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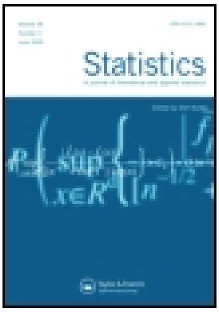

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CAPABILITY MEASURES FOR m-DEPENDENT STATIONARY PROCESSES

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CAPABILITY MEASURES FOR *m*-DEPENDENT STATIONARY PROCESSES

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Process capability indices, providing numerical measures on process potential and process performance, have received substantial research attention. Most research assumes that the process is normally distributed and the process data are independent. In real-world applications such as chemical, soft drinks, or tobacco/cigarette manufacturing processes, process data are often auto-correlated. In this paper, we consider the capability indices C_p, C_{pk}, C_{pm}, C_{pmk} for strictly *m*-dependent stationary processes. We investigate the statistical properties of their natural estimators. We derive the asymptotic distributions, and establish confidence intervals so that capability testing can be performed.

Keywords: Process capability index; Auto-correlated process; Asymptotic distribution; Strictly *m*-dependent stationary process

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1 INTRODUCTION

Process capability indices, providing numerical measures on whether a production process is capable of reproducing items meeting the quality requirements preset by the designers, have received substantial research attention. Those indices have been widely used in supplier selections, as well as in applications of statistical process control to continuously improve the quality and productivity. Examples include Sullivan (1984), Kane (1986), Chan *et al.* (1988), Chou and Owen (1990), Spiring (1991), Rodriguez (1992), Pearn *et al.* (1992), Vännman (1995), and many others. Most research assumes that the process is normally distributed and the process data are independent.

In real-world applications, however, process data are auto-correlated in many cases, particularly, for continuous manufacturing process such as chemical, soft drinks, or tobacco/cigarette manufacturing processes. Shore (1997), and Chow *et al.* (1999) investigated various effects that ignoring auto-correlation may have on estimating the process mean and process standard deviation. Zhang (1996; 1998) considered the estimator of C_p and C_{pk} for processes with auto-correlated data. Under the assumption that the process is discrete, stationary Gaussian, Zhang (1998) used the method of statistical differentials to obtain the approximate

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expectation and variance of the two estimators $\hat{C}_p$ and $\hat{C}_{pk}$ in terms of the auto-correlation function.

In this paper, we consider the four basic indices C_p, C_{pk}, C_{pm} and C_{pmk} for strictly m-dependent stationary processes, where each observation is correlated with its preceding and exceeding m sample data. We investigate the statistical properties of the natural estimators of the four indices, and derive their asymptotic distributions. Consequently, approximate interval estimation and capability testing can be performed for strictly m-dependent stationary processes, particularly, for those with normal and near-normal distributions. We also study the small sample performances of normal approximation and Edgeworth expansion via simulation.

2 THE STRICTLY *m*-DEPENDENT STATIONARY PROCESS

For continuous manufacturing processes, such as chemical, soft drinks, or tobacco/cigaratte manufacturing processes, if the process is under stable condition (under statistical control) and the sampling interval is not too small, then the sampled data can be regarded as taken from a stationary process (see Zhang, 1998). In this section, we consider a strictly m-dependent stationary process $\{X_n\}$ with common mean $EX_i = \mu$ and auto-covariances $E(X_i - \mu)(X_j - \mu)$, which we denote as $\mu_2(i-j)$. From now on, we will assume that joint density function f_{ij} of X_i and X_j satisfies $f_{ij}(a,b) = f_{ij}(b,a)$. Therefore, we have $\mu_2(i-j) = \mu_2(j-i)$ for all $|i-j| \leq m$ and $\mu_2(|i-j|) = 0$ when $|i-j| > m$. It is clear that

$$(X_i^2 - (\mu_2(0) + \mu^2))(X_j - \mu)$$

and

$$(X_i - \mu)(X_j^2 - (\mu_2(0) + \mu^2))$$

have common expectation,

$$E[(X_i^2 - (\mu_2(0) + \mu^2))(X_j - \mu)] = E[(X_i - \mu)(X_j^2 - (\mu_2(0) + \mu^2))]$$

which we denote as $\mu_3(i-j) = \mu_3(j-i)$ with $\mu_3(|i-j|) = 0$ for $|i-j| > m$. Similarly,

$$(X_i^2 - (\mu_2(0) + \mu^2))(X_j^2 - (\mu_2(0) + \mu^2))$$

and

$$(X_j^2 - (\mu_2(0) + \mu^2))(X_i^2 - (\mu_2(0) + \mu^2))$$

have common expectation,

$$E[(X_i^2 - (\mu_2(0) + \mu^2))(X_j^2 - (\mu_2(0) + \mu^2))] = E[(X_j^2 - (\mu_2(0) + \mu^2))(X_i^2 - (\mu_2(0) + \mu^2))],$$

which we denote as $\mu_4(i-j) = \mu_4(j-i)$ with $\mu_4(|i-j|) = 0$ for $|i-j| > m$.

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LEMMA 1 *Let $\{X_n\}$ be a strictly m-dependent stationary process with common mean μ, auto-covariance $\mu_2(j)$, and higher moments $\mu_3(j)$, $\mu_4(j)$, which are finite for $|j| \le m$. Then*

(a) $\mathrm{Var}(\sum_{i=1}^n X_i) = \sum_{|j|\le m}(n-|j|)\mu_2(j)$,
(b) $\mathrm{Var}(\sum_{i=1}^n X_i^2) = \sum_{|j|\le m}(n-|j|)\mu_4(j)$,
(c) $\mathrm{Cov}(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2) = \sum_{|j|\le m}(n-|j|)\mu_3(j)$,
(d) $1/n\sum_{i=1}^n X_i \xrightarrow{P} \mu$,
(e) $1/n\sum_{i=1}^n X_i^2 \xrightarrow{P} \mu_2(0)+\mu^2$.

Proof

(a) Since there are $n-|k|$ possible pairs (i,j) such that $i-j=k$, then

$$\mathrm{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n\sum_{j=1}^n \mu_2(i-j) = \sum_{k=-m}^{m}(n-|k|)\mu_2(k) = \sum_{|j|\le m}(n-|j|)\mu_2(j).$$

(b) and (c) can be derived similarly as that in (a).
(d) By Chebyshev's inequality, we have

$$P\left(\left|\frac{\sum X_i}{n}-\mu\right| > \varepsilon\right) \le \frac{\mathrm{Var}((\sum_{i=1}^n X_i)/n)}{\varepsilon^2} = \frac{\mathrm{Var}(\sum_{i=1}^n X_i)}{n^2\varepsilon^2} = \frac{\sum_{|j|\le m}(n-|j|)\mu_2(j)}{n^2\varepsilon^2} \to 0.$$

(e) can be proved similarly as that in (d). ■

LEMMA 2 *Let $\{X_n\}$ be a strictly m-dependent stationary process with common mean μ, auto-covariance $\mu_2(j)$, and higher moments $\mu_3(j)$, $\mu_4(j)$, which are finite for $|j| \le m$ with $\lambda_1^2\sum_{|j|\le m}\mu_2(j) + \lambda_2^2\sum_{|j|\le m}\mu_4(j) + 2\lambda_1\lambda_2\sum_{|j|\le m}\mu_3(j) \ne 0$, where λ_1 and λ_2 are two constants not equaling to zero simultaneously. Then*

$$\sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^n X_i, \frac{1}{n}\sum_{i=1}^n X_i^2\right) - (\mu, \mu_2(0)+\mu^2)\right] \xrightarrow{L} N(0,\Sigma), \tag{1}$$

where

$$\Sigma = \begin{pmatrix} \sum_{|j|\le m}\mu_2(j) & \sum_{|j|\le m}\mu_3(j) \\ \sum_{|j|\le m}\mu_3(j) & \sum_{|j|\le m}\mu_4(j) \end{pmatrix}.$$

Proof Denote $Y_n = \lambda_1(X_n-\mu) + \lambda_2[X_n^2 - (\mu_2(0)+\mu^2)]$. Then, $\{Y_n\}$ is a strictly m-dependent stationary process. By Lemma 1(a) and Lemma 1(c),

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^n Y_i\right) &= \mathrm{Var}\left(\lambda_1\sum_{i=1}^n X_i + \lambda_2\sum_{i=1}^n X_i^2\right) \\ &= \lambda_1^2\mathrm{Var}\left(\sum_{i=1}^n X_i\right) + \lambda_2^2\mathrm{Var}\left(\sum_{i=1}^n X_i^2\right) + 2\lambda_1\lambda_2\mathrm{Cov}\left\{\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2\right\} \\ &= \lambda_1^2\sum_{|j|\le m}(n-|j|)\mu_2(j) + \lambda_2^2\sum_{|j|\le m}(n-|j|)\mu_4(j) + 2\lambda_1\lambda_2\sum_{|j|\le m}(n-|j|)\mu_3(j).\end{aligned}$$

Therefore,

$$\begin{aligned} n\mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] &= \lambda_1^2 \sum_{|j|\le m}\left(1-\frac{|j|}{n}\right)\mu_2(j) + \lambda_2^2 \sum_{|j|\le m}\left(1-\frac{|j|}{n}\right)\mu_4(j) \\ &\quad + 2\lambda_1\lambda_2\left\{\sum_{|j|\le m}\left(1-\frac{|j|}{n}\right)\mu_3(j)\right\} \\ &\to \lambda_1^2 \sum_{|j|\le m}\mu_2(j) + \lambda_2^2 \sum_{|j|\le m}\mu_4(j) + 2\lambda_1\lambda_2 \sum_{|j|\le m}\mu_3(j). \end{aligned}$$

Applying the central limit theorem (Brockwell and Davis, 1987, p. 206) on the strictly m-dependent stationary process, $\sum_{i=1}^{n} Y_i/n$, then,

$$\sqrt{n}\left\{\frac{\lambda_1}{n}\sum_{i=1}^{n}(X_i-\mu) + \frac{\lambda_2}{n}\sum_{i=1}^{n}[X_i^2 - (\mu_2(0)+\mu^2)]\right\}$$

is asymptotically normal with mean 0 and variance

$$\lambda_1^2 \sum_{|j|\le m}\mu_2(j) + \lambda_2^2 \sum_{|j|\le m}\mu_4(j) + 2\lambda_1\lambda_2 \sum_{|j|\le m}\mu_3(j).$$

The result now follows directly by the Cramer and Wold (1936) argument. ■

3 ESTIMATING PROCESS CAPABILITY

Let $\{X_n\}$ denote a sequence of strictly m-dependent stationary process with common mean μ, and common variance $\sigma^2 = \mu_2(0)$.

