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Sortability of vector partitions

F.K. Hwang^a, J.S. Lee^a, Y.C. Liu^{a,*}, U.G. Rothblum^b

^aDepartment of Applied Mathematics, National Chiao Tung University, Hsinchu 30050, Taiwan, ROC

^bDavidson Faculty of IE and MS, TECHNION-Israel Institute of Technology, Haifa 3200, Israel

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Abstract

Some combinatorial properties of partitioning points in \mathbb{R}^1 have recently been well studied. We extend this study to points in \mathbb{R}^d with the hope to benefit clustering and optimal partition problems in \mathbb{R}^d .

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1. Introduction

Let P be a set of distinct points in \mathbb{R}^d and let $\pi = \{\pi_1, \dots, \pi_p\}$ be an (unordered) partition of P into p disjoint nonempty parts. Let $\text{Conv}(\pi_i)$ denote the convex hull of π_i . Then π is called a *disjoint partition* by Barnes et al. [3] if $\text{Conv}(\pi_i) \cap \text{Conv}(\pi_j) = \emptyset$ for all i and j . This “disjointness” will be referred to as a *partition property*. There are other partition properties studied in the literature [3–5] which will be brought up in Section 2.

Let Q denote a partition property such as disjointness. If Q defines a small class of partitions, then it is advantageous to prove the existence of an optimal partition (with respect to some objective function) in the class Q , for then we need only to search the class Q for an optimal partition.

A partition property Q is k -consistent if for any partition not satisfying Q , there exists a set of k parts not satisfying Q among themselves. Define k - Q -sorting of a partition π as the operation of rearranging k parts of π that do not satisfy Q (as a partition of their union) into a new partition that satisfies Q . Often we can prove that

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* Corresponding author.

E-mail address: u8722518@math.nctu.edu.tw (F.K. Hwang).

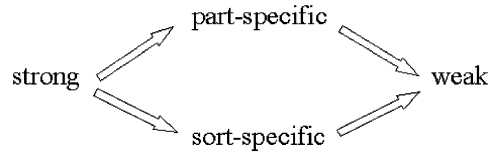


Fig. 1. implications among different levels of sortability.

any k parts can be k - Q -sorted without increasing the cost (assume the problem is to minimize the cost of a partition). Is this sufficient to conclude the existence of a Q -optimal partition? Surely, if we start with an optimal partition not in Q , we can keep on performing k - Q -sorting while optimality is preserved. But there is no guarantee that the iteration process will end with a Q -partition since a loop may be entered. When the local k - Q -sorting does guarantee ending with a global Q -partition satisfying Q in a finite number of iterations, the property Q is called k -*sortable* by Hwang et al. [9]. To prove k -sortability, it suffices to identify a partition statistics which decreases in every k - Q -sorting.

The notion of k -sortability was further expanded by Chang et al. [7]. Note that for a partition not in Q , there may be more than one set of k parts not in Q . So we have a choice of which k parts to Q -sort. Furthermore, for a given set of k parts not in Q , there may exist more than one Q -sorting. Q is called *strongly k -sortable* if choosing any k parts not in Q , and any Q -sorting will eventually lead to a partition in Q . Q is called *part-specific k -sortable* if there always exists a set of k parts such that any Q -sorting will do. Q is called *sort-specific k -sortable* if for any choice of k parts not in Q , there always exists a Q -sorting which works. Q is called *weakly k -sortable* if there always exists a set of k parts not Q and a Q -sorting which works for that k parts. Clearly, we have the implications among different levels of sortability represented in Fig. 1. Evidently, sufficient condition for the four sortability classes is the existence of a statistics that decreases under the corresponding k - Q -sortings.

The *size* of the partition $\pi = \{\pi_1, \dots, \pi_p\}$ is p , and $\{|\pi_1|, \dots, |\pi_p|\}$ is the *shape* of the partition π . Sortability is considered among three types of families of partitions: the *size family* if the size is invariant, the *shape family* if the shape is invariant and the *open family* if there is no constraint (meaning we can sort k parts into k' parts). It is known [7] that if the level is strong or part-specific, then k -open sortable implies k -size sortable implies k -shape sortable. If the level is sort-specific or weak, then the reverse holds.

