_{\text{I}} \geqslant c_0\} = P_{C_{\text{I}}}\left\{t(g,\delta) \geqslant 3\sqrt{mn}c_0/b_g\right\}.$$

Size of the test is given as the following, $\alpha(c_0) = \sup_{C_{\text{I}} \in H_0} \beta\{C_{\text{I}}, \psi_{C_0}(\underset{\sim}{x})\} = \beta\{c_0, \psi_{C_0}(\underset{\sim}{x})\}$. Table 3 (Panels A–D) display powers of the test at various $C_{\text{I}} = 1.00(0.02)1.70$ for H_0: $C_{\text{PU}} \leqslant 1.00$, $C_{\text{I}} = 1.33(0.02)2.03$ for H_0: $C_{\text{PU}} \leqslant 1.33$, $C_{\text{I}} = 1.67(0.02)2.37$ for H_0: $C_{\text{PU}} \leqslant 1.67$, and $C_{\text{I}} = 2.00(0.02)2.70$ for H_0: $C_{\text{PU}} \leqslant 2.00$, based on multiple samples under $n = 5$, $m = 15(5)30$, and $\alpha = 0.050$, 0.025, 0.010. Figs. 1–4 plot the power curves for the test, H_0: $C_{\text{PU}} \leqslant C$, with $C = 1.00$, 1.33, 1.67, and 2.00. The SAS programs used for calculating the critical values and the power are listed below.

Table 2
Critical values c_0 for $C_{PU} = 1.00$ (Panel A), 1.33 (Panel B), 1.67 (Panel C), 2.00 (Panel D) based on multiple samples with $n = 3(1)6$, $m = 5(1)40$, and $\alpha = 0.05, 0.025, 0.01$

m	*n*											
	3			4			5			6		
	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$
Panel A												
5	1.497	1.653	1.866	1.389	1.499	1.645	1.329	1.417	1.531	1.289	1.364	1.460
6	1.443	1.575	1.754	1.348	1.444	1.568	1.295	1.372	1.471	1.261	1.326	1.409
7	1.402	1.519	1.673	1.318	1.403	1.512	1.270	1.339	1.426	1.239	1.298	1.372
8	1.371	1.475	1.612	1.294	1.371	1.469	1.251	1.313	1.392	1.222	1.276	1.342
9	1.346	1.441	1.564	1.275	1.345	1.435	1.235	1.292	1.364	1.208	1.257	1.319
10	1.325	1.412	1.525	1.259	1.324	1.406	1.221	1.275	1.342	1.196	1.242	1.300
11	1.307	1.389	1.493	1.245	1.306	1.383	1.210	1.260	1.322	1.186	1.230	1.283
12	1.292	1.368	1.466	1.234	1.291	1.363	1.200	1.247	1.306	1.177	1.219	1.269
13	1.279	1.351	1.442	1.223	1.278	1.345	1.191	1.236	1.292	1.170	1.209	1.257
14	1.267	1.335	1.422	1.214	1.266	1.330	1.184	1.227	1.279	1.163	1.200	1.264
15	1.257	1.322	1.404	1.206	1.256	1.317	1.177	1.218	1.268	1.157	1.193	1.236
16	1.248	1.310	1.388	1.199	1.246	1.305	1.171	1.210	1.258	1.152	1.186	1.228
17	1.239	1.299	1.373	1.193	1.238	1.294	1.165	1.203	1.249	1.147	1.180	1.220
18	1.232	1.289	1.360	1.187	1.230	1.284	1.160	1.197	1.241	1.143	1.174	1.213
19	1.225	1.280	1.349	1.181	1.223	1.275	1.156	1.191	1.234	1.138	1.169	1.207
20	1.219	1.272	1.338	1.176	1.217	1.267	1.151	1.186	1.227	1.135	1.165	1.201
21	1.213	1.264	1.328	1.172	1.211	1.260	1.147	1.181	1.221	1.131	1.160	1.195
22	1.207	1.257	1.319	1.167	1.206	1.253	1.144	1.176	1.215	1.128	1.156	1.190
23	1.202	1.251	1.310	1.163	1.201	1.246	1.140	1.172	1.210	1.125	1.153	1.186
24	1.198	1.244	1.303	1.160	1.196	1.240	1.137	1.168	1.205	1.122	1.149	1.181
25	1.193	1.239	1.295	1.156	1.191	1.235	1.134	1.164	1.200	1.120	1.146	1.177
26	1.189	1.234	1.289	1.153	1.187	1.230	1.132	1.161	1.196	1.117	1.143	1.174
27	1.185	1.229	1.282	1.150	1.183	1.225	1.129	1.157	1.192	1.115	1.140	1.170
28	1.182	1.224	1.276	1.147	1.180	1.220	1.126	1.154	1.188	1.113	1.137	1.167
29	1.178	1.220	1.271	1.144	1.176	1.216	1.124	1.151	1.184	1.111	1.135	1.163
30	1.175	1.215	1.265	1.141	1.173	1.212	1.122	1.149	1.181	1.109	1.132	1.160
31	1.172	1.211	1.260	1.139	1.170	1.208	1.120	1.146	1.178	1.107	1.130	1.157
32	1.169	1.208	1.255	1.137	1.167	1.204	1.118	1.144	1.174	1.105	1.128	1.155
33	1.166	1.204	1.251	1.134	1.164	1.201	1.116	1.141	1.172	1.103	1.126	1.152
34	1.163	1.201	1.247	1.132	1.162	1.197	1.114	1.139	1.169	1.102	1.124	1.150
35	1.161	1.198	1.243	1.130	1.159	1.194	1.112	1.137	1.166	1.100	1.122	1.147
36	1.158	1.194	1.239	1.128	1.157	1.191	1.111	1.135	1.164	1.099	1.120	1.145
37	1.156	1.192	1.235	1.127	1.154	1.188	1.109	1.133	1.161	1.097	1.118	1.143
38	1.154	1.189	1.231	1.125	1.152	1.185	1.108	1.131	1.159	1.096	1.116	1.141