Let LSL, USL be the lower and the upper specification limits, respectively. Let $d = (\mathrm{USL} - \mathrm{LSL})/2$ be half length of the specification interval, $M = (\mathrm{USL} + \mathrm{LSL})/2$ be the midpoint of the specification tolerance, and T be the target value. Consider the following process capability indices investigated by Kane (1986), Chan *et al.* (1988), and Pearn *et al.* (1992):

$$\begin{aligned} C_p &= \frac{\mathrm{USL}-\mathrm{LSL}}{6\sigma} = \frac{d}{3\sigma}, \\ C_{pk} &= \min\left\{\frac{\mathrm{USL}-\mu}{3\sigma}, \frac{\mu-\mathrm{LSL}}{3\sigma}\right\} = \frac{d-|\mu-M|}{3\sigma}, \\ C_{pm} &= \frac{\mathrm{USL}-\mathrm{LSL}}{6\sqrt{\sigma^2+(T-\mu)^2}} = \frac{d}{3\sqrt{\sigma^2+(T-\mu)^2}}, \\ C_{pmk} &= \min\left\{\frac{\mathrm{USL}-\mu}{3\sqrt{\sigma^2+(T-\mu)^2}}, \frac{\mu-\mathrm{LSL}}{3\sqrt{\sigma^2+(T-\mu)^2}}\right\} = \frac{d-|\mu-M|}{3\sqrt{\sigma^2+(T-\mu)^2}}. \end{aligned}$$

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Given unknown process mean μ and the process variance σ^2, the following natural estimators of the four basic indices are often applied,

$$\hat{C}_p = \frac{d}{3\hat{\sigma}}, \quad \hat{C}_{pk} = \frac{d - |\hat{\mu} - M|}{3\hat{\sigma}}, \quad \hat{C}_{pm} = \frac{d}{3\sqrt{\hat{\sigma}^2 + (T - \hat{\mu})^2}}, \quad \hat{C}_{pmk} = \frac{d - |\hat{\mu} - M|}{3\sqrt{\hat{\sigma}^2 + (T - \hat{\mu})^2}},$$

where $\hat{\mu} = \sum_{i=1}^{n} X_i/n$ and $\hat{\sigma}^2 = S^2 = \sum_{i=1}^{n} X_i^2/n - (\sum_{i=1}^{n} X_i/n)^2$ are conventional estimators of μ and σ^2 respectively.

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4 ASYMPTOTIC DISTRIBUTIONS OF THE ESTIMATORS

The exact distributions of the four estimators, and formula of their expected values and variances are analytically intractable. But, for large sample sizes, we may obtain their limiting distributions.

THEOREM 1 *Let $\{X_n\}$ denote a strictly m-dependent stationary process which satisfying the assumptions in Lemma* 2. *Then*

$$\sqrt{n}(\hat{C}_p - C_p) \xrightarrow{L} N(0, D_1 \Sigma D_1'),$$

where $D_1 = (\mu d/3(\mu_2(0))^{3/2}, -d/6(\mu_2(0))^{3/2})$.

Proof We first define

$$g_1(a, b) = \begin{cases} \dfrac{d}{3\sqrt{b - a^2}} & \text{if } a \in (\text{LSL}, \text{USL}), \\ 0 & \text{otherwise}, \end{cases}$$

$$D_1 = \left(\frac{\partial g_1(a, b)}{\partial a}, \frac{\partial g_1(a, b)}{\partial b} \right) \bigg|_{(a,b) = (\mu, \mu_2(0) + \mu^2)}.$$

Then, $D_1 = (\mu d/3(\mu_2(0))^{3/2}, -d/6(\mu_2(0))^{3/2})$. By Lemma 2 and Proposition 6.4.3 in Brockwell and Davis (1987), we have

$$\sqrt{n}(\hat{C}_p - C_p) = \sqrt{n} \left[g_1 \left(\sum_{i=1}^{n} \frac{X_i}{n}, \sum_{i=1}^{n} \frac{X_i^2}{n} \right) - g_1(\mu, \mu_2(0) + \mu^2) \right]$$
$$\xrightarrow{L} N(0, D_1 \Sigma D_1'). \qquad \blacksquare$$

If we denote Σ as

$$\Sigma = \begin{pmatrix} \sum_{|j| \le m} \mu_2(j) & \sum_{|j| \le m} \mu_3(j) \\ \sum_{|j| \le m} \mu_3(j) & \sum_{|j| \le m} \mu_4(j) \end{pmatrix} = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_3 \end{pmatrix},$$

and $\Sigma^{1*} = D_1 \Sigma D_1'$, then Theorem 1 can be rewritten as

$$\sqrt{n}(\hat{C}_p - C_p) \xrightarrow{L} N(0, \Sigma^{1*}), \quad \text{where } \Sigma^{1*} = \left[\frac{\mu^2\Sigma_1 - \mu\Sigma_2 + \Sigma_3/4}{\mu_2(0)^2}\right] C_p^2.$$

Therefore, an approximate $100(1-\alpha)\%$ confidence interval of C_p may be established as the following:

$$\left(\hat{C}_p - Z_{\alpha/2}\sqrt{\frac{\widehat{\Sigma^{1*}}}{n}}, \hat{C}_p + Z_{\alpha/2}\sqrt{\frac{\widehat{\Sigma^{1*}}}{n}}\right),$$

where

$$\widehat{\Sigma^{1*}} = \left[\frac{\bar{X}^2\hat{\Sigma}_1 - \bar{X}\hat{\Sigma}_2 + \hat{\Sigma}_3/4}{S^4}\right]\hat{C}_p^2,$$

$\hat{\Sigma}_1$, $\hat{\Sigma}_2$, $\hat{\Sigma}_3$ are sample auto-covariances of Σ_1, Σ_2, and Σ_3, respectively, and $Z_{\alpha/2}$ is the upper $\alpha/2$ quantile of standard normal distribution. Let $\mu_k = E(X-\mu)^k$ be the kth central moment. Then, for the case with $m = 0$, that is, the process data are independent, we have

$$\Sigma = \Sigma_0 = \begin{pmatrix} \mu_2(0) & \mu_3(0) \\ \mu_3(0) & \mu_4(0) \end{pmatrix} = \begin{pmatrix} \sigma^2 & \mu_3 + 2\mu\sigma^2 \\ \mu_3 + 2\mu\sigma^2 & \mu_4 + 4\mu\mu_3 + 4\sigma^2\mu^2 - \sigma^4 \end{pmatrix}.$$

COROLLARY 1.1 *Let $\{X_n\}$ be an independent ($m = 0$ in this case) stationary process with common mean μ, auto-covariance $\mu_2(j)$, and higher moments $\mu_3(j)$, $\mu_4(j)$, which are finite. Then,*

$$\sqrt{n}(\hat{C}_p - C_p) \xrightarrow{L} N(0, \Sigma_0^1), \quad \text{where } \Sigma_0^1 = \frac{(\mu_4 - \sigma^4)}{4\sigma^4} C_p^2;$$

We note that for $m = 0$, the results just obtained are consistent with part (a) of the Theorem in Chan *et al.* (1990) for processes with independent data.

COROLLARY 1.2 *Let $\{X_n\}$ be an independent ($m = 0$ in this case) Gaussian stationary process with common mean μ, auto-covariance $\mu_2(j)$, and higher moments $\mu_3(j)$, $\mu_4(j)$ which are finite. Then,*

$$\sqrt{n}(\hat{C}_p - C_p) \xrightarrow{L} N(0, \Sigma_{0N}^1), \quad \text{where } \Sigma_{0N}^1 = \frac{1}{2} C_p^2.$$

THEOREM 2 *Let $\{X_n\}$ be a strictly m-dependent stationary process satisfying the assumptions in Lemma 2. Then*

$$\sqrt{n}(\hat{C}_{pm} - C_{pm}) \xrightarrow{L} N(0, D_2 \Sigma D_2'),$$

where $D_2 = (Td/3[\mu_2(0) + (\mu - T)^2]^{3/2}, -d/6[\mu_2(0) + (\mu - T)^2]^{3/2})$, *and* Σ *is given in Lemma 2.*

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Proof Let

$$g_2(a,b) = \begin{cases} \dfrac{d}{3\sqrt{b-a^2+(T-a)^2}} & \text{if } a \in (\mathrm{LSL}, \mathrm{USL}), \\ 0 & \text{otherwise,} \end{cases}$$

$$D_2 = \left(\frac{\partial g_2(a,b)}{\partial a}, \frac{\partial g_2(a,b)}{\partial b}\right)\Bigg|_{(a,b)=(\mu,\mu_2(0)+\mu^2)},$$

then

$$D_2 = (Td/3[\mu_2(0)+(\mu-T)^2]^{3/2}, -d/6[\mu_2(0)+(\mu-T)^2]^{3/2}).$$

By Lemma 2 above, and Proposition 6.4.3 in Brockwell and Davis (1987),

$$\sqrt{n}(\hat{C}_{pm} - C_{pm}) = \sqrt{n}\left[g_2\left(\sum_{i=1}^{n}\frac{X_i}{n}, \sum_{i=1}^{n}\frac{X_i^2}{n}\right) - g_2(\mu, \mu_2(0)+\mu^2)\right] \xrightarrow{L} N(0, D_2\Sigma D_2'). \qquad \blacksquare$$