In general, we refer to the (l, k, t) sortability of a property Q where l refers to level and t refers to type. The (l, k, t) sortability for a list of partition properties of points in \mathbb{R}^1 is completely determined in [7]. However, the problem of partitions over \mathbb{R}^d is much harder. In this paper, we study some partition properties in \mathbb{R}^d .

2. Partition properties in \mathbb{R}^d

Throughout, let d and n be given positive integers. We let $\pi = \{\pi_1, \dots, \pi_p\}$ denote a partition of n points in \mathbb{R}^d into p parts.

A finite set $A \subseteq \mathbb{R}^d$ is said to *penetrate* another finite set $B \subseteq \mathbb{R}^d$ if $A \cap \text{Conv}(B) \neq \emptyset$; in this case we write $A \rightarrow B$.

We study the following properties of partition π .

- (i) *Acyclic (AC)*: There do not exist $k \geq 2$ parts such that $\pi_1 \rightarrow \pi_2 \rightarrow \dots \rightarrow \pi_k \rightarrow \pi_1$.
- (ii) *Disjoint (D)*: For all i, j , $\text{Conv}(\pi_i) \cap \text{Conv}(\pi_j) = \emptyset$.
- (iii) *Nonpenetrating (NP)*: For all i, j , $\pi_i \not\rightarrow \pi_j$.
- (iv) *Noncrossing (NC)*: For all i, j , either $\text{Conv}(\pi_i) \cap \text{Conv}(\pi_j) = \emptyset$ or $\pi_i \subseteq \text{Conv}(\pi_j)$ and $\text{Conv}(\pi_i) \cap \pi_j = \emptyset$ or vice versa.
- (v) *Cone disjoint (CD)*: For all i, j , $\text{Cone}(\pi_i) \cap \text{Cone}(\pi_j) = \emptyset$ (where for a subset $A \subseteq \mathbb{R}^d$, $\text{Cone}(A)$ is the (convex) cone spanned by A with the vertex at the origin).
- (vi) *Sphere noncrossing (SN)*: For any two parts π_i and π_j there exists a sphere $S \subset \mathbb{R}^d$ such that one part is within S and the other outside of S .
- (vii) *Monopoly (M)*: At most one part has more than one point.

It is noted that points in \mathbb{R}^1 are linearly ordered and partition properties in \mathbb{R}^1 are usually studied with respect to the partitioned indices. When $d > 1$, we must consider partitions of the vector themselves.

The one-dimensional partition property “nestedness” can be extended to \mathbb{R}^d by the following characterization:

A partition is nested if it is noncrossing and for any two parts, the convex hull of one of them is contained in the convex hull of the other. However, nested partition are not very interesting in \mathbb{R}^d for $d > 1$ since a point set can have no nested partition for any partition shape, for example, this is the case for the vertices of any polytope (but in \mathbb{R}^1 , there exists a nested partition for any shape). Therefore, we will not consider nested partitions.

We note that over \mathbb{R}^1 , D and NP reduce to consecutiveness (see [7,9]). Also, \mathbb{R}^1 has exactly two cones—the nonnegative reals and the nonpositive reals, hence CD is relevant only if $p \leq 2$. Finally, over \mathbb{R}^1 , NC , AC and SN coincide.

A hyperplane H is said to *separate* two sets A and B if A and B are on different sides of H and neither intersects H . Evidently, D is equivalent to the assertion that every pair of part of π can be separated by a hyperplane.