39	1.152	1.186	1.228	1.123	1.150	1.183	1.106	1.129	1.156	1.095	1.115	1.139
40	1.150	1.183	1.225	1.121	1.148	1.180	1.105	1.127	1.154	1.093	1.113	1.137
Panel B												
5	1.975	2.179	2.458	1.833	1.977	2.166	1.754	1.869	2.017	1.703	1.800	1.924
6	1.905	2.078	2.311	1.781	1.905	2.066	1.711	1.811	1.939	1.666	1.751	1.858
7	1.852	2.004	2.206	1.741	1.851	1.993	1.679	1.768	1.881	1.638	1.714	1.810
8	1.812	1.947	2.126	1.710	1.810	1.937	1.653	1.734	1.836	1.616	1.685	1.772
9	1.779	1.902	2.063	1.685	1.777	1.893	1.632	1.707	1.800	1.598	1.662	1.741
10	1.751	1.865	2.013	1.665	1.749	1.856	1.615	1.684	1.771	1.582	1.642	1.716
11	1.728	1.834	1.970	1.647	1.726	1.825	1.600	1.665	1.746	1.569	1.626	1.695
12	1.709	1.808	1.935	1.632	1.706	1.799	1.588	1.649	1.725	1.558	1.611	1.677
13	1.692	1.785	1.904	1.618	1.689	1.777	1.577	1.635	1.707	1.549	1.599	1.661
14	1.676	1.765	1.877	1.607	1.674	1.757	1.567	1.622	1.690	1.540	1.588	1.647
15	1.663	1.747	1.854	1.596	1.660	1.740	1.558	1.611	1.676	1.532	1.578	1.635
16	1.651	1.732	1.833	1.587	1.648	1.724	1.550	1.601	1.663	1.525	1.570	1.624
17	1.640	1.717	1.814	1.579	1.637	1.710	1.543	1.592	1.652	1.519	1.562	1.614
18	1.630	1.704	1.797	1.571	1.627	1.697	1.536	1.584	1.641	1.513	1.554	1.604
19	1.621	1.693	1.782	1.564	1.618	1.686	1.530	1.576	1.632	1.508	1.548	1.596
20	1.613	1.682	1.768	1.557	1.610	1.675	1.525	1.569	1.623	1.503	1.542	1.589
21	1.606	1.672	1.755	1.551	1.603	1.666	1.520	1.563	1.615	1.499	1.536	1.582
22	1.598	1.663	1.743	1.546	1.596	1.657	1.515	1.557	1.607	1.495	1.531	1.575
23	1.592	1.655	1.732	1.541	1.589	1.648	1.511	1.551	1.601	1.491	1.526	1.569
24	1.586	1.647	1.722	1.536	1.583	1.640	1.507	1.546	1.594	1.487	1.522	1.563
25	1.580	1.639	1.713	1.531	1.577	1.633	1.503	1.541	1.588	1.484	1.518	1.558
26	1.575	1.633	1.704	1.527	1.572	1.627	1.499	1.537	1.582	1.481	1.514	1.553
27	1.570	1.626	1.696	1.523	1.567	1.620	1.496	1.533	1.577	1.478	1.510	1.549
28	1.565	1.620	1.688	1.519	1.562	1.614	1.493	1.529	1.572	1.475	1.506	1.544
29	1.561	1.614	1.681	1.516	1.558	1.609	1.490	1.525	1.568	1.472	1.503	1.540
30	1.556	1.609	1.674	1.513	1.553	1.603	1.487	1.521	1.563	1.470	1.500	1.536
31	1.552	1.604	1.667	1.509	1.549	1.598	1.484	1.518	1.559	1.467	1.497	1.533
32	1.548	1.599	1.661	1.506	1.546	1.593	1.482	1.515	1.555	1.465	1.494	1.529
33	1.545	1.594	1.655	1.503	1.542	1.589	1.479	1.512	1.551	1.463	1.491	1.526
34	1.541	1.590	1.650	1.501	1.539	1.585	1.477	1.509	1.547	1.461	1.489	1.523
35	1.538	1.586	1.644	1.498	1.535	1.580	1.475	1.506	1.544	1.459	1.486	1.520
36	1.535	1.582	1.639	1.496	1.532	1.577	1.473	1.503	1.541	1.457	1.484	1.517
37	1.532	1.578	1.634	1.493	1.529	1.573	1.470	1.501	1.538	1.455	1.482	1.514
38	1.529	1.574	1.630	1.491	1.526	1.569	1.469	1.498	1.534	1.453	1.480	1.511
39	1.526	1.571	1.625	1.489	1.524	1.566	1.467	1.496	1.532	1.452	1.478	1.509
40	1.524	1.567	1.621	1.487	1.521	1.562	1.465	1.494	1.529	1.450	1.476	1.506

Table 2 (*continued*)

m	*n*											
	3			4			5			6		
	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$
Panel C												
5	2.471	2.725	3.072	2.293	2.471	2.707	2.195	2.337	2.521	2.131	2.251	2.404
6	2.383	2.599	2.889	2.228	2.382	2.583	2.142	2.265	2.424	2.085	2.190	2.323
7	2.318	2.507	2.757	2.179	2.316	2.492	2.101	2.212	2.352	2.050	2.145	2.263
8	2.267	2.436	2.658	2.141	2.264	2.422	2.070	2.170	2.297	2.023	2.109	2.216
9	2.226	2.380	2.580	2.110	2.223	2.367	2.044	2.136	2.252	2.000	2.080	2.178
10	2.193	2.334	2.517	2.084	2.189	2.322	2.022	2.108	2.216	1.982	2.056	2.147
11	2.164	2.296	2.465	2.062	2.160	2.284	2.004	2.085	2.185	1.966	2.035	2.121
12	2.140	2.263	2.420	2.043	2.136	2.251	1.988	2.064	2.159	1.952	2.018	2.098
13	2.118	2.234	2.382	2.027	2.114	2.223	1.975	2.047	2.136	1.940	2.002	2.079
14	2.100	2.209	2.349	2.012	2.095	2.199	1.963	2.031	2.116	1.929	1.989	2.062
15	2.083	2.187	2.320	2.000	2.079	2.177	1.952	2.017	2.098	1.920	1.977	2.046
16	2.068	2.168	2.294	1.988	2.064	2.158	1.942	2.005	2.082	1.911	1.966	2.033
17	2.054	2.150	2.271	1.977	2.050	2.140	1.933	1.994	2.068	1.903	1.956	2.020
18	2.042	2.134	2.250	1.968	2.038	2.125	1.925	1.983	2.055	1.896	1.947	2.009
19	2.031	2.120	2.231	1.959	2.027	2.110	1.918	1.974	2.043	1.890	1.939	1.999
20	2.021	2.106	2.213	1.951	2.017	2.097	1.911	1.966	2.032	1.884	1.932	1.989
21	2.011	2.094	2.197	1.944	2.007	2.085	1.905	1.958	2.022	1.878	1.925	1.981
22	2.003	2.083	2.183	1.937	1.999	2.074	1.899	1.950	2.013	1.873	1.918	1.973
23	1.995	2.072	2.169	1.931	1.990	2.064	1.893	1.944	2.004	1.869	1.912	1.965
24	1.987	2.063	2.156	1.925	1.983	2.054	1.888	1.937	1.997	1.864	1.907	1.958
25	1.980	2.054	2.145	1.919	1.976	2.045	1.884	1.931	1.989	1.860	1.902	1.952
26	1.973	2.045	2.134	1.914	1.969	2.037	1.879	1.926	1.982	1.856	1.897	1.946
27	1.967	2.037	2.123	1.909	1.963	2.029	1.875	1.921	1.976	1.852	1.892	1.940
28	1.961	2.029	2.114	1.904	1.957	2.022	1.871	1.916	1.969	1.849	1.888	1.935
29	1.956	2.022	2.105	1.900	1.952	2.015	1.867	1.911	1.964	1.846	1.884	1.930
30	1.950	2.016	2.096	1.896	1.946	2.008	1.864	1.907	1.958	1.843	1.880	1.925
31	1.945	2.009	2.088	1.892	1.941	2.002	1.861	1.902	1.953	1.840	1.876	1.920
32	1.941	2.003	2.080	1.888	1.937	1.996	1.857	1.898	1.948	1.837	1.873	1.916
33	1.936	1.998	2.073	1.884	1.932	1.990	1.854	1.895	1.943	1.834	1.869	1.912
34	1.932	1.992	2.066	1.881	1.928	1.985	1.851	1.891	1.939	1.832	1.866	1.908
35	1.928	1.987	2.059	1.878	1.924	1.980	1.849	1.888	1.934	1.829	1.863	1.904
36	1.924	1.982	2.053	1.875	1.920	1.975	1.846	1.884	1.930	1.827	1.860	1.901
37	1.920	1.977	2.047	1.872	1.916	1.970	1.844	1.881	1.926	1.825	1.858	1.897
38	1.917	1.973	2.041	1.869	1.913	1.966	1.841	1.878	1.923	1.822	1.855	1.894