If we let $\Sigma^{2*} = D_2\Sigma D_2'$, then Theorem 2 can be rewritten as

$$\sqrt{n}(\hat{C}_{pm} - C_{pm}) \xrightarrow{L} N(0, \Sigma^{2*}),$$

where

$$\Sigma^{2*} = \left[\frac{T^2\Sigma_1 - T\Sigma_2 + \Sigma_3/4}{[\mu_2(0)+(T-\mu)^2]^2}\right]C_{pm}^2.$$

Therefore, an approximate $100(1-\alpha)\%$ confidence interval of C_{pm} may be established as the following:

$$\left(\hat{C}_{pm} - Z_{\alpha/2}\sqrt{\frac{\widehat{\Sigma^{2*}}}{n}}, \hat{C}_{pm} + Z_{\alpha/2}\sqrt{\frac{\widehat{\Sigma^{2*}}}{n}}\right),$$

where

$$\widehat{\Sigma^{2*}} = \left[\frac{T^2\hat{\Sigma}_1 - T\hat{\Sigma}_2 + \hat{\Sigma}_3/4}{[S^2+(T-\bar{X})^2]^2}\right]\hat{C}_{pm}^2.$$

COROLLARY 2.1 *Let $\{X_n\}$ be an independent ($m = 0$ in this case) stationary process with common mean μ, auto-covariance $\mu_2(j)$, and higher moments $\mu_3(j)$, $\mu_4(j)$ which are finite. Then,*

$$\sqrt{n}(\hat{C}_{pm} - C_{pm}) \xrightarrow{L} N(0, \Sigma_0^2),$$

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where

$$\Sigma_0^2 = \left[\frac{(T-\mu)^2\sigma^2 - (T-\mu)\mu_3 + (\mu_4 - \sigma^4)/4}{[\sigma^2 + (T-\mu)^2]^2}\right] C_{pm}^2.$$

When $m = 0$, the above results coincide with part (c) of the Theorem in Chan *et al.* (1990) for processes with independent data.

COROLLARY 2.2 *Let* $\{X_n\}$ *be an independent stationary Gaussian process* ($m = 0$ *in this case*) *with common mean* μ, *auto-covariance* $\mu_2(j)$, *and higher moments* $\mu_3(j)$, $\mu_4(j)$ *which are finite. Then*,

$$\sqrt{n}(\hat{C}_{pm} - C_{pm}) \xrightarrow{L} N(0, \Sigma_{0N}^2),$$

where

$$\Sigma_{0N}^2 = \left[\frac{(T-\mu)^2\sigma^2 + \sigma^4/2}{[\sigma^2 + (T-\mu)^2]^2}\right] C_{pm}^2.$$

THEOREM 3 *Let* $\{X_n\}$ *denote a strictly m-dependent stationary process satisfying the assumptions in Lemma 2. Define*

$$\operatorname{sgn}(M-\mu) = \begin{cases} -1, & \text{if } M < \mu \\ +1, & \text{if } M > \mu \end{cases}$$

Then

$$\sqrt{n}(\hat{C}_{pk} - C_{pk}) \xrightarrow{L} \begin{cases} N(0, D_3 \Sigma D_3'), & \text{if } \mu \neq M, \\ \dfrac{W_1 d}{3\mu_2(0)} - \dfrac{|W_2|}{3\sqrt{\mu_2(0)}}, & \text{if } \mu = M, \end{cases}$$

where

$$D_3' = \begin{pmatrix} \dfrac{\operatorname{sgn}(M-\mu)}{3\sqrt{\mu_2(0)}} + \dfrac{\mu(d - [\operatorname{sgn}(M-\mu)](M-\mu))}{3\mu_2(0)^{3/2}} \\ -\dfrac{[d - [\operatorname{sgn}(M-\mu)](M-\mu)]}{6(\mu_2(0))^{3/2}} \end{pmatrix},$$

and (W_1, W_2) *are* $N((0,0), \Phi^* \Sigma \Phi^{*'})$, *with*

$$\Phi^* = \begin{pmatrix} \dfrac{\mu}{\sqrt{\mu_2(0)}} & 1 \\ -\dfrac{1}{2\sqrt{\mu_2(0)}} & 0 \end{pmatrix}.$$

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Proof Define

$$g_3(a, b) = \begin{cases} \dfrac{d - |M - a|}{3\sqrt{b - a^2}} & \text{if } a \in (\text{LSL}, \text{USL}), \\ 0 & \text{otherwise.} \end{cases}$$

Then $C_{pk} = g_3(\mu, \mu_2(0) + \mu^2)$, $\hat{C}_{pk} = g_3(\sum_{i=1}^n X_i/n, \sum_{i=1}^n X_i^2/n)$ and

$$\sqrt{n}(\hat{C}_{pk} - C_{pk}) = \sqrt{n}\left[g_3\left(\sum_{i=1}^n \frac{X_i}{n}, \sum_{i=1}^n \frac{X_i^2}{n}\right) - g_3(\mu, \mu_2(0) + \mu^2)\right].$$

Case 1 If LSL $< \mu < M$, LSL $< a < M$, then

$$g_3(a, b) = \frac{d - M + a}{3\sqrt{b - a^2}},$$

$$D_3 = \left(\frac{\partial g_3(a, b)}{\partial a}, \frac{\partial g_3(a, b)}{\partial b}\right)\bigg|_{(a,b)=(\mu,\mu_2(0)+\mu^2)}.$$

Then

$$D_3' = \begin{pmatrix} \dfrac{\text{sgn}(M - \mu)}{3\sqrt{\mu_2(0)}} + \dfrac{\mu(d - [\text{sgn}(M - \mu)](M - \mu))}{3\mu_2(0)^{3/2}} \\ -\dfrac{(d - [\text{sgn}(M - \mu)](M - \mu))}{6(\mu_2(0))^{3/2}} \end{pmatrix}.$$

By Lemma 2 above, and Proposition 6.4.3. in Brockwell and Davis (1987),

$$\sqrt{n}(\hat{C}_{pk} - C_{pk}) = \sqrt{n}\left[g_3\left(\sum_{i=1}^n \frac{X_i}{n}, \sum_{i=1}^n \frac{X_i^2}{n}\right) - g_3(\mu, \mu_2(0) + \mu^2)\right] \xrightarrow{L} N(0, D_3 \Sigma D_3').$$

Case 2 If $M < \mu <$ USL, $M < a <$ USL, then

$$g_3(a, b) = \frac{d + M - a}{3\sqrt{b - a^2}}.$$

The proof can be carried out using the same technique as that in Case 1.

Case 3 If $\mu = M$

$$\sqrt{n}(\hat{C}_{pk} - C_{pk}) = \sqrt{n}\left[g_3\left(\sum_{i=1}^{n}\frac{X_i}{n}, \sum_{i=1}^{n}\frac{X_i^2}{n}\right) - g_3(\mu, \mu_2(0) + \mu^2)\right]$$

$$= \sqrt{n}\left[\frac{d - |\sum_{i=1}^{n} X_i/n - M|}{3\sqrt{\sum_{i=1}^{n} X_i^2/n - (\sum_{i=1}^{n} X_i/n)^2}} - \frac{d}{3\sqrt{\mu_2(0)}}\right]$$

$$= \sqrt{n}\left[\frac{d}{3}\frac{\sqrt{\mu_2(0)} - \sqrt{\sum_{i=1}^{n} X_i^2/n - (\sum_{i=1}^{n} X_i/n)^2}}{\sqrt{\sum_{i=1}^{n} X_i^2/n - (\sum_{i=1}^{n} X_i/n)^2}\sqrt{\mu_2(0)}}\right.$$

$$\left. - \frac{|\sum_{i=1}^{n} X_i/n - \mu|}{3\sqrt{\sum_{i=1}^{n} X_i^2/n - (\sum_{i=1}^{n} X_i/n)^2}}\right].$$

Let

$$(V_{1n}, V_{2n}) = \left(\frac{d}{3}\frac{1}{\sqrt{\sum_{i=1}^{n} X_i^2/n - (\sum_{i=1}^{n} X_i/n)^2}\sqrt{\mu_2(0)}}, \frac{1}{3\sqrt{\sum_{i=1}^{n} X_i^2/n - (\sum_{i=1}^{n} X_i/n)^2}}\right),$$

$$(W_{1n}, W_{2n}) = \left(\sqrt{n}\left(\sqrt{\mu_2(0)} - \sqrt{\sum_{i=1}^{n}\frac{X_i^2}{n} - \left(\sum_{i=1}^{n}\frac{X_i}{n}\right)^2}\right), \sqrt{n}\left(\sum_{i=1}^{n}\frac{X_i}{n} - \mu\right)\right).$$

Then

$$(V_{1n}, V_{2n}) \xrightarrow{P} \left(\frac{d}{3\mu_2(0)}, \frac{1}{3\sqrt{\mu_2(0)}}\right) \triangleq (V_1, V_2).$$

Let $(k_1(a,b), k_2(a,b)) = (\sqrt{\mu_2(0)} - \sqrt{b - a^2}, a - \mu)$. Define the following and evaluate it at $(\mu, \mu_2(0) + \mu^2)$. Then

$$\Phi^* = \begin{pmatrix} \frac{\partial k_1(a,b)}{\partial a} & \frac{\partial k_2(a,b)}{\partial a} \\ \frac{\partial k_1(a,b)}{\partial b} & \frac{\partial k_2(a,b)}{\partial b} \end{pmatrix} = \begin{pmatrix} \frac{\mu}{\sqrt{\mu_2(0)}} & 1 \\ -\frac{1}{2\sqrt{\mu_2(0)}} & 0 \end{pmatrix}.$$

