Journal of Combinatorial Optimization, 7, 79–86, 2003

Any Maximal Planar Graph with Only One Separating Triangle is Hamiltonian*

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Received December 27, 2000; Revised December 27, 2000; Accepted December 28, 2000

Abstract. A graph is hamiltonian if it has a hamiltonian cycle. It is well-known that Tutte proved that any 4-connected planar graph is hamiltonian. It is also well-known that the problem of determining whether a 3-connected planar graph is hamiltonian is NP-complete. In particular, Chvátal and Wigderson had independently shown that the problem of determining whether a maximal planar graph is hamiltonian is NP-complete. A classical theorem of Whitney says that any maximal planar graph with no separating triangles is hamiltonian, where a separating triangle is a triangle whose removal separates the graph. Note that if a planar graph has separating triangles, then it can not be 4-connected and therefore Tutte's result can not be applied. In this paper, we shall prove that any maximal planar graph with only one separating triangle is still hamiltonian.

Keywords: planar graph, maximal planar graph, hamiltonian cycle, separating triangle, NP-complete

1. Introduction

Our terminology and notation in graphs are standard; see Chartrand and Lensniak (1981) and West (1996), except as indicated. Graphs discussed in this paper are assumed simple and finite. A graph is *hamiltonian* if it has a hamiltonian cycle. A graph is *planar* if it can be drawn in the plane with no two edges crossing. A *plane graph* is a graph drawn in the plane with no two edges crossing. Unless otherwise specified, a planar graph means the plane embedding of the graph. A planar graph divides the plane into regions, which are called *faces*. The unbounded region is called the *exterior face*; the other faces are called *interior faces*. An edge is a *boundary edge* if it is on the exterior face. A graph is *maximal planar* if it is planar and no edge can be added without losing planarity. Note that any face of a maximal planar graph is a triangle. A *triangulation* is a 2-connected planar graph in which all faces (except possibly the exterior face) are triangles. A triangle of a planar graph is a *separating triangle* if it does not form the boundary of a face. That is, a separating triangle has vertices both inside it and outside it; therefore its removal separates the graph. For example, the planar graph in figure 1 has three separating triangles: *ACE*, *BDC*, and *FBA*.

It is well-known that the problem of determining whether a 3-connected planar graph is hamiltonian is NP-complete (Garey et al., 1976). In particular, Chvátal and Wigderson had independently shown that the problem of determining whether a maximal planar graph is

*This research was partially supported by the National Science Council of the Republic of China under grant no. NSC89-2115-M009-026.

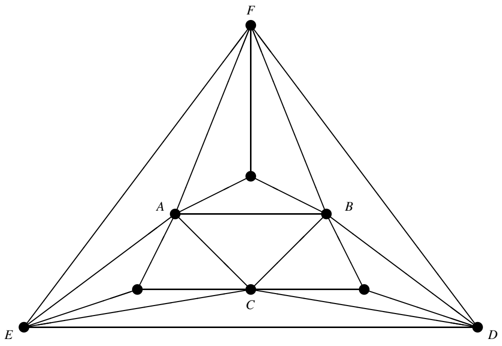


Figure 1. An illustration of separating triangles.

hamiltonian is NP-complete (Chvátal, 1985). On the other hand, Whitney (1931) proved that any maximal planar graph with no separating triangles is hamiltonian; Asano et al. (1984) proposed a linear-time algorithm for finding hamiltonian cycles in such planar graphs. Tutte (1956) proved that any 4-connected planar graph is hamiltonian; Chiba and Nishizeki (1989) proposed a linear-time algorithm for finding hamiltonian cycles in these planar graphs. Note that Tutte's result can be viewed as a generalization of Whitney's result since any maximal planar graph with at least five vertices and with no separating triangles is 4-connected (we prove this in Section 4). Dillencourt (1990) also generalized Whitney's result, but in a different way: he relaxed the requirement that the triangulation must be maximal planar but still insisted that the triangulation has no separating triangles.

All the results of Whitney (1931), Tutte (1956), and Dillencourt (1990) insist that the given planar graph has no separating triangles. Note that if a planar graph has separating triangles, then it can not be 4-connected and therefore Tutte's result can not be applied. In this paper, we shall prove that any maximal planar graph with only one separating triangle is still hamiltonian. Note that there exist non-hamiltonian maximal planar graphs with 7 separating triangles.