39	1.913	1.968	2.036	1.866	1.909	1.962	1.839	1.875	1.919	1.820	1.852	1.891
40	1.910	1.964	2.031	1.864	1.906	1.958	1.837	1.872	1.916	1.818	1.850	1.888
Panel D												
5	2.954	3.256	3.670	2.741	2.953	3.234	2.624	2.793	3.011	2.548	2.690	2.873
6	2.849	3.106	3.451	2.663	2.847	3.086	2.560	2.707	2.896	2.493	2.618	2.776
7	2.771	2.996	3.295	2.605	2.768	2.977	2.512	2.644	2.810	2.452	2.564	2.704
8	2.711	2.912	3.177	2.560	2.707	2.895	2.475	2.594	2.744	2.419	2.521	2.649
9	2.662	2.845	3.084	2.523	2.657	2.829	2.444	2.554	2.692	2.392	2.487	2.604
10	2.622	2.790	3.008	2.492	2.617	2.775	2.419	2.521	2.648	2.370	2.458	2.567
11	2.588	2.744	2.946	2.466	2.583	2.729	2.397	2.493	2.612	2.351	2.434	2.536
12	2.559	2.705	2.893	2.444	2.553	2.691	2.378	2.468	2.580	2.335	2.413	2.509
13	2.533	2.671	2.848	2.424	2.528	2.658	2.362	2.447	2.553	2.320	2.395	2.486
14	2.511	2.642	2.808	2.407	2.505	2.628	2.347	2.429	2.529	2.308	2.378	2.465
15	2.491	2.616	2.773	2.392	2.486	2.603	2.334	2.412	2.508	2.296	2.364	2.447
16	2.473	2.592	2.742	2.378	2.468	2.580	2.323	2.398	2.490	2.286	2.351	2.431
17	2.457	2.571	2.715	2.365	2.452	2.559	2.312	2.384	2.473	2.277	2.340	2.416
18	2.443	2.552	2.690	2.354	2.437	2.540	2.303	2.372	2.457	2.269	2.329	2.403
19	2.429	2.535	2.667	2.344	2.424	2.523	2.294	2.361	2.443	2.261	2.319	2.390
20	2.417	2.519	2.646	2.334	2.412	2.508	2.286	2.351	2.430	2.254	2.311	2.379
21	2.406	2.504	2.627	2.325	2.401	2.493	2.279	2.342	2.418	2.247	2.302	2.369
22	2.396	2.491	2.610	2.317	2.390	2.480	2.272	2.333	2.407	2.241	2.295	2.359
23	2.386	2.479	2.594	2.310	2.381	2.468	2.265	2.325	2.397	2.236	2.288	2.351
24	2.377	2.467	2.579	2.302	2.372	2.457	2.259	2.317	2.388	2.230	2.281	2.342
25	2.369	2.456	2.565	2.296	2.363	2.446	2.254	2.310	2.379	2.225	2.275	2.335
26	2.361	2.446	2.552	2.290	2.356	2.436	2.248	2.304	2.371	2.221	2.269	2.327
27	2.353	2.436	2.539	2.284	2.348	2.427	2.244	2.297	2.363	2.216	2.264	2.321
28	2.346	2.428	2.528	2.278	2.341	2.418	2.239	2.292	2.356	2.212	2.259	2.314
29	2.340	2.419	2.517	2.273	2.335	2.410	2.234	2.286	2.349	2.208	2.254	2.308
30	2.333	2.411	2.507	2.268	2.328	2.402	2.230	2.281	2.342	2.205	2.249	2.302
31	2.327	2.404	2.497	2.263	2.323	2.394	2.226	2.276	2.336	2.201	2.245	2.297
32	2.322	2.396	2.488	2.259	2.317	2.387	2.222	2.271	2.330	2.198	2.241	2.292
33	2.316	2.390	2.479	2.255	2.312	2.381	2.219	2.267	2.324	2.195	2.237	2.287
34	2.311	2.383	2.471	2.251	2.307	2.374	2.215	2.262	2.319	2.192	2.233	2.282
35	2.306	2.377	2.463	2.247	2.302	2.368	2.212	2.258	2.314	2.189	2.229	2.278
36	2.302	2.371	2.456	2.243	2.297	2.362	2.209	2.254	2.309	2.186	2.226	2.274
37	2.297	2.365	2.448	2.240	2.293	2.357	2.206	2.251	2.304	2.183	2.223	2.270
38	2.293	2.360	2.442	2.236	2.288	2.352	2.203	2.247	2.300	2.181	2.219	2.266
39	2.289	2.355	2.435	2.233	2.284	2.346	2.200	2.244	2.296	2.178	2.216	2.262
40	2.285	2.350	2.429	2.230	2.280	2.342	2.198	2.240	2.292	2.176	2.214	2.258