Theorem 2.1. *The relations between the above seven properties are characterized in Fig. 2.*

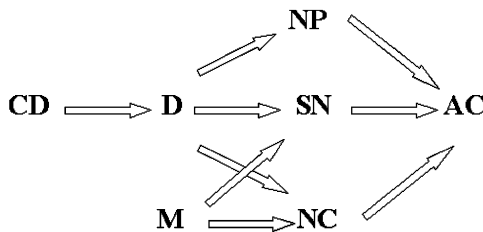
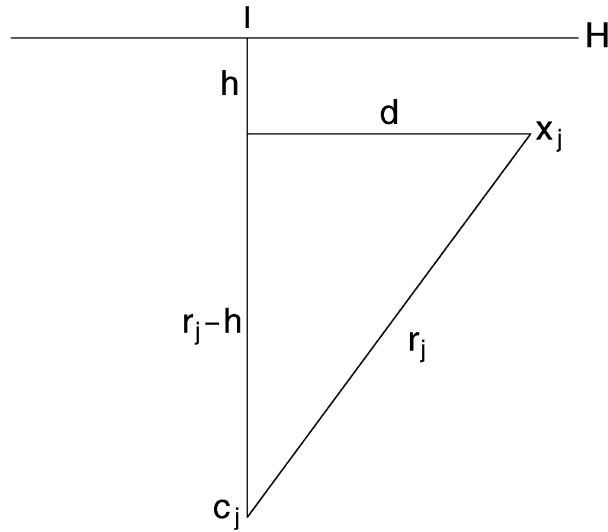
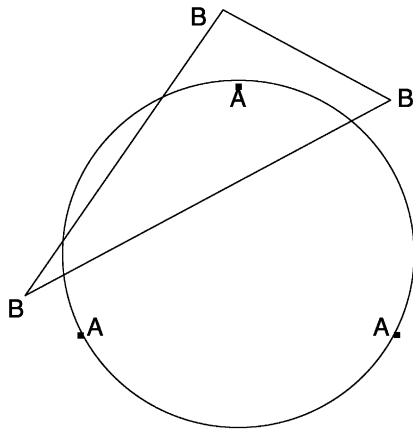


Fig. 2. Relation among 7 properties in \mathbb{R}^d .

Fig. 3. Solving for r_j .Fig. 4. $SN \not\Rightarrow NC$.

Proof. Most implications and nonimplications are pretty straightforward. We only show some not so obvious ones.

(i) $D \Rightarrow SN$. Suppose two parts A and B satisfy D . Then there exists a hyperplane H separating A and B . Let l be a point in H and L a half-line perpendicular to H and containing l . Then for any point x_j on the same side of H as L , there exists a unique point c_j on L such that $\|c_j - x_j\| = \|c_j - l\| \equiv r_j$ (c_j can be obtained by solving for r_j in $r_j^2 - (r_j - h)^2 = d^2$, where solution is $r_j = (d^2 + h^2)/2h$, see Fig. 3). Let $j^* = \arg \max r_j$, $r_* = r_{j^*}$ and $c_* = c_{j^*}$. The sphere with center c_* and radius r_* separates A from B .

(ii) $SN \not\Rightarrow NC$. See Fig. 4.

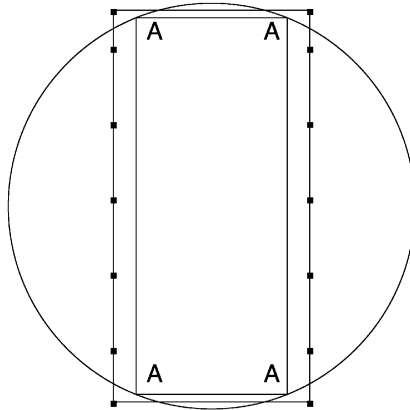


Fig. 5. $NC \not\Rightarrow SN$.

(iii) $NC \not\Rightarrow SN$. See Fig. 5 in which only four marked points are in part *A*, others are in part *B*. \square

3. Enumeration of partitions over \mathbb{R}^d

Let the maximal number of p -partitions over a set of n vectors in \mathbb{R}^d satisfying property Q be denoted $\#_Q(d, p)$. Hwang et al. [8] showed that $\#_D(d, p) = O(n^{d \binom{p}{2}})$; thus, D is polynomially bounded in n for fixed d and p . Recently, Aviran et al. [2] improved the upper bound to $O(n^{d-1})$ for $p=2$ and to $O(n^p)$ for $d=2$. Alon and Onn [1] confirmed that $O(n^{d \binom{p}{2}})$ is best for $p \geq 3$ and $d \geq 3$.