2. Preliminaries

Since Whitney's result (Whitney, 1931) plays a crucial role in (Asano et al., 1984; Chiba and Nishizeki, 1989; Dillencourt, 1990) and this paper, we state this result first. Let G be a triangulation, let R be the exterior face of G, and let A and B be two vertices on R. We say that (G, R, A, B) satisfies *Whitney's condition* (*Condition* (W) for short) if (G, R, A, B) satisfies Conditions (W1) and (W2) described below. We say that (G, R, A, B) satisfies *Condition* (*W1*) if G has no separating triangles. We say that (G, R, A, B) satisfies *Condition* (*W2*) if either

(W2a) $a_0, a_1, \ldots, a_m$ is the path from A to B and $b_0, b_1, \ldots, b_n$ is the path from B to A ($a_0 = b_n = A$, $b_0 = a_m = B$), and there is no chord (a chord is an edge joining two non-consecutive vertices of a cycle) of the form $a_i a_j$ or $b_i b_j$, or

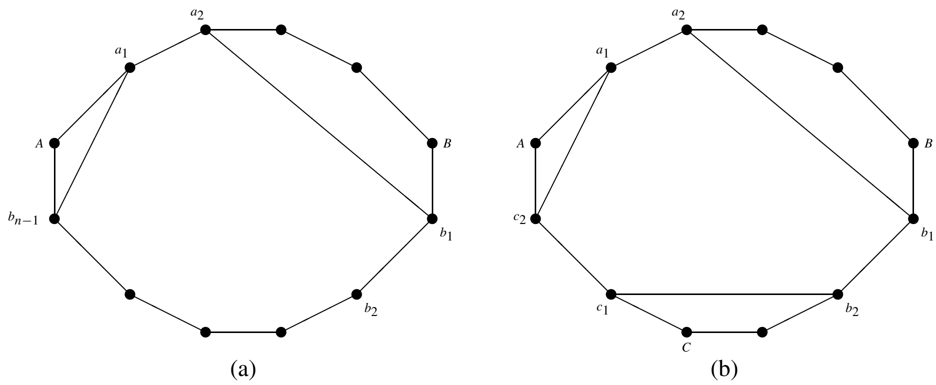


Figure 2. An illustration of Conditions (W2a) and (W2b). Note that only vertices on the exterior faces are shown.

(W2b) $a_0, a_1, \ldots, a_m$ is the path from A to B, $b_0, b_1, \ldots, b_n$ is the path from B to C, $c_0, c_1, \ldots, c_k$ is the path from C to A for some vertex C on R ($a_0 = c_k = A$, $b_0 = a_m = B$, $c_0 = b_n = C$), and there is no chord of the form $a_i a_j$, $b_i b_j$, or $c_i c_j$.

For example, in figure 2(a), the chords of the exterior face are $a_1 b_{n-1}$ and $a_2 b_1$; since there is no chord of the form $a_i a_j$ or $b_i b_j$, (G, R, A, B) satisfies Condition (W2a). In figure 2(b), the chords of the exterior face are $a_1 c_2$, $a_2 b_1$, and $b_2 c_1$; since there is no chord of the form $a_i a_j$, $b_i b_j$, or $c_i c_j$, (G, R, A, B) satisfies Condition (W2b).

Whitney (1931) proved that

Lemma 1 (*Whitney's Lemma*). *Let G be a triangulation, let R be the exterior face of G, and let A and B be two vertices on R. If (G, R, A, B) satisfies Condition (W), then G has a hamiltonian path from A to B.*

Theorem 2 (*Whitney's Theorem*). *If G is a maximal planar graph with no separating triangles, then G is hamiltonian.*

The following lemma will be used heavily in this paper.

Lemma 3. *Let G be a maximal planar graph with no separating triangles, let R be the exterior face of G, and let A and B be two vertices on R. Then (G, R, A, B) satisfies Condition (W).*

Proof: Since G has no separating triangles, (G, R, A, B) satisfies Condition (W1). Since R is a triangle, it has no chords; therefore (G, R, A, B) satisfies Condition (W2). Thus, (G, R, A, B) satisfies Condition (W). $\square$

3. What we are afraid of?

Let G be a maximal planar graph with only one separating triangle ABC. Let G_{in} (G_{out}) be the subgraph of G derived by deleting all the vertices outside (inside) the separating triangle ABC. Then, both G_{in} and G_{out} are maximal planar graphs with no separating triangles.