Table 3

Power of the test for H_0: $C_{PU} \leqslant 1.00$ given $C_1 = 1.00(0.02)1.70$ (Panel A), $C_{PU} \leqslant 1.33$ given $C_1 = 1.33(0.02)2.03$ (Panel B), $C_{PU} \leqslant 1.67$ given $C_1 = 1.67(0.02)2.37$ (Panel C), $C_{PU} \leqslant 2.00$ given $C_1 = 2.00(0.02)2.70$ (Panel D) based on multiple samples of $n = 5$, $m = 15(5)30$, and $\alpha = 0.050, 0.025, 0.010$

C_1	m											
	15			20			25			30		
	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$
Panel A												
1.00	0.050	0.025	0.010	0.050	0.025	0.010	0.050	0.025	0.010	0.050	0.025	0.010
1.02	0.072	0.037	0.016	0.076	0.040	0.017	0.080	0.042	0.018	0.084	0.045	0.019
1.04	0.100	0.054	0.024	0.111	0.061	0.028	0.122	0.068	0.031	0.132	0.075	0.035
1.06	0.135	0.076	0.035	0.156	0.091	0.043	0.176	0.105	0.051	0.196	0.119	0.059
1.08	0.177	0.105	0.051	0.211	0.129	0.065	0.244	0.153	0.080	0.276	0.178	0.095
1.10	0.227	0.140	0.071	0.276	0.177	0.094	0.323	0.214	0.119	0.369	0.252	0.145
1.12	0.284	0.182	0.097	0.349	0.235	0.132	0.411	0.287	0.170	0.469	0.339	0.210
1.14	0.346	0.231	0.129	0.428	0.301	0.179	0.503	0.370	0.233	0.572	0.436	0.289
1.16	0.412	0.286	0.167	0.510	0.374	0.236	0.595	0.459	0.307	0.669	0.536	0.379
1.18	0.481	0.346	0.212	0.590	0.452	0.300	0.681	0.549	0.389	0.755	0.634	0.475
1.20	0.550	0.411	0.263	0.667	0.532	0.371	0.758	0.636	0.476	0.828	0.723	0.573
1.22	0.617	0.477	0.320	0.737	0.610	0.447	0.824	0.717	0.564	0.884	0.800	0.665
1.24	0.681	0.544	0.380	0.798	0.683	0.524	0.876	0.788	0.648	0.826	0.862	0.749
1.26	0.739	0.610	0.444	0.850	0.750	0.600	0.917	0.846	0.726	0.955	0.909	0.819
1.28	0.792	0.672	0.508	0.892	0.808	0.672	0.946	0.893	0.793	0.974	0.943	0.876
1.30	0.837	0.730	0.572	0.924	0.857	0.737	0.967	0.929	0.850	0.986	0.966	0.919
1.32	0.875	0.781	0.634	0.949	0.896	0.796	0.980	0.954	0.894	0.993	0.981	0.949
1.34	0.906	0.827	0.693	0.966	0.927	0.845	0.989	0.972	0.929	0.996	0.990	0.970
1.36	0.931	0.866	0.747	0.979	0.950	0.886	0.994	0.983	0.954	0.998	0.995	0.983
1.38	0.951	0.898	0.795	0.987	0.967	0.918	0.997	0.990	0.971	0.999	0.997	0.991
1.40	0.966	0.924	0.837	0.992	0.979	0.943	0.998	0.995	0.982	1.000	0.999	0.995
1.42	0.976	0.945	0.873	0.996	0.987	0.961	0.999	0.997	0.990	1.000	0.999	0.998
1.44	0.984	0.960	0.903	0.998	0.992	0.975	1.000	0.999	0.994	1.000	1.000	0.999
1.46	0.990	0.972	0.927	0.999	0.995	0.984	1.000	0.999	0.997	1.000	1.000	0.999
1.48	0.993	0.981	0.946	0.999	0.997	0.990	1.000	1.000	0.998	1.000	1.000	1.000
1.50	0.996	0.987	0.961	1.000	0.999	0.994	1.000	1.000	0.999	1.000	1.000	1.000
1.52	0.998	0.992	0.973	1.000	0.999	0.996	1.000	1.000	1.000	1.000	1.000	1.000
1.54	0.999	0.995	0.981	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000
1.56	0.999	0.997	0.987	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
1.58	1.000	0.998	0.991	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
1.60	1.000	0.999	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.62	1.000	0.999	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.64	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

1.66	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.68	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.70	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Panel B												
1.33	0.050	0.025	0.010	0.050	0.025	0.010	0.050	0.025	0.010	0.050	0.025	0.010
1.35	0.066	0.034	0.014	0.069	0.036	0.015	0.072	0.038	0.016	0.075	0.039	0.017
1.37	0.086	0.046	0.020	0.094	0.051	0.022	0.102	0.056	0.025	0.109	0.060	0.027
1.39	0.110	0.061	0.027	0.125	0.070	0.032	0.138	0.079	0.037	0.151	0.088	0.042
1.41	0.139	0.079	0.037	0.161	0.094	0.045	0.183	0.109	0.054	0.204	0.124	0.063
1.43	0.172	0.101	0.049	0.204	0.124	0.062	0.236	0.147	0.076	0.266	0.170	0.091
1.45	0.209	0.127	0.063	0.253	0.159	0.083	0.295	0.192	0.104	0.336	0.225	0.127
1.47	0.251	0.157	0.081	0.307	0.201	0.110	0.361	0.245	0.140	0.412	0.288	0.172
1.49	0.296	0.191	0.103	0.365	0.248	0.141	0.430	0.304	0.182	0.491	0.359	0.225
1.51	0.345	0.230	0.128	0.427	0.300	0.179	0.502	0.369	0.232	0.570	0.435	0.287
1.53	0.396	0.272	0.157	0.490	0.356	0.221	0.573	0.437	0.288	0.646	0.512	0.356
1.55	0.449	0.317	0.190	0.553	0.416	0.269	0.642	0.507	0.350	0.716	0.589	0.430
1.57	0.502	0.366	0.227	0.614	0.477	0.321	0.706	0.576	0.415	0.779	0.663	0.505
1.59	0.556	0.416	0.267	0.673	0.538	0.377	0.764	0.643	0.483	0.833	0.730	0.580
1.61	0.608	0.468	0.311	0.727	0.599	0.435	0.815	0.706	0.551	0.877	0.790	0.653
1.63	0.658	0.519	0.357	0.777	0.657	0.495	0.858	0.763	0.618	0.912	0.841	0.719
1.65	0.705	0.571	0.405	0.820	0.711	0.554	0.894	0.813	0.680	0.939	0.883	0.779
1.67	0.749	0.621	0.455	0.858	0.761	0.612	0.923	0.856	0.738	0.959	0.916	0.831
1.69	0.789	0.669	0.505	0.890	0.805	0.668	0.945	0.891	0.790	0.974	0.942	0.873
1.71	0.825	0.714	0.554	0.916	0.844	0.720	0.962	0.920	0.835	0.983	0.961	0.908
1.73	0.857	0.756	0.603	0.937	0.877	0.767	0.974	0.942	0.873	0.990	0.974	0.935
1.75	0.884	0.794	0.650	0.954	0.905	0.810	0.983	0.959	0.905	0.994	0.984	0.955
1.77	0.907	0.828	0.695	0.967	0.928	0.847	0.989	0.972	0.930	0.997	0.990	0.970
1.79	0.927	0.859	0.737	0.977	0.946	0.879	0.993	0.981	0.950	0.998	0.994	0.981
1.81	0.943	0.885	0.775	0.984	0.961	0.906	0.996	0.988	0.965	0.999	0.996	0.988
1.83	0.957	0.908	0.810	0.989	0.972	0.928	0.998	0.992	0.976	0.999	0.998	0.993
1.85	0.967	0.927	0.842	0.993	0.980	0.946	0.999	0.995	0.984	1.000	0.999	0.996
1.87	0.976	0.943	0.869	0.995	0.986	0.960	0.999	0.997	0.989	1.000	0.999	0.997
1.89	0.982	0.956	0.893	0.997	0.991	0.971	1.000	0.998	0.993	1.000	1.000	0.999
1.91	0.987	0.966	0.914	0.998	0.994	0.979	1.000	0.999	0.996	1.000	1.000	0.999
1.93	0.991	0.974	0.931	0.999	0.996	0.985	1.000	0.999	0.997	1.000	1.000	1.000
1.95	0.993	0.981	0.946	0.999	0.997	0.990	1.000	1.000	0.998	1.000	1.000	1.000
1.97	0.995	0.986	0.958	1.000	0.998	0.993	1.000	1.000	0.999	1.000	1.000	1.000
1.99	0.997	0.990	0.968	1.000	0.999	0.995	1.000	1.000	0.999	1.000	1.000	1.000
2.01	0.998	0.993	0.975	1.000	0.999	0.997	1.000	1.000	1.000	1.000	1.000	1.000
2.03	0.999	0.995	0.981	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000