Theorem 3.1. $\#_{CD}(d, p) = O(n^{(d-1) \binom{p}{2}})$.

Proof. The proof is similar to the disjoint case [8]. First consider $p=2$. Then any choice of $d-1$ vectors plus the origin determines a hyperplane which partition the n vectors into two parts (points on the hyperplane are treated as in the disjoint case). Therefore there are $O(n^{d-1})$ 2-partitions. For general p , we have to perform the 2-partition for all $\binom{p}{2}$ choices of pairs of parts. Hence Theorem 3.1. \square

For $d=2$, we give a better upper bound (except for $p=2$).

Theorem 3.2. $\#_{CD}(2, p) \leq \binom{n}{p}$ for $p \geq 2$.

Proof. Ordered the n vectors according to their angles in the polar representation. Let $i_1 \leq i_2 \leq \dots \leq i_n$ denote their indices. We observe that each CD p -partition is determined by indices $j_1 < \dots < j_p$ where $\pi_t = \{i_{j_t+1}, \dots, i_{j_{t+1}}\}$ with $j_{p+1} \equiv j_1$ (but some selection does not yield a CD partition when $i_{j_{t+1}} - i_{j_t+1} > 180^\circ$ for some t). Since the number of choices is bounded by $\binom{n}{p}$, Theorem 3.2 follows. \square

Theorem 3.3. $\#_M(d, p) = \binom{n}{p-1}$ for $n > p$.

Proof. Each choice of $p - 1$ points induces a distinct M p -partition with each point in the chosen set corresponding to a singleton part. \square

Theorem 3.4. $\#_{\text{SN}}(d, p) = O((n + 1)^{d \binom{p}{2}})$.

Proof. Suppose π is an SN p -partition of the set of n distinct d vectors $P = \{a^1, \dots, a^n\}$. Then for every distinct $i, j \in \{1, \dots, p\}$ there is a vector c_{ij} and scalar r_{ij} such that the sphere $S_{ij} = \{x \in \mathbb{R}^d: (c_{ij})^T x = r_{ij}\}$ separates π_i and π_j . Consider such i, j, c_{ij}, r_{ij} and S_{ij} , and without loss of generality assume that

$$\pi_i \subseteq \{x \in \mathbb{R}^d: \|c_{ij} - x\| < r_{ij}\} \quad \text{and} \quad \pi_j \subseteq \{x \in \mathbb{R}^d: \|c_{ij} - x\| \geq r_{ij}\},$$

that is, for $x \in \pi_i$ and $y \in \pi_j$,

$$\|c_{ij}\|^2 - 2(c_{ij})^T x + \|x\|^2 < (r_{ij})^2 < \|c_{ij}\|^2 - 2(c_{ij})^T y + \|y\|^2.$$

For such x and y we then have that

$$(*) \quad (-2(c_{ij})^T, 1) \begin{pmatrix} x \\ \|x\|^2 \end{pmatrix} < (-2(c_{ij})^T, 1) \begin{pmatrix} y \\ \|y\|^2 \end{pmatrix}.$$

Let α_{ij} be the average of the maximum over $x \in \pi_i$ of the left-hand side of (*) and the minimum over $y \in \pi_j$ of the right-hand side of (*). Then

$$(-2(c_{ij})^T, 1) \begin{pmatrix} x \\ \|x\|^2 \end{pmatrix} < \alpha_{ij} < (-2(c_{ij})^T, 1) \begin{pmatrix} y \\ \|y\|^2 \end{pmatrix}$$

for every $x \in \pi_i$ and $y \in \pi_j$.