Consider G_{in}. A, B, C form the exterior face, say R, of G_{in}. By Lemma 3, (G_{in}, R, A, B) satisfies Condition (W). By Whitney's Lemma, G_{in} has a hamiltonian path P_{in} from A to B. Then

$$P_{in} = A, P_{in}(A, C), C, P_{in}(C, B), B,$$

where $P_{in}(A, C)$ ($P_{in}(C, B)$) is the subpath of P_{in} between A and C (C and B).

Next consider G_{out}. A, B, C form an interior face, say R', of G_{out}. Note that a plane graph can always be embedded in the plane so that a given face of the graph becomes the exterior face; see Nishizeki and Chiba (1988). Therefore, we can embed G_{out} in the plane so that R' becomes the exterior face of G_{out}. By Lemma 3, (G_{out}, R', B, C) satisfies Condition (W). By Whitney's Lemma, G_{out} has a hamiltonian path P_{out} from B to C. Then

$$P_{out} = B, P_{out}(B, A), A, P_{out}(A, C), C,$$

where $P_{out}(B, A)$ ($P_{out}(A, C)$) is the subpath of P_{out} between B and A (A and C).

If $P_{out}(A, C) = \emptyset$, then

$$A, P_{in}(A, C), C, P_{in}(C, B), B, P_{out}(B, A), A$$

is a hamiltonian cycle of G. Similarly, if $P_{in}(A, C) = \emptyset$, then

$$A, P_{out}(A, C), C, P_{in}(C, B), B, P_{out}(B, A), A$$

is a hamiltonian cycle of G. What we are afraid of is that

$$P_{in}(A, C) \neq \emptyset \quad \text{and} \quad P_{out}(A, C) \neq \emptyset.$$

Then it is impossible to use P_{in} and P_{out} to derive a hamiltonian cycle of G.

4. The main result

We first prove a lemma mentioned in the introduction. A *vertex cut* of a connected graph $G = (V, E)$ is a subset V' of V such that its removal disconnects G. A *minimal vertex cut* is a vertex cut such that no proper subset of it is also a vertex cut.

Lemma 4. *Any maximal planar graph with at least five vertices and with no separating triangles is* 4-*connected.*

Proof: Whitney (1932) proved that a maximal planar graph G with at least four vertices is 3-connected. Hakimi and Schmeichel (1978) proved that if V' is a minimal vertex cut of a maximal planar graph G, then the subgraph of G induced by V' is a cycle without chords. (Recall that a chord is an edge joining two non-consecutive vertices of a cycle.) Let V' be a minimal vertex cut of G. Since G is 3-connected, $|V'| \geq 3$. Suppose $|V'| = 3$. Then by Hakimi and Schmeichel (1978), the subgraph of G induced by V' is a triangle; this contradicts the assumption that G has no separating triangles. Therefore, $|V'| \geq 4$. This proves that G is 4-connected. □

We now prove a variation of Whitney's Lemma.

Lemma 5. *Let G be a maximal planar graph with no separating triangles, let R be the exterior face of G, and let A, B, C be the three vertices on R. Then G has a hamiltonian path from A to B passing through the edge AC.*

Proof: If G has only three vertices, then this lemma is clearly true. In the following, assume that G has at least four vertices. Let $P_0, P_1, \ldots, P_r$ ($P_0 = B$, $P_r = C$) be the sequence of vertices adjacent to A such that each AP_i is the immediate counter-clockwise edge of AP_{i-1} around A; see figure 3. Note that $P_i P_{i+1}$ is an edge of G for all i, $0 \leq i \leq r-1$.

Since G has no separating triangles, the following three properties hold:

1. G has no edge of the form $P_i P_j$ ($0 \leq i < i+2 \leq j \leq r$), since otherwise $P_i P_j A$ is a separating triangle.
2. G has no edge of the form BP_i ($2 \leq i \leq r-1$), since otherwise $BP_i A$ is a separating triangle.
3. G has no edge of the form CP_i ($1 \leq i \leq r-2$), since otherwise $P_i CA$ is a separating triangle.