By Lemma 2, we have $(W_{1n}, W_{2n}) \xrightarrow{L} (W_1, W_2)$, where (W_1, W_2) are $N((0,0), \Phi^*\Sigma\Phi^{*'})$. Therefore, $(V_{1n}W_{1n}, V_{2n}W_{2n}) \xrightarrow{L} (V_1W_1, V_2W_2)$. Since $\sqrt{n}(\hat{C}_{pk} - C_{pk}) = V_{1n}W_{1n} - |V_{2n}W_{2n}|$ is a continuous function of $V_{1n}W_{1n}$ and $V_{2n}W_{2n}$, by the theorem on page 24 of Serfling (1980), we have

$$\sqrt{n}(\hat{C}_{pk} - C_{pk}) = V_{1n}W_{1n} - |V_{2n}W_{2n}| \xrightarrow{L} \frac{W_1 d}{3\mu_2(0)} - \frac{|W_2|}{3\sqrt{\mu_2(0)}}. \qquad \blacksquare$$

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Let $\Sigma^{3*} = D_3 \Sigma D_3'$, and $\Psi = \Phi^* \Sigma \Phi^{*'}$. Then, Theorem 3 can be rewritten as

$$\sqrt{n}(\hat{C}_{pk} - C_{pk}) \xrightarrow{L} \begin{cases} N(0, \Sigma^{3*}), & \text{if } \mu \neq M, \\ \dfrac{W_1 d}{3\mu_2(0)} - \dfrac{|W_2|}{3\sqrt{\mu_2(0)}}, & \text{if } \mu = M. \end{cases}$$

$$\Sigma^{3*} = \frac{\Sigma_1}{9\mu_2(0)} + [\operatorname{sgn}(M-\mu)]\left[\frac{2\mu\Sigma_1 - \Sigma_2}{3\mu_2(0)^{3/2}}\right] C_{pk} + \left[\frac{\mu^2\Sigma_1 - \mu\Sigma_2 + \Sigma_3/4}{\mu_2(0)^2}\right] C_{pk}^2,$$

$(W_1, W_2) \sim N((0,0), \Psi)$, and

$$\Psi = \begin{pmatrix} \dfrac{\mu^2\Sigma_1}{\mu_2(0)} + \dfrac{2\Sigma_2\mu}{\sqrt{\mu_2(0)}} + \Sigma_3 & \dfrac{-\mu\Sigma_1}{2\mu_2(0)} - \dfrac{\Sigma_2}{2\sqrt{\mu_2(0)}} \\ \dfrac{-\mu\Sigma_1}{2\mu_2(0)} - \dfrac{\Sigma_2}{2\sqrt{\mu_2(0)}} & \dfrac{\Sigma_1}{4\mu_2(0)} \end{pmatrix}.$$

Therefore, an approximate $100(1-\alpha)\%$ confidence interval of C_{pk} for $\mu \neq M$ may be established as the following:

$$\left(\hat{C}_{pk} - Z_{\alpha/2}\sqrt{\frac{\widehat{\Sigma^{3*}}}{n}},\ \hat{C}_{pk} + Z_{\alpha/2}\sqrt{\frac{\widehat{\Sigma^{3*}}}{n}}\right),$$

where

$$\widehat{\Sigma^{3*}} = \frac{\hat{\Sigma}_1}{9S^2} + \operatorname{sgn}(M-\bar{X})\left[\frac{2\bar{X}\hat{\Sigma}_1 - \hat{\Sigma}_2}{3S^3}\right]\hat{C}_{pk} + \left[\frac{\bar{X}^2\hat{\Sigma}_1 - \bar{X}\hat{\Sigma}_2 + \hat{\Sigma}_3/4}{S^4}\right]\hat{C}_{pk}^2.$$

COROLLARY 3.1 *Let $\{X_n\}$ be an independent stationary process ($m = 0$ in this case) with common mean μ, auto-covariance $\mu_2(j)$, and higher moments $\mu_3(j)$, $\mu_4(j)$ which are finite. Then,*

$$\sqrt{n}(\hat{C}_{pk} - C_{pk}) \xrightarrow{L} \begin{cases} N(0, \Sigma_0^3), & \text{if } \mu \neq M, \\ \dfrac{W_1 d}{3\sigma^2} - \dfrac{|W_2|}{3\sigma}, & \text{if } \mu = M. \end{cases}$$

$$\Sigma_0^3 = \frac{1}{9} - [\operatorname{sgn}(M-\mu)]\frac{\mu_3}{3\sigma^3} C_{pk} + \frac{\mu_4 - \sigma^4}{4\sigma^4} C_{pk}^2;$$

$$(W_1, W_2) \sim N((0,0), \Psi_0^3), \quad \Psi_0^3 = \begin{pmatrix} A_1 & B_1 \\ B_1 & C_1 \end{pmatrix},$$

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where

$$A_1 = \mu^2 + \frac{2(\mu_3 + 2\sigma^2\mu)\mu}{\sigma} + [\mu_4 + 4\mu\mu_3 + 4\sigma^2\mu^2 - \sigma^4],$$
$$B_1 = -\frac{\mu}{2} - \frac{(\mu_3 + 2\sigma^2\mu)}{2\sigma},$$
$$C_1 = \frac{n}{4}.$$

When $m = 0$, it is easy to see that the results in Corollary 3.1 is identical to part (b) of the Theorem in Chan *et al.* (1990) for processes with independent data.

COROLLARY 3.2 *Let* $\{X_n\}$ *be an independent Gaussian stationary process* ($m = 0$ *in this case*) *with common mean* μ, *auto-covariance* $\mu_2(j)$, *and higher moments* $\mu_3(j)$, $\mu_4(j)$ *which are finite. Then*,

$$\sqrt{n}(\hat{C}_{pk} - C_{pk}) \xrightarrow{L} \begin{cases} N(0, \Sigma_{0N}^3), & \text{if } \mu \neq M, \\ \dfrac{W_1 d}{3\sigma^2} - \dfrac{|W_2|}{3\sigma}, & \text{if } \mu = M. \end{cases}$$

$$\Sigma_{0N}^3 = \frac{1}{9} + \frac{1}{2} C_{pk}^2,$$

$$(W_1, W_2) \sim N((0,0), \Psi_{0N}^3), \quad \Psi_{0N}^3 = \begin{pmatrix} A_2 & B_2 \\ B_2 & C_2 \end{pmatrix},$$

where

$$A_2 = \mu^2 + 4\sigma\mu^2 + 4\sigma^2\mu^2 + 2\sigma^4,$$
$$B_2 = -\frac{\mu}{2} - \sigma\mu,$$
$$C_2 = \frac{n}{4}.$$

Remark If μ approaches m from above or below, then both limiting distributions in Theorem 3 are normal with mean 0 but different variances.

By the same argument as the proof of Theorem 3, we have the following theorem.

THEOREM 4 *Let* $\{X_n\}$ *denote a strictly m-dependent stationary process which satisfying the assumptions in Lemma* 2. *Then*

$$\sqrt{n}(\hat{C}_{pmk} - C_{pmk}) \xrightarrow{L} \begin{cases} N(0, D_4 \Sigma D_4'), & \text{if } \mu \neq M, \\ \dfrac{W_1 d}{3[\mu_2(0) + (\mu - T)^2]} - \dfrac{|W_2|}{3\sqrt{\mu_2(0) + (\mu - T)^2}}, & \text{if } \mu = M. \end{cases}$$

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where

$$D_4' = \begin{pmatrix} \dfrac{\mathrm{sgn}(M-\mu)}{3\sqrt{\mu_2(0)+(\mu-T)^2}} + \dfrac{T(d-[\mathrm{sgn}(M-\mu)](M-\mu))}{3[\mu_2(0)+(\mu-T)^2]^{3/2}} \\ -\dfrac{d-[\mathrm{sgn}(M-\mu)](M-\mu)}{6(\mu_2(0)+(\mu-T)^2)^{3/2}} \end{pmatrix}.$$

(W_1, W_2) *are* $N((0,0), \Phi^{**'}\Sigma\Phi^{**})$, *and*

$$\Phi^{**} = \begin{pmatrix} \dfrac{T}{\sqrt{\mu_2(0)+(\mu-T)^2}} & 1 \\ -\dfrac{1}{2\sqrt{\mu_2(0)+(\mu-T)^2}} & 0 \end{pmatrix}.$$

Let $\Sigma^{4*} = D_4\Sigma D_4'$, and $\Psi^{**} = \Phi^{**}\Sigma\Phi^{**'}$. Theorem 4 can be rewritten as

$$\sqrt{n}(\hat{C}_{pmk} - C_{pmk}) \xrightarrow{L} \begin{cases} N(0, \Sigma^{4*}), & \text{if } \mu \neq M, \\ \dfrac{W_1 d}{3[\mu_2(0)+(\mu-T)^2]} - \dfrac{|W_2|}{3\sqrt{\mu_2(0)+(\mu-T)^2}}, & \text{if } \mu = M. \end{cases}$$

$$\Sigma^{4*} = \frac{\Sigma_1}{9[\mu_2(0)+(T-\mu)^2]} + \mathrm{sgn}(M-\mu)\left[\frac{2T\Sigma_1 - \Sigma_2}{3[\mu_2(0)+(T-\mu)^2]^{3/2}}\right]C_{pmk} + \left[\frac{T^2\Sigma_1 - T\Sigma_2 + \Sigma_3/4}{[\mu_2(0)+(T-\mu)^2]^2}\right]C_{pmk}^2;$$

$$(W_1, W_2) \sim N((0,0), \Psi^{**}), \text{ and } \Psi^{**} = \begin{pmatrix} A_3 & B_3 \\ B_3 & C_3 \end{pmatrix},$$

where

$$A_3 = \frac{T^2\Sigma_1}{\mu_2(0)+(\mu-T)^2} + \frac{2\Sigma_2 T}{\sqrt{\mu_2(0)+(\mu-T)^2}} + \Sigma_3,$$

$$B_3 = \frac{-T\Sigma_1}{2[\mu_2(0)+(\mu-T)^2]} - \frac{\Sigma_2}{2\sqrt{\mu_2(0)+(\mu-T)^2}},$$

$$C_3 = \frac{\Sigma_1}{4[\mu_2(0)+(\mu-T)^2]}.$$

Therefore, an approximate $100(1-\alpha)\%$ confidence interval of C_{pmk} for $\mu \neq M$ may be established as the following:

$$\left(\widehat{C_{pmk}} - Z_{\alpha/2}\sqrt{\frac{\widehat{\Sigma^{4*}}}{n}},\ \widehat{C_{pmk}} + Z_{\alpha/2}\sqrt{\frac{\widehat{\Sigma^{4*}}}{n}}\right),$$