Table 3 (*continued*)

C_1	m											
	15			20			25			30		
	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$
Panel C												
1.67	0.050	0.025	0.010	0.050	0.025	0.010	0.050	0.025	0.010	0.050	0.025	0.010
1.69	0.063	0.032	0.013	0.065	0.034	0.014	0.068	0.035	0.015	0.070	0.036	0.015
1.71	0.078	0.041	0.017	0.084	0.045	0.019	0.090	0.048	0.021	0.095	0.051	0.023
1.73	0.096	0.052	0.023	0.107	0.059	0.026	0.116	0.065	0.029	0.126	0.071	0.033
1.75	0.117	0.065	0.029	0.133	0.075	0.035	0.148	0.086	0.040	0.163	0.096	0.046
1.77	0.140	0.080	0.037	0.163	0.095	0.046	0.185	0.111	0.055	0.207	0.126	0.064
1.79	0.167	0.098	0.047	0.198	0.119	0.059	0.228	0.141	0.072	0.257	0.163	0.086
1.81	0.196	0.118	0.058	0.236	0.147	0.076	0.275	0.176	0.094	0.321	0.206	0.114
1.83	0.228	0.141	0.071	0.278	0.178	0.095	0.326	0.216	0.120	0.371	0.254	0.147
1.85	0.263	0.166	0.087	0.323	0.213	0.118	0.380	0.261	0.151	0.434	0.308	0.186
1.87	0.300	0.195	0.105	0.371	0.252	0.144	0.437	0.310	0.187	0.498	0.366	0.231
1.89	0.340	0.226	0.125	0.421	0.294	0.175	0.495	0.362	0.227	0.562	0.427	0.281
1.91	0.381	0.259	0.148	0.471	0.339	0.208	0.553	0.417	0.271	0.624	0.490	0.336
1.93	0.423	0.295	0.174	0.523	0.386	0.245	0.609	0.473	0.319	0.683	0.552	0.394
1.95	0.466	0.333	0.202	0.573	0.435	0.285	0.663	0.530	0.371	0.738	0.614	0.454
1.97	0.509	0.372	0.232	0.622	0.485	0.328	0.714	0.585	0.424	0.787	0.672	0.515
1.99	0.552	0.413	0.265	0.670	0.535	0.374	0.761	0.640	0.479	0.830	0.726	0.576
2.01	0.595	0.454	0.300	0.714	0.584	0.421	0.803	0.691	0.534	0.867	0.776	0.635
2.03	0.636	0.496	0.336	0.756	0.631	0.468	0.840	0.738	0.589	0.898	0.819	0.691
2.05	0.675	0.538	0.374	0.793	0.677	0.517	0.872	0.782	0.641	0.923	0.857	0.742
2.07	0.713	0.580	0.414	0.827	0.720	0.564	0.900	0.821	0.691	0.943	0.889	0.789
2.09	0.748	0.620	0.454	0.857	0.760	0.611	0.922	0.855	0.737	0.959	0.916	0.830
2.11	0.781	0.659	0.494	0.884	0.796	0.656	0.941	0.884	0.780	0.971	0.937	0.865
2.13	0.811	0.696	0.534	0.906	0.829	0.699	0.956	0.909	0.818	0.980	0.954	0.895
2.15	0.838	0.731	0.574	0.925	0.858	0.739	0.967	0.930	0.851	0.986	0.967	0.920
2.17	0.863	0.764	0.613	0.941	0.884	0.776	0.976	0.946	0.880	0.991	0.977	0.940
2.19	0.884	0.795	0.651	0.954	0.906	0.810	0.983	0.960	0.905	0.994	0.984	0.956
2.21	0.904	0.823	0.687	0.965	0.924	0.841	0.988	0.970	0.926	0.996	0.989	0.968
2.23	0.920	0.848	0.722	0.974	0.940	0.867	0.992	0.978	0.943	0.998	0.993	0.977
2.25	0.935	0.871	0.754	0.980	0.953	0.891	0.994	0.985	0.957	0.999	0.995	0.984
2.27	0.947	0.891	0.784	0.985	0.964	0.911	0.996	0.989	0.968	0.999	0.997	0.989
2.29	0.957	0.909	0.812	0.989	0.972	0.929	0.998	0.992	0.976	0.999	0.998	0.993
2.31	0.966	0.924	0.837	0.992	0.979	0.943	0.998	0.995	0.983	1.000	0.999	0.995
2.33	0.973	0.938	0.860	0.995	0.984	0.955	0.999	0.997	0.988	1.000	0.999	0.997
2.35	0.979	0.949	0.881	0.996	0.988	0.965	0.999	0.998	0.991	1.000	1.000	0.998
2.37	0.984	0.959	0.899	0.997	0.992	0.973	1.000	0.999	0.994	1.000	1.000	0.999