See figure 3. Let G' be the subgraph of G derived by deleting A. Then G' is a triangulation. Let R' be the exterior face of G'. Then $R' = B, P_1, \ldots, P_{r-1}, C$. Since G has no separating triangles, G' also has no separating triangles; thus (G', R', C, B) satisfies Condition (W1). By (1)–(3), R' has no chords and therefore (G', R', C, B) satisfies Condition (W2). Thus, (G', R', C, B) satisfies Condition (W). By Lemma 1, G' has a hamiltonian path P from C

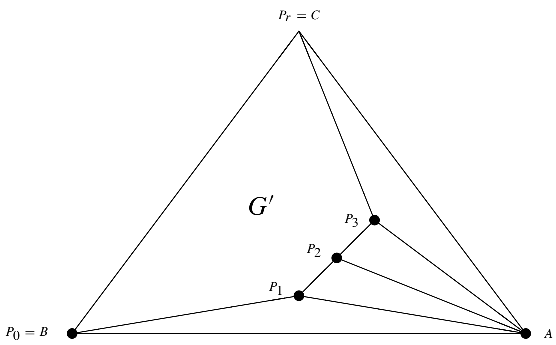
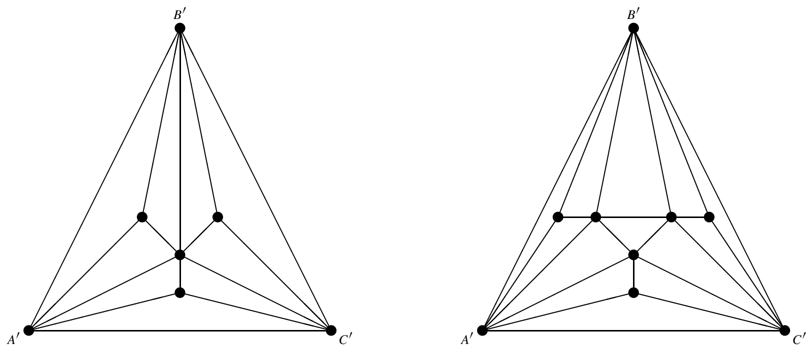


Figure 3. An illustration of the proof of Lemma 5.

to B. The path P together with the edges AC form a hamiltonian path of G from A to B passing through the edge AC. □

We classify maximal planar graphs here. Let G be a maximal planar graph. Note that unless G is a triangle, then it is impossible for G to have a hamiltonian cycle passing through all of its three boundary edges. We say that G is *hamiltonian* (*non-hamiltonian*) *for any two boundary edges* if for any two boundary edges, G has (does not have) a hamiltonian cycle passing through them. We say that G is *hamiltonian* (*non-hamiltonian*) *for any boundary edge* if for any boundary edge, G has (does not have) a hamiltonian cycle passing through it. A K_4 (the complete graph with four vertices) is hamiltonian for any two boundary edges. Note that not every hamiltonian maximal planar graph is hamiltonian for any two boundary edges. For example, the graph in figure 1 has a hamiltonian cycle passing through both *DF* and *EF*, but it does not have a hamiltonian cycle passing through both *DE* and *DF* (or *DE* and *EF*). The two graphs in figure 4 are even worse: although they are hamiltonian, they are non-hamiltonian for any two boundary edges.

What makes a maximal planar graph with separating triangles non-hamiltonian? We introduce the definitions of "hamiltonian for any two boundary edges", "non-hamiltonian for any two boundary edges", "hamiltonian for any boundary edge", and "non-hamiltonian for any boundary edge" because we suspect they affect the hamiltonicity of maximal planar graphs with separating triangles. Suppose *ABC* is a separating triangle of a maximal planar graph G. Then, the graph G_{in} derived by deleting all the vertices outside the separating triangle *ABC* is still a maximal planar graph. If G_{in} is "non-hamiltonian for any boundary edge," then G is certainly non-hamiltonian. "Non-hamiltonian for any two boundary edges" maximal planar graphs had been used as building blocks for constructing non-hamiltonian maximal planar graphs. For example, in Dillencourt (1996), Dillencourt used the two graphs in figure 4 as building blocks to construct non-hamiltonian maximal planar graphs. (This approach was also used in Nishizeki (1980).) In Dillencourt (1996), A', B', C' in figure 4(a) (or figure 4(b)) were identified with A, B, C in figure 1 (that is, the interior face *ABC* of figure 1 was replaced with figure 4(a) (or figure 4(b))); the resultant graph



(a) (b)

Figure 4. Two "non-hamiltonian for any two boundary edges" maximal planar graphs.

is a non-hamiltonian maximal planar graph. It is not difficult to verify that the above two non-hamiltonian maximal planar graphs each has 7 separating triangles.