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where

$$\widehat{\Sigma^{4*}} = \frac{\hat{\Sigma}_1}{9[S^2 + (T - \bar{X})^2]} + \operatorname{sgn}(M - \bar{X})\left[\frac{2T\hat{\Sigma}_1 - \hat{\Sigma}_2}{3[S^2 + (T - \bar{X})^2]^{3/2}}\right]\hat{C}_{pmk} + \left[\frac{T^2\hat{\Sigma}_1 - T\hat{\Sigma}_2 + \hat{\Sigma}_3/4}{[S^2 + (T - \bar{X})^2]^2}\right]\hat{C}^2_{pmk}.$$

COROLLARY 4.1 *Let $\{X_n\}$ be an independent stationary process ($m = 0$ in this case) with common mean μ, auto-covariance $\mu_2(j)$, and higher moments $\mu_3(j)$, $\mu_4(j)$ which are finite. Then,*

$$\sqrt{n}(\hat{C}_{pmk} - C_{pmk}) \xrightarrow{L} \begin{cases} N(0, \Sigma_0^4), & \text{if } \mu \neq M, \\ \dfrac{W_1 d}{3[\sigma^2 + (\mu - T)^2]} - \dfrac{|W_2|}{3\sqrt{\sigma^2 + (\mu - T)^2}}, & \text{if } \mu = M, \end{cases}$$

$$\Sigma_0^4 = \frac{\sigma^2}{9[\sigma^2 + (T - \mu)^2]} + \operatorname{sgn}(M - \mu)\left[\frac{2(T - \mu)\sigma^2 - \mu_3}{3[\sigma^2 + (T - \mu)^2]^{3/2}}\right]C_{pmk} + \left[\frac{(T - \mu)^2\sigma^2 - (T - \mu)\mu_3 + (\mu_4 - \sigma^4)/4}{[\sigma^2 + (T - \mu)^2]^2}\right]C^2_{pmk};$$

$$(W_1, W_2) \sim N((0, 0), \Psi_0^4), \ \textit{and} \ \Psi^{**} = \begin{pmatrix} A_4 & B_4 \\ B_4 & C_4 \end{pmatrix},$$

where

$$A_4 = \frac{\sigma^2 T^2}{\sigma^2 + (\mu - T)^2} + \frac{2(\mu_3 + 2\sigma^2\mu)T}{\sqrt{\sigma^2 + (\mu - T)^2}} + [\mu_4 + 4\mu\mu_3 + 4\sigma^2\mu^2 - \sigma^4],$$

$$B_4 = \frac{-\sigma^2 T}{2[\sigma^2 + (\mu - T)^2]} - \frac{(\mu_3 + 2\sigma^2\mu)}{2\sqrt{\sigma^2 + (\mu - T)^2}},$$

$$C_4 = \frac{\sigma^2}{4[\sigma^2 + (\mu - T)^2]}.$$

We note that for $m = 0$, the results reduce to Theorem 2.2 in Chen and Hsu (1995) for processes with independent data.

If the process data are independent, and the process is normally distributed, the result stated in Corollary 4.1 can be further reduced to the following.

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COROLLARY 4.2 *Let $\{X_n\}$ be an independent stationary Gaussian process ($m = 0$ in this case) with common mean μ, auto-covariance $\mu_2(j)$, and higher moments $\mu_3(j)$, $\mu_4(j)$ which are finite. Then,*

$$\sqrt{n}(\hat{C}_{pmk} - C_{pmk}) \xrightarrow{L} \begin{cases} N(0, \Sigma_{0N}^4), & \text{if } \mu \neq M, \\ \dfrac{W_1 d}{3[\sigma^2 + (\mu - T)^2]} - \dfrac{|W_2|}{3\sqrt{\sigma^2 + (\mu - T)^2}}, & \text{if } \mu = M. \end{cases}$$

$$\Sigma_{0N}^4 = \frac{\sigma^2}{9[\sigma^2 + (T - \mu)^2]} + \operatorname{sgn}(M - \mu)\left[\frac{2(T - \mu)\sigma^2}{3[\sigma^2 + (T - \mu)^2]^{3/2}}\right] C_{pmk} + \left[\frac{(T - \mu)^2\sigma^2 + \sigma^4/2}{[\sigma^2 + (T - \mu)^2]^2}\right] C_{pmk}^2;$$

$$(W_1, W_2) \sim N((0, 0), \Psi_{0N}^4), \text{ and } \Psi_{0N}^4 = \begin{pmatrix} A_5 & B_5 \\ B_5 & C_5 \end{pmatrix},$$

where

$$A_5 = \frac{\sigma^2 T^2}{\sigma^2 + (\mu - T)^2} + \frac{4\sigma^2 \mu T}{\sqrt{\sigma^2 + (\mu - T)^2}} + [3\sigma^4 + 4\sigma^2\mu^2 - \sigma^4],$$

$$B_5 = \frac{-\sigma^2 T}{2[\sigma^2 + (\mu - T)^2]} - \frac{\sigma^2 \mu}{\sqrt{\sigma^2 + (\mu - T)^2}},$$

$$C_5 = \frac{\sigma^2}{4[\sigma^2 + (\mu - T)^2]}.$$

5 SIMULATION STUDY

Central limit theorem results are often very useful when sample size is large. It is then interesting to study its small sample behaviour. This section is devoted to a small sample simulation comparison between central limit theorem and Edgeworth expansion.

We first generate a sequence of i.i.d. $N(0, 1)$ random variables Z_i, $i = 1, 2, \ldots, n$. Define $X_i = Z_i + Z_{i+1}$, then X_i, $i = 1, 2, \ldots, n$, is a sequence of 1-dependent random variables. It is then easy to see that

$$\Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 12 \end{pmatrix}.$$

Each standardized empirical distribution of the four indices is compared with the standard normal distribution and Edgeworth expansion (see Section 2.3 of Hall, 1992). The simulation results (see Figs. 1–4) indicate that Edgeworth expansion works better than normal approximation when sample size is 100. But the difference is not significant when $n = 200, 300, 400$. In Figure 1, Edgcp* means Edgeworth expansion, empcp* means the standardized empirical when the sample size is *, and nor means the standard normal for capability index C_p. Notations used in the rest of the figures are defined in similar way for the other three indices.

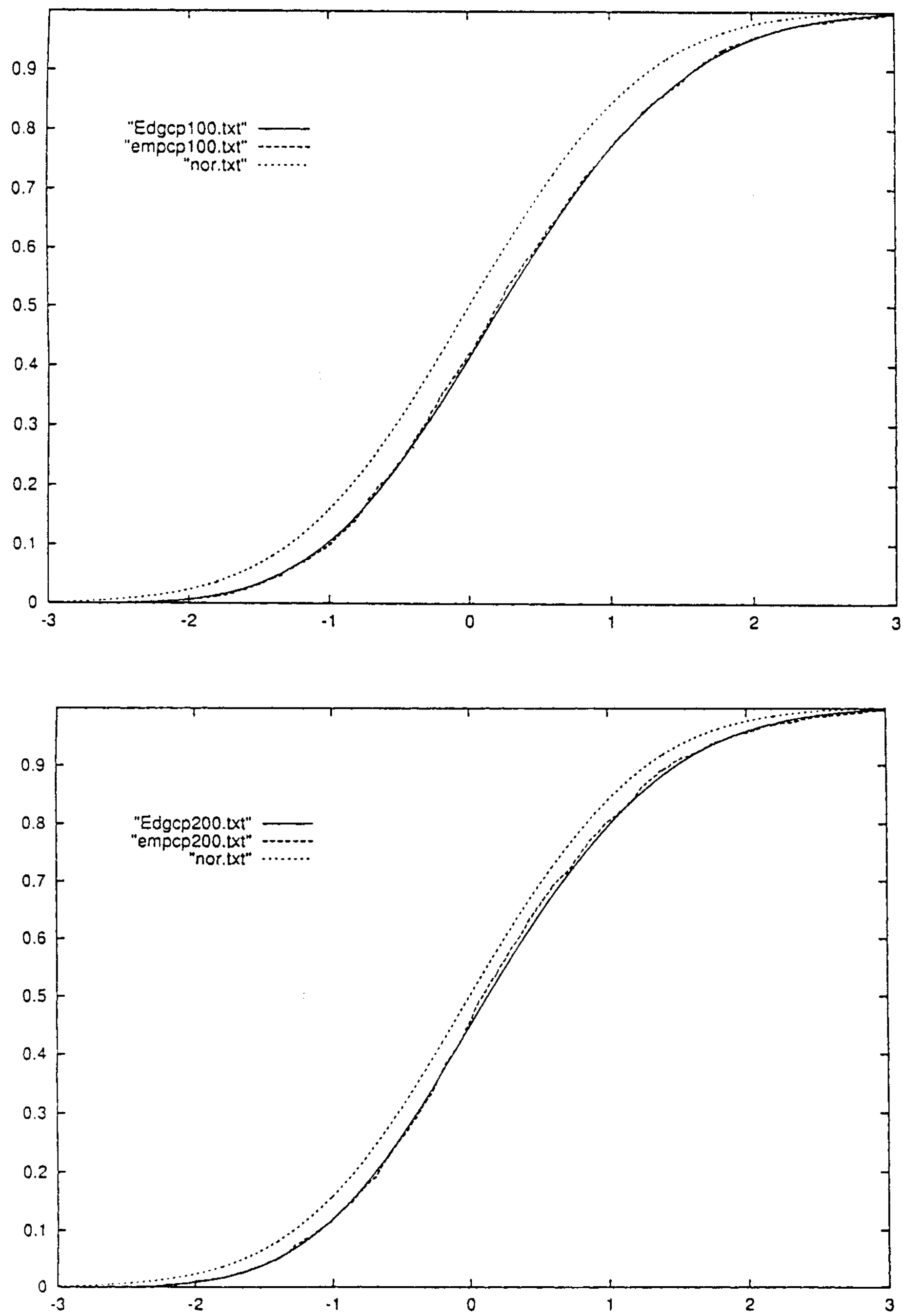

FIGURE 1 Comparison of standardized empirical distribution, the Edgeworth expansion and standard normal for $\hat{C}_p$ with sample sizes 100, 200, 300 and 400.