Panel D

2.00	0.050	0.025	0.010	0.050	0.025	0.010	0.05	0.025	0.010	0.050	0.025	0.010
2.02	0.061	0.031	0.013	0.063	0.032	0.013	0.065	0.033	0.014	0.066	0.034	0.014
2.04	0.073	0.038	0.016	0.078	0.041	0.017	0.082	0.044	0.019	0.086	0.046	0.020
2.06	0.087	0.047	0.020	0.095	0.052	0.023	0.103	0.056	0.025	0.110	0.061	0.027
2.08	0.103	0.056	0.025	0.116	0.064	0.029	0.128	0.072	0.033	0.139	0.079	0.037
2.10	0.122	0.068	0.031	0.139	0.079	0.037	0.156	0.091	0.043	0.172	0.102	0.050
2.12	0.142	0.081	0.038	0.165	0.097	0.046	0.188	0.112	0.055	0.209	0.128	0.065
2.14	0.164	0.096	0.046	0.194	0.117	0.058	0.223	0.138	0.070	0.251	0.159	0.084
2.16	0.188	0.112	0.055	0.226	0.139	0.071	0.262	0.166	0.088	0.297	0.194	0.106
2.18	0.215	0.131	0.066	0.260	0.165	0.087	0.304	0.199	0.109	0.346	0.233	0.132
2.20	0.243	0.151	0.078	0.297	0.193	0.104	0.348	0.234	0.133	0.398	0.276	0.163
2.22	0.273	0.173	0.092	0.336	0.223	0.125	0.395	0.273	0.160	0.451	0.323	0.197
2.24	0.305	0.198	0.107	0.376	0.257	0.147	0.443	0.315	0.191	0.505	0.372	0.236
2.26	0.338	0.224	0.124	0.418	0.292	0.173	0.492	0.359	0.225	0.559	0.424	0.278
2.28	0.372	0.252	0.143	0.461	0.330	0.201	0.541	0.405	0.262	0.612	0.476	0.324
2.30	0.407	0.282	0.164	0.504	0.369	0.231	0.589	0.452	0.301	0.662	0.529	0.372
2.32	0.443	0.313	0.187	0.547	0.409	0.264	0.635	0.500	0.343	0.710	0.582	0.422
2.34	0.480	0.345	0.211	0.589	0.451	0.299	0.680	0.547	0.387	0.754	0.632	0.473
2.36	0.516	0.379	0.237	0.630	0.493	0.335	0.722	0.594	0.433	0.794	0.681	0.525
2.38	0.552	0.413	0.265	0.670	0.534	0.374	0.761	0.639	0.479	0.830	0.726	0.576
2.40	0.588	0.448	0.294	0.707	0.576	0.413	0.797	0.683	0.525	0.861	0.768	0.626
2.42	0.623	0.483	0.324	0.743	0.616	0.453	0.829	0.724	0.571	0.889	0.806	0.673
2.44	0.657	0.518	0.356	0.776	0.655	0.494	0.858	0.762	0.616	0.912	0.840	0.718
2.46	0.689	0.553	0.388	0.806	0.693	0.534	0.883	0.797	0.659	0.931	0.869	0.760
2.48	0.720	0.588	0.422	0.834	0.728	0.574	0.905	0.828	0.701	0.947	0.895	0.797
2.50	0.750	0.622	0.455	0.859	0.761	0.613	0.923	0.856	0.739	0.960	0.917	0.831
2.52	0.777	0.654	0.489	0.881	0.792	0.651	0.939	0.881	0.775	0.970	0.935	0.861
2.54	0.803	0.686	0.523	0.900	0.820	0.688	0.952	0.903	0.808	0.978	0.950	0.887
2.56	0.827	0.716	0.557	0.918	0.846	0.722	0.963	0.921	0.837	0.984	0.962	0.910
2.58	0.848	0.745	0.590	0.932	0.869	0.755	0.971	0.937	0.864	0.988	0.971	0.929
2.60	0.868	0.772	0.623	0.945	0.889	0.785	0.978	0.950	0.887	0.992	0.979	0.944
2.62	0.886	0.797	0.654	0.956	0.907	0.813	0.984	0.961	0.907	0.994	0.984	0.957
2.64	0.902	0.821	0.685	0.964	0.923	0.839	0.988	0.970	0.925	0.996	0.989	0.967
2.66	0.917	0.842	0.714	0.972	0.937	0.862	0.991	0.977	0.939	0.997	0.992	0.975
2.68	0.929	0.862	0.742	0.978	0.948	0.882	0.994	0.982	0.952	0.998	0.994	0.982
2.70	0.940	0.880	0.768	0.983	0.958	0.901	0.995	0.987	0.962	0.999	0.996	0.987

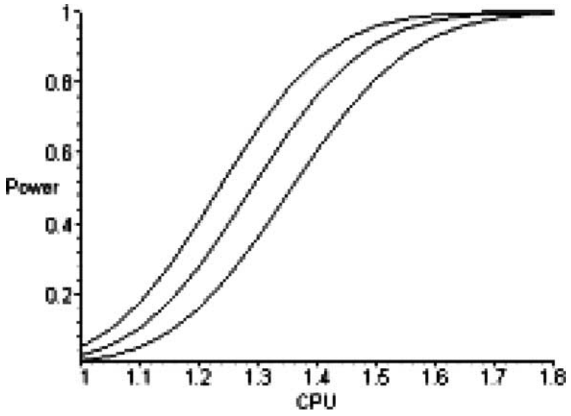

Fig. 1. Power of H_0: $C_{PU} \leqslant 1.00$ vs. H_1: $C_{PU} = 1.00(0.05)2.00$; $m = 10$, $n = 5$ under $\alpha = 0.05, 0.025, 0.01$ (from top to bottom).