We now prove a stronger version of Whitney's Theorem.

Theorem 6. *If G is a maximal planar graph with no separating triangles, then G is hamiltonian for any two boundary edges.*

Proof: Let R be the exterior face of G and let A, B, C be the three vertices on R. By Lemma 5, G has a hamiltonian path P from A to B with AC being an edge of it. The path P together with the edge BA form a hamiltonian cycle of G passing through both AB and AC. By similar arguments, G has a hamiltonian cycle passing through both BA and CB and a hamiltonian cycle passing through both CB and BA. Thus, G is hamiltonian for any two boundary edges. □

We now prove the main result.

Theorem 7. *If G is a maximal planar graph with only one separating triangle, then G is hamiltonian.*

Proof: Let A, B, C be the vertices form this unique separating triangle. Let G_{in} (G_{out}) be the subgraph of G derived by deleting all the vertices outside (inside) the separating triangle ABC. Then, both G_{in} and G_{out} are maximal planar graphs with no separating triangles. Consider G_{in}. A, B, C form the exterior face of G_{in}. By Theorem 6, G_{in} has a hamiltonian cycle C_{in} passing through both BA and AC. Then

$$C_{in} = B, A, C, P_{in}(C, B), B,$$

where $P_{in}(C, B)$ is the subpath of C_{in} between C and B. Consider G_{out}. A, B, C form an interior face, say R', of G_{out}. Embed G_{out} in the plane so that R' becomes the exterior face of G_{out}. By Theorem 6, G_{out} has a hamiltonian cycle C_{out} passing through both AC and CB. Then

$$C_{out} = A, C, B, P_{out}(B, A), A,$$

where $P_{out}(B, A)$ is the subpath of C_{out} between B and A. Then

$$A, C, P_{in}(C, B), B, P_{out}(B, A), A$$

a hamiltonian cycle of G. □

References

T. Asano, S. Kikuchi, and N. Saito, "A linear algorithm for finding hamiltonian cycles in 4-connected maximal planar graphs," *Disc. Appl. Math.*, vol. 7, pp. 1–15, 1984.

G. Chartrand and L. Lensniak, *Graph and Digraphs*, Wadsworth: Monterey, CA, 1981.

N. Chiba and T. Nishizeki, "The Hamilton cycle problem is linear-time solvable for 4-connected planar graphs," *J. Algorithms*, vol. 10, pp. 187–211, 1989.

V. Chvátal, "Hamiltonian cycles," *The Traveling Salesman Problem*, John Wiley & Sons: NY, 1985, p. 426.

M.B. Dillencourt, "Hamiltonian cycles in planar triangulations with no separating triangles," *J. Graph Theory*, vol. 14, pp. 31–49, 1990.

M.B. Dillencourt, "Polyhedra of small order and their hamiltonian properties," *J. Comb. Theory, Series B*, vol. 66, pp. 87–122, 1996.

M.R. Garey, D.S. Johnson, and R.E. Tarjan, "The planar Hamiltonian circuit problem is NP-complete," *SIAM J. Comput*., vol. 5, pp. 704–714, 1976.

S.L. Hakimi and E.F. Schmeichel, "On the connectivity of maximal planar graphs," *J. Graph Theory*, vol. 2, pp. 307–314, 1978.

T. Nishizeki, "A 1-tought nonhamiltonian maximal planar graph," *Disc. Math.*, vol. 30, pp. 305–307, 1980.

T. Nishizeki and N. Chiba, *Planar Graphs: Theory and Algorithms*, North-Holland Mathematics Studies 140, *Annals of Discrete Mathematics*, vol. 32, p. 7, 1988.

W.T. Tutte, "A theorem on planar graphs," *Trans. Amer. Math. Soc.*, vol. 82, pp. 99–116, 1956.

D.B. West, *Introduction to Graph Theory*, Prentice Hall: Upper Saddle River, NJ, 1996.

H. Whitney, "A theorem on graphs," *Ann. Math.*, vol. 32, pp. 378–390, 1931.

H. Whitney, "Congruent graphs and the connectivity of graphs," *Amer. J. Math.*, vol. 54, pp. 150–168, 1932.