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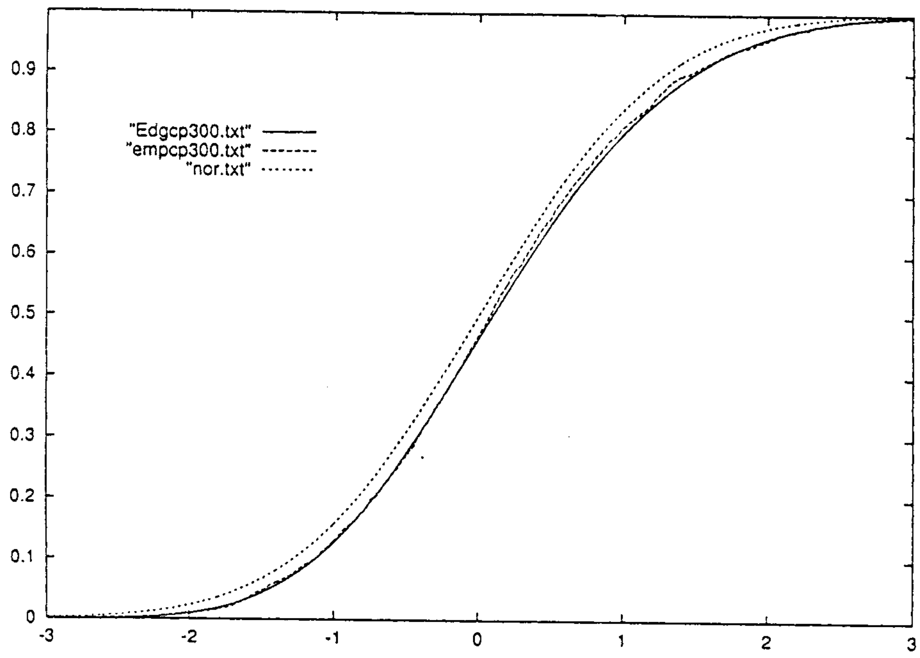

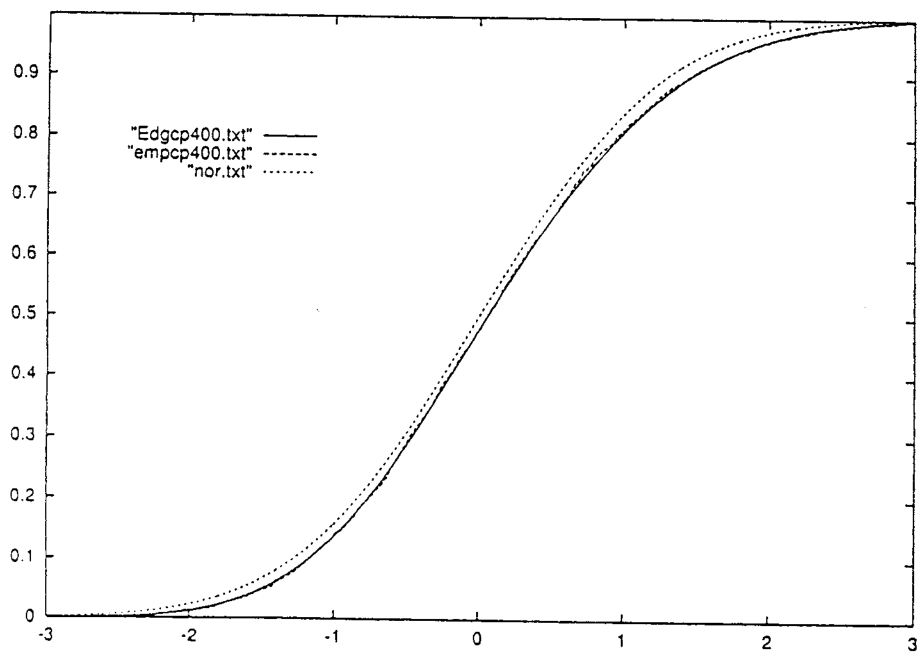

FIGURE 1 (*Continued*)

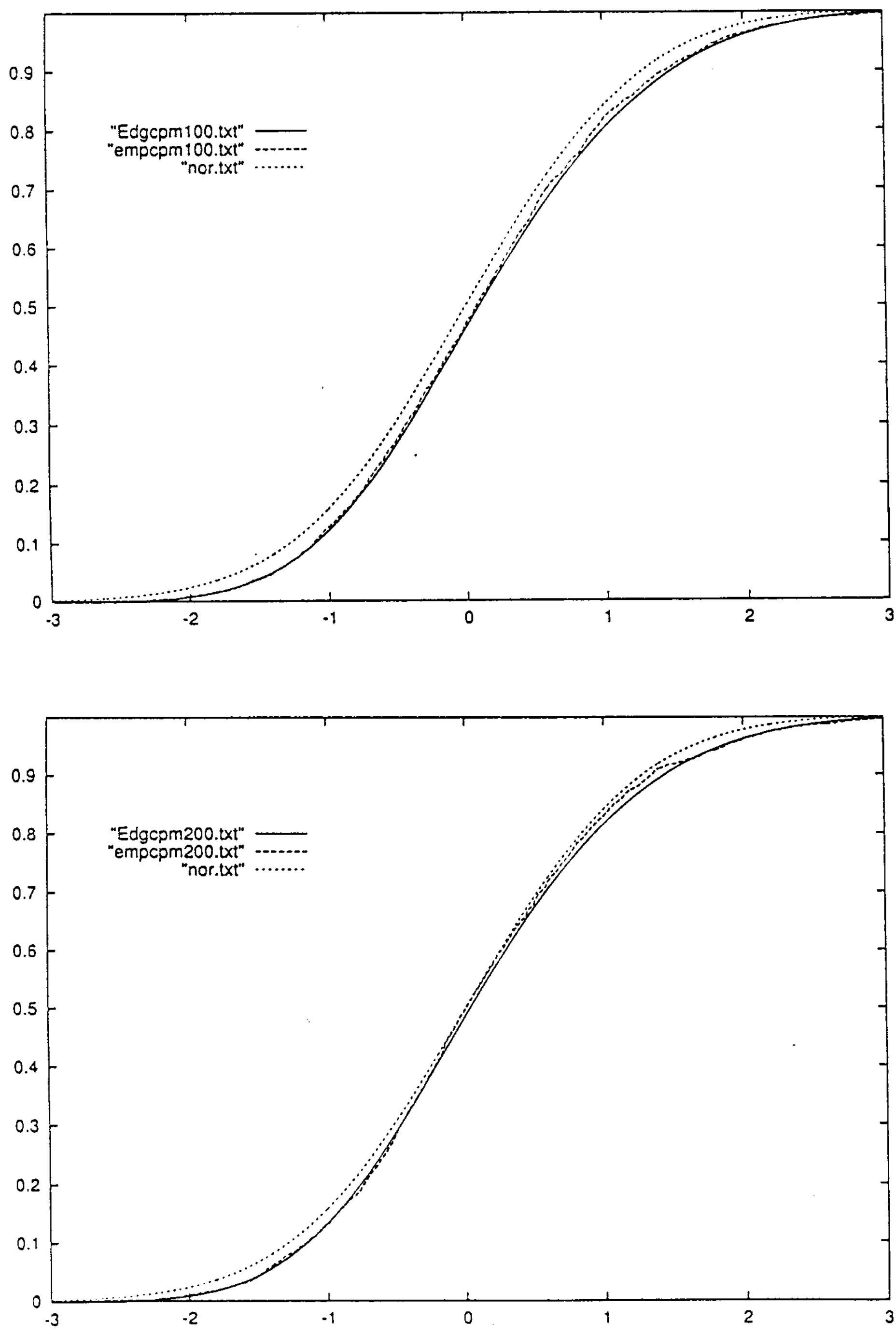

FIGURE 2 Comparison of standardized empirical distribution, the Edgeworth expansion and standard normal for $\hat{C}_{pm}$ with sample sizes 100, 200, 300 and 400.