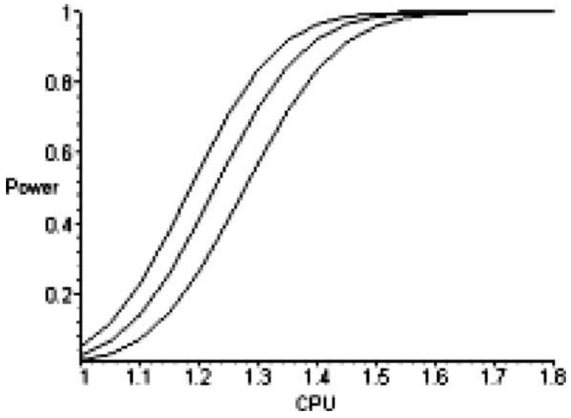

Fig. 2. Power of H_0: $C_{PU} \leqslant 1.00$ vs. H_1: $C_{PU} = 1.00(0.05)2.00$; $m = 15$, $n = 5$ under $\alpha = 0.05, 0.025, 0.01$ (from top to bottom).

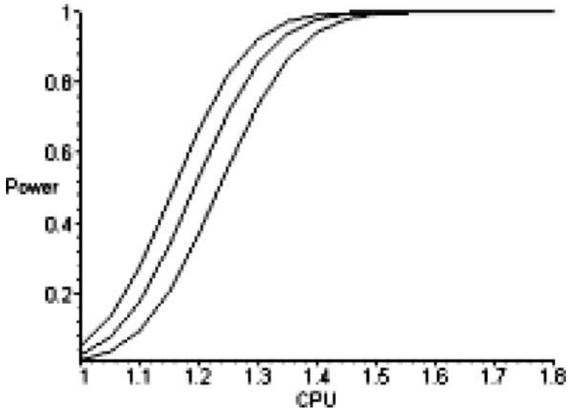

Fig. 3. Power of H_0: $C_{PU} \leqslant 1.00$ vs. H_1: $C_{PU} = 1.00(0.05)2.00$; $m = 20$, $n = 5$ under $\alpha = 0.05, 0.025, 0.01$ (from top to bottom).

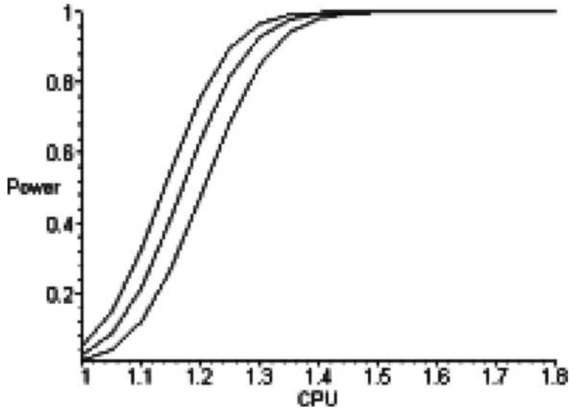

Fig. 4. Power of H_0: $C_{PU} \leqslant 1.00$ vs. H_1: $C_{PU} = 1.00(0.05)2.00$; $m = 25$, $n = 5$ under $\alpha = 0.05, 0.025, 0.01$ (from top to bottom).

SAS program for the critical values

```
/* Critical values of CPU (or CPL) */;
OPTIONS REPLACE PAGESIZE = 58
INESIZE = 78 NODATE;
DATA CPU;
   ALPHA1 = 0.01; ALPHA2 = 0.025;
   ALPHA3 = 0.05;
   N1 = 3; N2 = 4; C0 = 1;
   DO M = 5 TO 50;
   K1 = M * (N1 - 1); K2 = M * (N2 - 1);
   DELTA1 = 3 * (M * N1) * *0.5 * C0;
   DELTA2 = 3 * (M * N2) * *0.5 * C0;
   B1 = (2/K1) * *0.5 * GAMMA(K1/2)/
   GAMMA((K1 - 1)/2);
   B2 = (2/K2) * *0.5 * GAMMA(K2/2)/
   GAMMA((K2 - 1)/2);
   CR11 = (B1/3) * (M * N1) * * (-0.5) * TINV
   (1 - ALPHA1, K1, DELTA1);
   CR12 = (B1/3) * (M* N1) * *(-0.5) * TINV
   (1 - ALPHA2, K1, DELTA1);
   CR13 = (B1/3) * (M * N1) * * (-0.5) * TINV
   (1 - ALPHA3, K1, DELTA1);
   CR21 = (B2/3) * (M* N2) * *(-0.5) * TINV
   (1 - ALPHA1, K2, DELTA2);
   CR22 = (B2/3) * (M * N2) * *(-0.5) * TINV
   (1 -ALPHA2, K2, DELTA2);
   CR23 = (B2/3) * (M * N2) * *(-0.5) * TINV
   (1 -ALPHA3, K2, DELTA2);
OUTPUT;
END;
FORMAT CR11 CR12 CR13 CR21 CR22 CR23 5. 3;
PROC PRINT DATA = CPU;
VAR M CR11 CR12 CR13 CR21 CR22 CR23;
RUN;
```

SAS program for the power

```
/* Power of CPU or CPL */;
OPTIONS REPLACE PAGESIZE = 58 LINESIZE =
78 NODATE;
DATA P0;
  ALPHA1 = 0.05; ALPHA2 = 0.025; ALPHA3 =
  0.01;
  N = 5; C0 = 1;
  M1 = 15; M2 = 20;
  DO C1 = 1.00 BY 0. 02 TO 1. 70;
  K1 = M1 ∗ (N − 1); K2 = M2 ∗ (N − 1);
  DELTA10 = 3 ∗ (M1 ∗ N) ∗ ∗0.5 ∗ C0;
  DELTA20 = 3 ∗ (M2 ∗ N) ∗ ∗0.5 ∗ C0;
  DELTA11 = 3 ∗ (M1 ∗ N) ∗ ∗0.5 ∗ C1;
  DELTA21 = 3 ∗ (M2 ∗ N) ∗ ∗0.5 ∗ C1;
  D1 = TINV(1 − ALPHA1, K1, DELTA10);
  D2 = TINV(1 − ALPHA2, K1, DELTA10);
  D3 = TINV(1 − ALPHA3, K1, DELTA10);
  D4 = TINV(1 − ALPHA1, K2, DELTA20);
  D5 = TINV(1 − ALPHA2, K2, DELTA20);
  D6 = TINV(1 − ALPHA3, K2, DELTA20);
  OC1 = PROBT(D1, K1, DELTA11);
  OC2 = PROBT(D2, K1, DELTA11);
  OC3 = PROBT(D3, K1, DELTA11);
  OC4 = PROBT(D4, K2, DELTA21);
  OC5 = PROBT(D5, K2, DELTA21);
  OC6 = PROBT(D6, K2, DELTA21);
  P01 = 1 − OC1;      P02 = 1 − OC2;      P03 =
  1 − OC3;
  P04 = 1 − OC4;      P05 = 1 − OC5;      P06 =
  1 − OC6;
OUTPUT;
END;
FORMAT P01 P02 P03 P04 P05 P06 5. 3;
PROC PRINT DATA = P0;
VAR C1 P01 P02 P03 P04 P05 P06;
RUN;
```

3.1. Test procedure

To determine if the process meets the capability (quality) requirement, we first determine the capability requirement C, the α-risk, and the sample size. From Table 2 (Panels A–D), or run the computer program, we can find the critical value c_0 based on the α-risk, C, and m sub-samples of size n. If the estimated value $\widetilde{C}_{\mathrm{I}}^*$ is greater than the critical value c_0 then we conclude that the process meets the capability requirement. Otherwise, we do not have sufficient information to conclude that the process meets the preset capability requirement. In this case, we would tend to believe that the process is incapable. In the following, we develop a simple step-by-step procedure for the practitioners to use for their in-plant applications to obtain reliable decisions.