Every partition π over P defines a partition $\bar{\pi}$ over $\bar{P} = \{(\frac{x}{\|x\|^2}): x \in P\}$ with $\bar{\pi}_j = \{(\frac{x}{\|x\|^2}) \in \bar{P}: x \in \pi_j\}$ for $j = 1, \dots, p$, and the correspondence $\pi \rightarrow \bar{\pi}$ is clearly one-to-one and onto. Further, the above paragraph shows that if π is SN then $\bar{\pi}$ is D , hence the $\#_{\text{SN}}(d, p)$ is bounded by $\#_D(d, p)$ with $n \rightarrow n + 1$, as the latter is $O((n + 1)^{d \binom{p}{2}})$ (see [8]), our proof is complete. \square

Theorem 3.5. The number of NC 2-partitions in \mathbb{R}^2 with shape $(\lfloor \alpha n \rfloor, \lceil (1 - \alpha)n \rceil)$ for some constant α is exponential in n .

Proof. Consider the $n - 3$ points on a circle enclosed with a triangle abc in Fig. 6. Then the partition (π_1, π_2) where π_1 consists of any αn points on the circle satisfies NC, and there are $\binom{n-3}{\lfloor \alpha n \rfloor}$ of them. Note that the convex hull of points on a circle does not cover other points on the circle. \square

Theorem 3.6. The number of NP 2-partitions in \mathbb{R}^2 with shape $(\lfloor \alpha n \rfloor, \lceil (1 - \alpha)n \rceil)$ for some constant α is exponential in n .

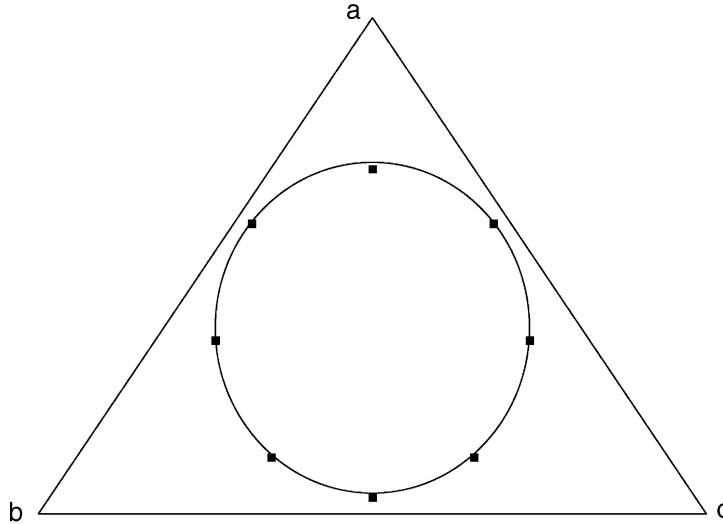


Fig. 6. Exponentially many NC partitions.

Proof. Consider n points on a circle. Then any 2-partition with shape $(\lfloor \alpha n \rfloor, \lfloor (1-\alpha)n \rfloor)$ is NP. \square

Corollary 3.7. *The number of NP and NC shape partitions in \mathbb{R}^d for some shape is exponential for $d \geq 2$. Consequently, the number of p -partitions in \mathbb{R}^d is exponential for all $p \geq 2$ and $d \geq 2$.*

Since $\text{NC} \Rightarrow \text{AC}$, we conclude that the number of AC p -partitions in \mathbb{R}^d is also exponential.

4. Consistency

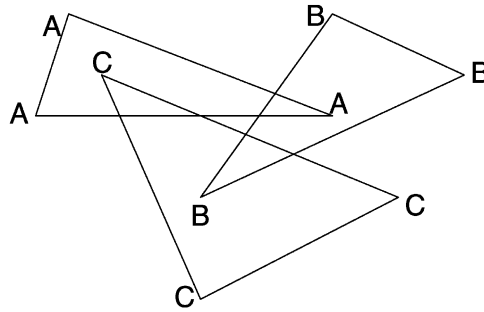
We next study the consistency issue.

Clearly, Q is k -consistent implies Q is k' -consistent for all $k' > k$. The minimum value of k such that Q is k -consistent is referred to as the *minimum consistency index* of Q .