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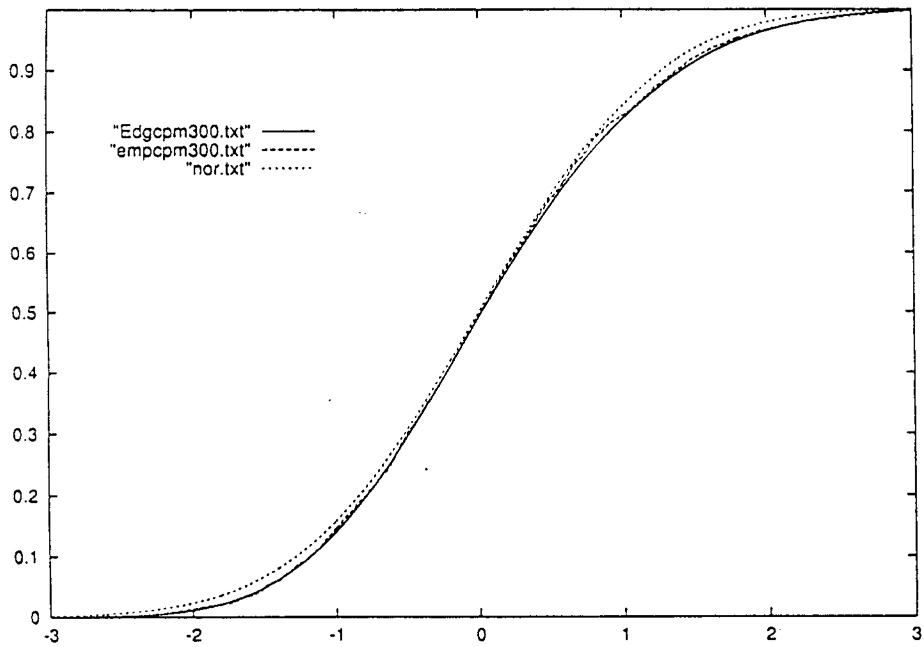


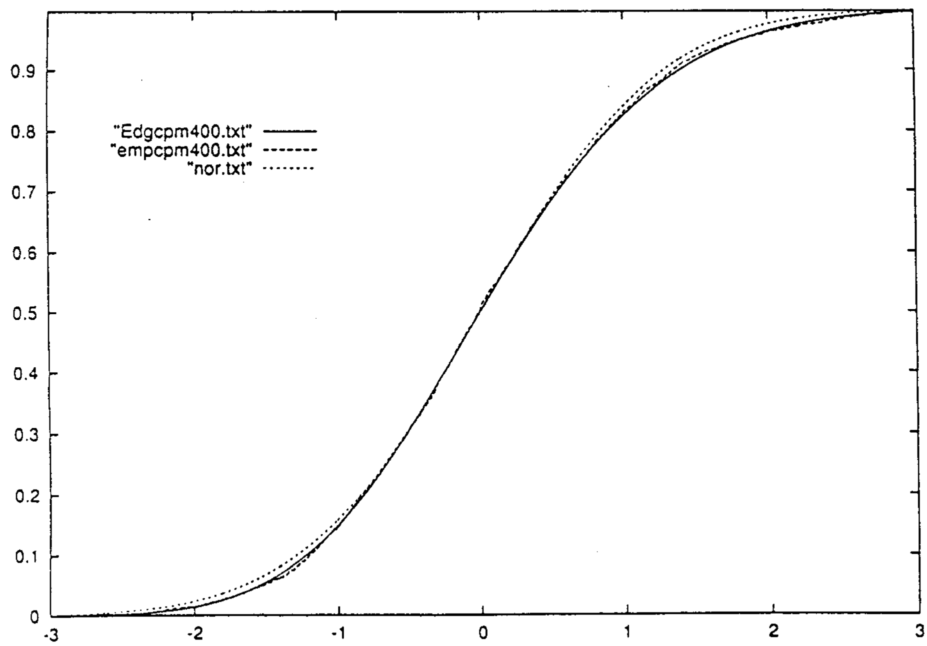
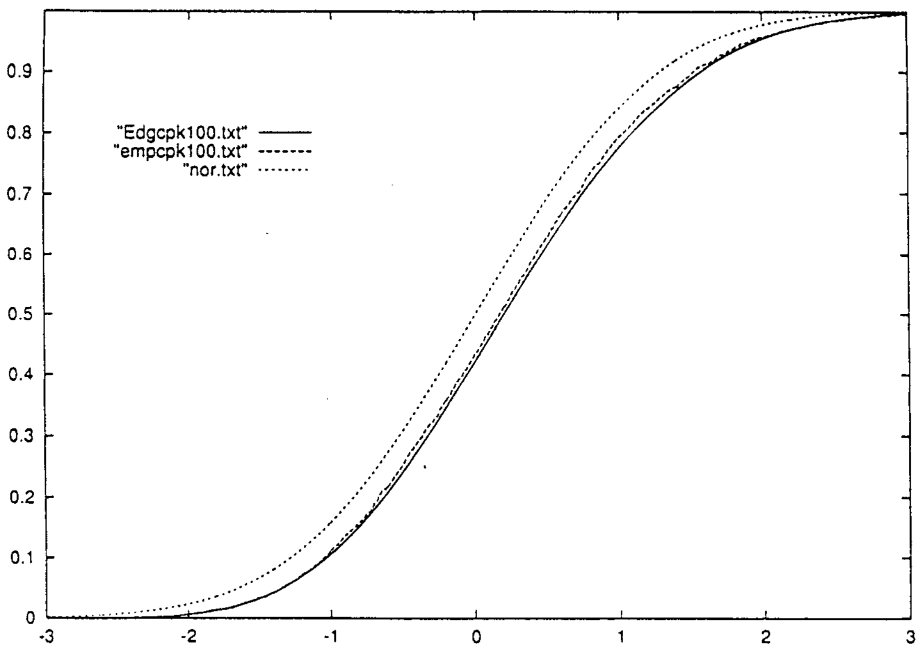


FIGURE 2 (*Continued*)



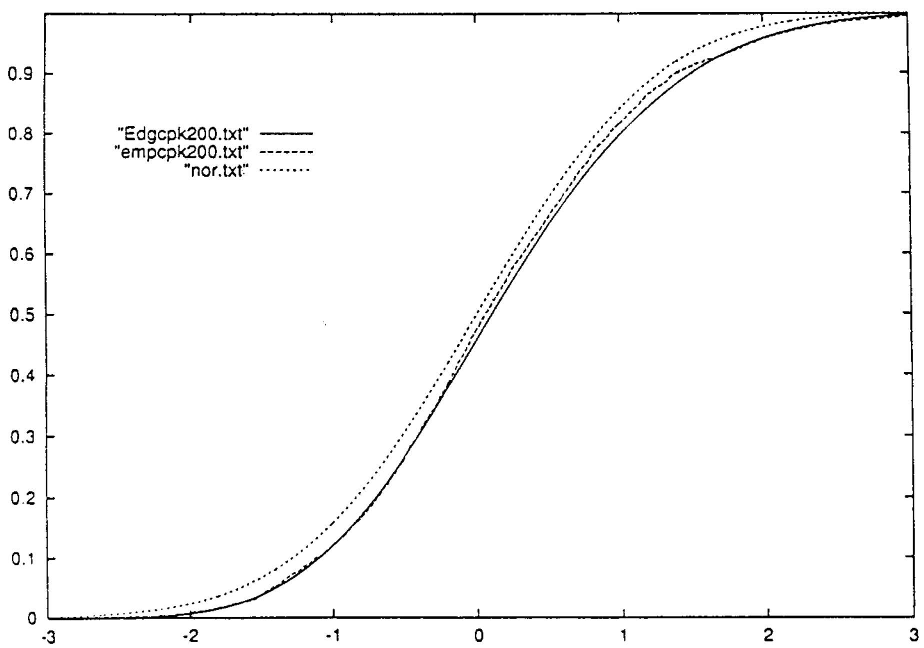


FIGURE 3 Comparison of standardized empirical distribution, the Edgeworth expansion and standard normal for $\hat{C}_{pk}$ with sample sizes 100, 200, 300 and 400.

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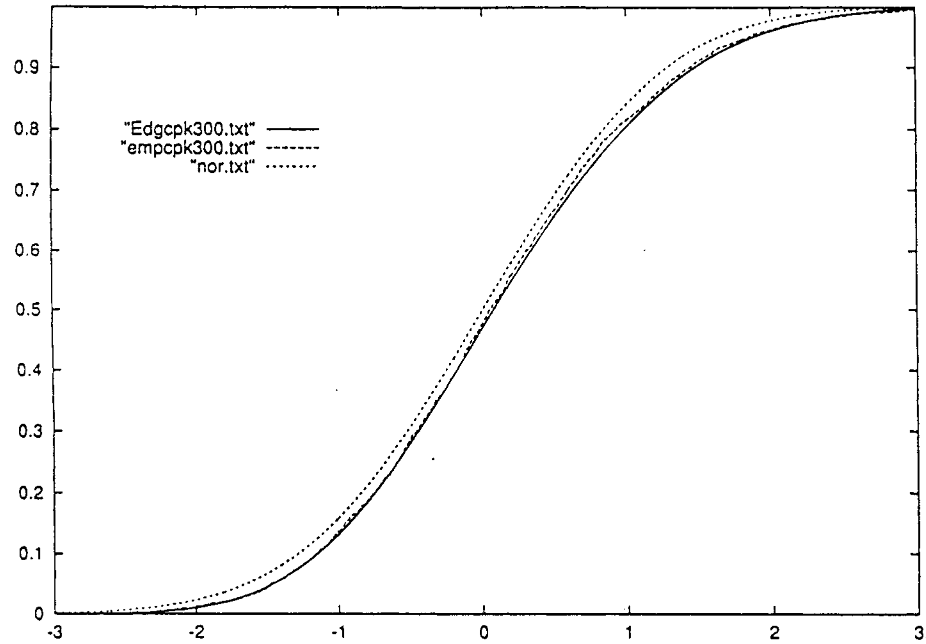