STEP 1: Determine the value of the capability requirement C (normally chosen from those in Table 1), the desired quality condition, and the α-risk (type-I error, normally set to 0.05, 0.025, or 0.01), the chance of incorrectly concluding a bad process (does not meet the preset capability requirement) as good one (meets the preset capability requirement).

STEP 2: Calculate the value of the estimator, $\widetilde{C}_{\mathrm{I}}^*$, from the sample.

STEP 3: Check the appropriate entry listed in Table 2 (Panels A–D) to find the corresponding critical value, c_0, based on α, C and m sub-samples of size n.

STEP 4: Conclude that the process meets the capability requirement if $\widetilde{C}_{\mathrm{I}}^*$ is greater than c_0. Otherwise, we do not have enough information to conclude that the process meets the capability requirement. In this case, we would conclude that the process is incapable.

4. An application example

The positive linear series pass voltage regulators are tailored for low-drop-out applications where low quiescent power is considered important. These regulators include the reverse voltage sensing that prevents current in the reverse direction. The regulators are fabricated with the BiCMOS technology, which is ideally suited for the low input-to-output differential applications. These products are specifically designed to provide well-regulated supply for low IC applications such as high-speed bus termination, low current logic supply, and VGA cards.

The products investigated here are low-drop-out 3 A linear regulators with a low dropout voltage and short-circuit protection. The 3-lead packages have preset outputs at 3.3 or 5.0 V. The output voltage is regulated to 1.5% at room temperature. The 5-lead packages regulate the output voltage programmed by an external resistor ratio. Short-circuit current is internally limited. The device responds a sustained over-current condition by turning off after a t_{ON} time delay. The device then stay off for a period, t_{OFF}, that is 32 times the t_{ON} delay. The device then begins pulsing on and off at the $t_{\mathrm{ON}}/(t_{\mathrm{ON}} + t_{\mathrm{OFF}})$ duty cycle of 3%. This drastically reduces the power dissipation during the short-circuited, which means that the heat sinks need only accommodating the normal operation.

The quiescent current is an essential product characteristic, which has significant impact to product quality. For the quiescent current of a particular model of low-drop-out 3 A linear regulators, the upper specification limit, USL, is set to 650 μA. The capability requirement for this particular model was preset to

Table 4
Sample data with twenty groups of size five

Sample no.	Observations in sample of size five				
1	637	643	638	639	639
2	637	639	647	641	637
3	639	641	641	639	641
4	634	637	640	640	638
5	640	640	638	640	639
6	640	640	641	638	639
7	639	641	641	643	643
8	641	640	638	643	640
9	643	634	639	639	643
10	642	642	639	642	642
11	639	641	639	638	642
12	640	641	640	642	642
13	636	640	638	638	638
14	640	641	635	636	638
15	642	636	638	640	639
16	638	636	643	640	641
17	643	637	643	641	639
18	637	639	640	639	639
19	639	641	639	642	638
20	640	639	639	638	641

satisfactory ($C_{PL} > 1.33$). Twenty samples of size five are collected from a stable process (under statistical control), which is displayed in Table 4.

In order to obtain the critical values, we first calculate the overall sample mean $\overline{\overline{X}} = 639.660$, and the pooled sample variance $S_p^2 = 4.505$. The estimator $\widetilde{C}^*_{PU} = b_{m(n-1)}(\mathrm{USL} - \overline{\overline{X}})/(3S_p) = 1.609$. With type I error α-risk set to 0.05, we find the critical value $c_0 = 1.506$ from Table 2 (Panel C) based on $C = 1.33$, $\alpha = 0.05$, and $n = 5$, $m = 20$. Since $\widetilde{C}^*_{PU} = 1.609$ is greater than the critical value $c_0 = 1.506$ in this case, we therefore conclude that the process meets the capability requirement "*satisfactory* ".

5. Conclusions

Process capability indices C_{PU} and C_{PL} have been widely used in the manufacturing industry to provide quantitative measures on process potential and performance, for processes with one-sided specification limits. Statistical properties of the estimators of C_{PU} and C_{PL} have been investigated extensively, but only restricted to cases with single samples. In this paper, we considered the estimation and capability testing of C_{PU} and C_{PL} based on multiple samples taken from an in-control process. We showed that the proposed estimators of C_{PU} and C_{PL} are the UMVUEs. A simple but practical procedure based on a hypothesis testing using the proposed UMVUE is developed. The engineers/practitioners can use the proposed procedure to determine whether a stable process meets the preset capability requirement, and make reliable decisions. We also presented an application example on the low-drop-out regulators, to illustrate how one may apply the proposed procedure to the actual data collected from real-world applications for reliable capability testing.

Acknowledgements

The authors would like to thank the anonymous referees for their helpful comments and suggestions, which significantly improved the paper.

References

[1] Kane VE. Process capability indices. J Qual Technol 1986; 18:41–52.

[2] Pearn WL, Lin GH, Chen KS. Distributional and inferential properties of the process accuracy and process precision indices. Commun Statist—Theor Meth 1998;27:985–1000.

[3] Chou YM, Owen DB. On the distributions of the estimated process capability indices. Commun Statist—Theor Meth 1989;18:4549–60.

[4] Pearn WL, Chen KS. One-sided capability indices C_{PU} and C_{PL}: decision making with sample information. Int J Qual Reliab Mgmt 2002;19:221–45.

[5] Lin PC, Pearn WL. Testing process capability for one-sided specification limit with application to the voltage level translator. Microelectron Reliab 2002;42(12):1975–83.

[6] Pearn WL, Shu MH. An algorithm for calculating the lower confidence bounds of C_{PU} and C_{PL} with application to low-drop-out regulators. Microelectron Reliab, in press.

[7] Kirmani S, Kocherlakota K, Kocherlakota S. Estimation of σ and the process capability index based on subsamples. Commun Statist—Theor Meth 1991;20:275–91.