Theorem 4.1. *The minimum consistency index is 2 for D, M, NP, SN, NC, CD, and ∞ for AC.*

Proof. The definitions of D, NP, SN, NC, CD use binary relations, hence they are 2-consistent. The proof for M is trivial.

Liu (private comm.) gave an example with $d=2$ depicted in Fig. 7 which shows that AC is not 2-consistent. By extending the above example to a $(k+1)$ -cycle of

Fig. 7. $A \rightarrow B \rightarrow C \rightarrow A$.

parts where each part penetrates the next part but no other, we can show that AC is not k -consistent for any finite k . \square

5. Sortability

We quote two results from [7].

Lemma 5.1. *If Q is not k -consistent, then Q is not (weak, k , open) sortable.*

Lemma 5.2. *Let Q' and Q be two partition properties such that $Q' \Rightarrow Q$. Then Q' is (sort-specific, k , t) sortable and Q is k -consistent together imply Q is (sort-specific, k , t) sortable.*

By Theorem 4.1 and Lemma 5.1, we know AC is not k -sortable for any l, k, t .

Points in \mathbb{R}^d can be linearly ordered. For example, let (t_1, \dots, t_d) be a permutation of $(1, \dots, d)$. Then we can order the points lexicographically in the order of the coordinates x_{t_1}, \dots, x_{t_d} . A consecutive sorting of points in this linear order will be called a *dimension sorting*. It is easily verified that a dimension sorting always induces a disjoint partition. This linear ordering allows us to inherit the sortability of consecutiveness in \mathbb{R}^1 . However, since the dimension sorting does not account for most of disjoint partition, we treat any result using the dimension sorting as sort-specific.

Theorem 5.3. *D is (sort-specific, k , shape) sortable for all $k \geq 2$.*

Proof. As the above paragraph mentioned, k -dimension-sortings $\forall k \geq 2$ can reach the goal. \square

By Lemma 5.2, we have

Corollary 5.4. *NP , SN and NC are (sort-specific, k , shape) sortable for all $k \geq 2$.*