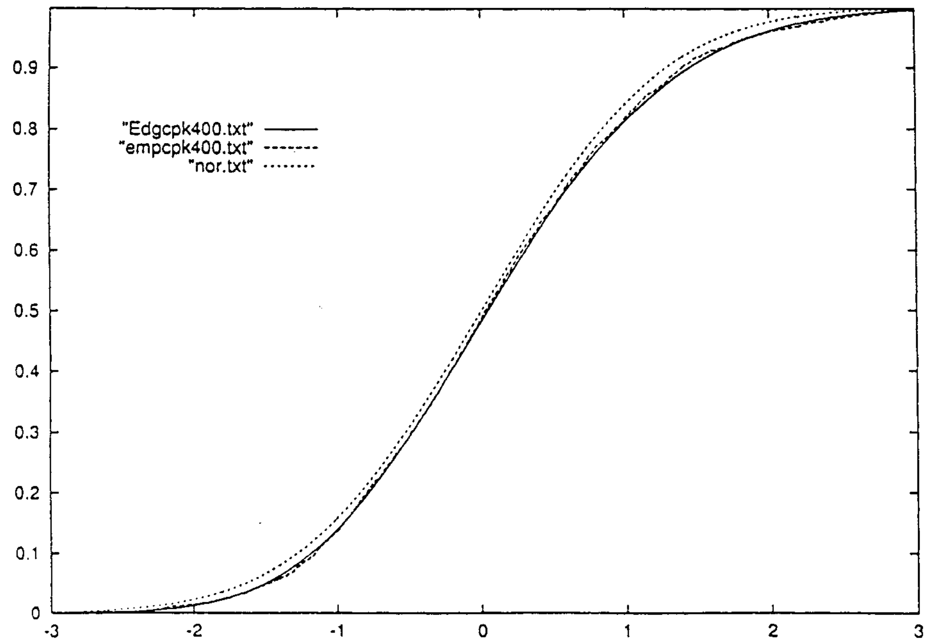

FIGURE 3 (*Continued*)

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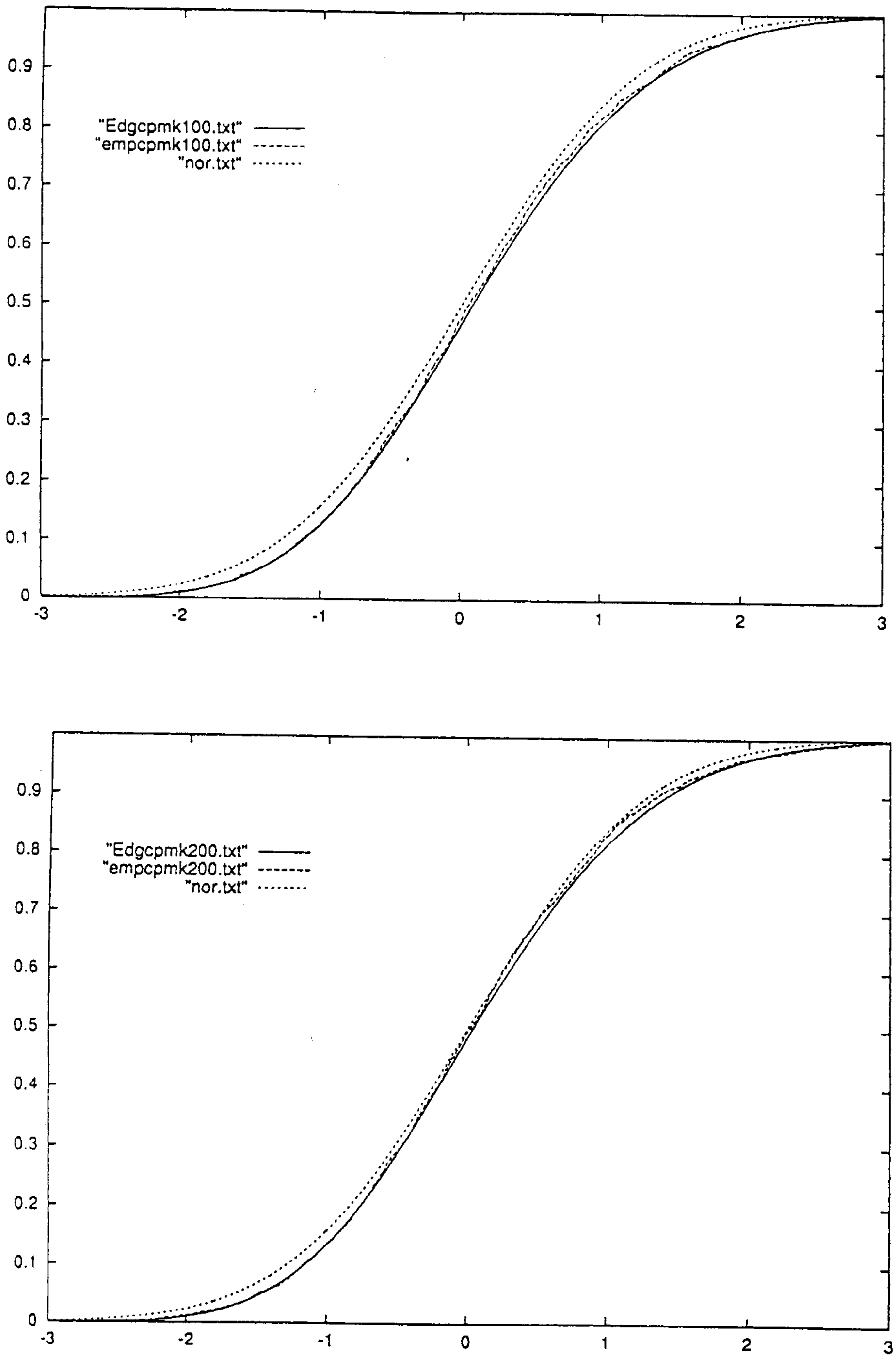
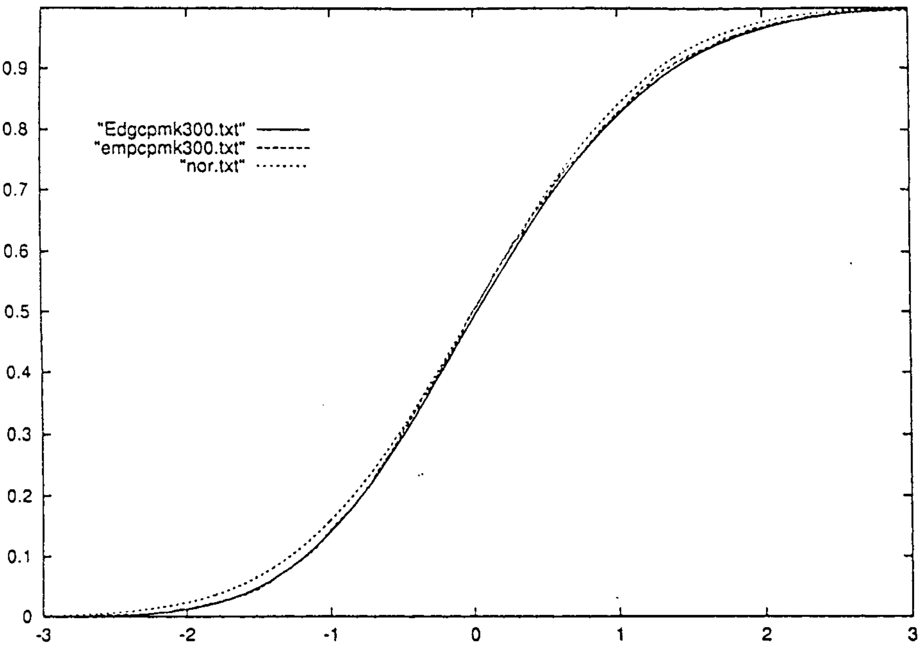


FIGURE 4 Comparison of standardized empirical distribution, the Edgeworth expansion and standard normal for $\hat{C}_{pmk}$ with sample sizes 100, 200, 300 and 400.



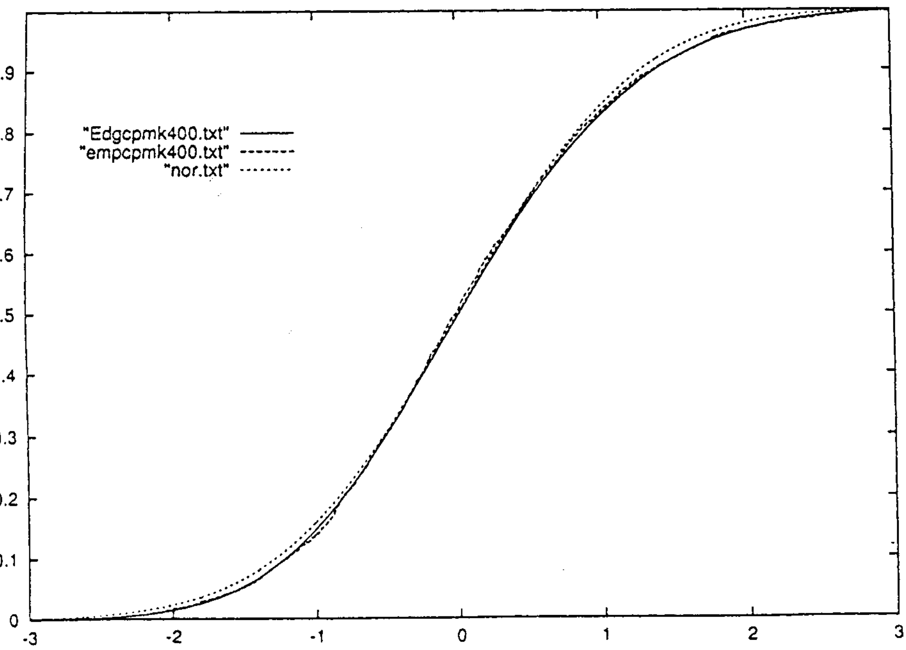


FIGURE 4 (*Continued*)

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6 CONCLUSIONS

Process capability indices, providing numerical measures on process potential and process performance, have received substantial research attention in quality control and quality assurance literatures recently. Most research, have assumed that the process is normally distributed and the process data are independent. But, in real-world applications such as chemical, soft drinks, or tobacco/cigarette manufacturing processes, process data are often auto-correlated. In this paper, we considered the capability indices C_p, C_{pk}, C_{pm}, C_{pmk} for strictly m-dependent stationary processes, where each observation is correlated with its preceding and exceeding m sample data and independent with other observations. We investigated the statistical properties of the natural estimators of the four indices. We derived the asymptotic distributions of their natural estimators, and established confidence intervals. Consequently, capability measures and testing can be performed for strictly m-dependent stationary processes, particularly, for those with normal and near-normal distributions. It is worth noting that when sample size is small, the performance of Edgeworth expansion could be better than the central limit theorem.

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