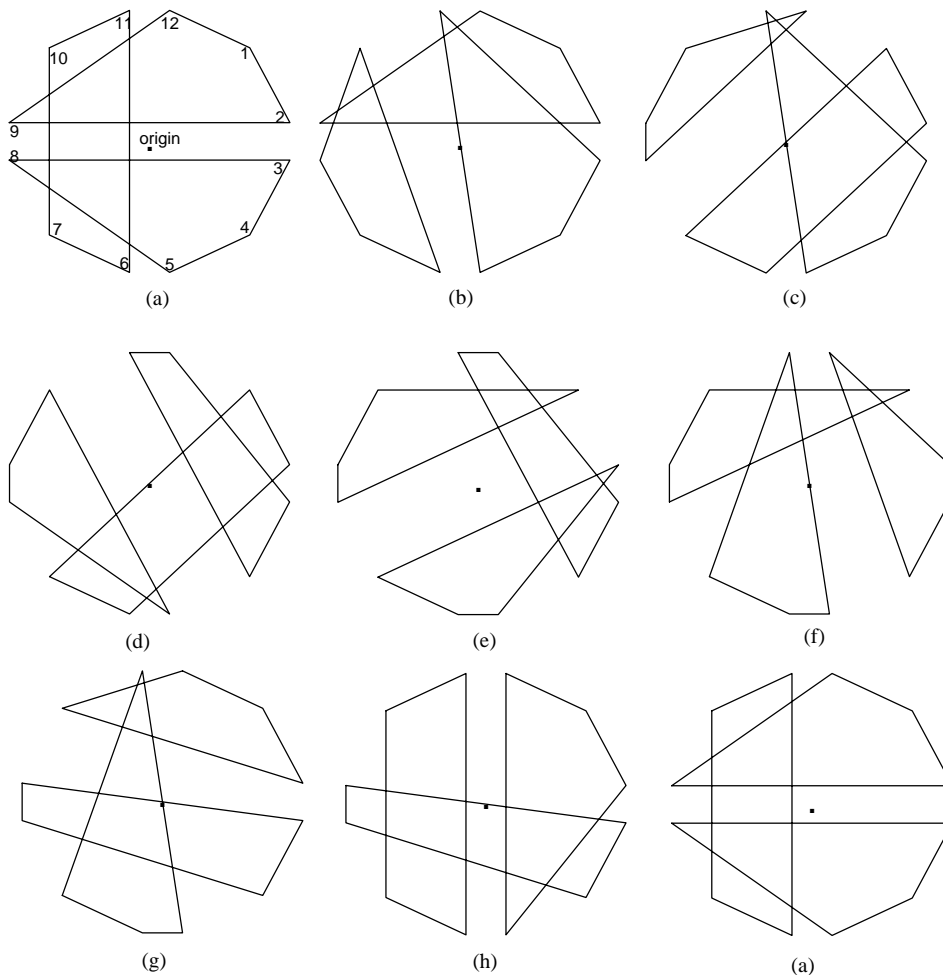


Fig. 8. D and CD are not (strong, 2, shape) sortable.

Note that if a property is not (l, k, t) sortable in \mathbb{R}^k , then it is not so in $\mathbb{R}^{k'}$ where $k' > k$. So in the following discussion, all counterexamples given in \mathbb{R}^2 work also in \mathbb{R}^d for general d .

Theorem 5.5. D is not (strong, 2, shape) sortable.

Proof. See Fig. 8 where partitions of 12 points in \mathbb{R}^2 are considered. The coordinates of points $1, \dots, 12$ are $(5, 5), (7, 1), (7, -1), (5, -5), (1, -7), (-1, -7), (-5, -5), (-7, -1), (-7, 1), (-5, 5), (-1, 7), (1, 7)$, respectively. \square

In the following, we talk about sortability of CD . Note that for five points spread evenly on a circle, no CD -partition exists for the shape $(4, 1)$. This phenomenon can

be generalized to d dimension. Thus when we discuss the shape k -sortability of CD, we assume that the shape is such that if k parts do not satisfy CD, then a CD- k -sorting always exists.

Theorem 5.6. *CD is not (strong, 2, shape) sortable.*

Proof. Note that in Fig. 8, the two nondisjoint parts in each resorting case do not satisfy CD either, and the D -sortings are also CD-sortings since those 12 points lie on a circle with radius equals to $5\sqrt{2}$ and the coordinates of center is $(0,0)$, the two disjoint parts after sorting can be separated by some diameter of the circle which can also be regarded as two rays with angle 180° among them. Hence each D -sorting in Fig. 8 is an CD-sorting. Consequently, Fig. 8 is also a counterexample against the (strong, 2, shape) sortability of CD. \square

Note that the boundary lines of a cone are rays. So if several points lie on the same ray which is a boundary line, then they must all belong to the same part. But sometimes this is impossible for shape-partitions. To avoid complication, we assume that no two points lie on the same ray.

Theorem 5.7. *CD is (sort-specific, k , shape) sortable for all $k \geq 2$.*

Proof. The argument is similar to that of D . The only difference is several preprocessing steps are needed.

First, we transform the rectangular coordinates of every point to its polar coordinates. Then, we project all points to the surface of a sphere.

After these two steps, we can then order these points lexicographically by their polar coordinates so that a consecutive sorting can induce a cone disjoint partition. \square

Theorem 5.8. *CD is not (part-specific, k , shape) sortable for all $k \geq 3$.*

Proof. Consider a shape-partition problem for five points lie on a circle with $p=4$ and shape $(2, 1, 1, 1)$. The class \overline{CD} of partition not satisfying CD can be characterized by the feature that the two elements of part 1 are not consecutive. Take an arbitrary partition in \overline{CD} , say, partition (a) in Fig. 9. There are two 3-parts, parts $(1, 2, 4)$ and parts $(1, 3, 4)$, not satisfying CD. When we sort parts $(1, 2, 4)$ of partition (a), we may obtain partition (b); and when we sort parts $(1, 3, 4)$, we may obtain partition (c). Both partition (b) and (c) are in \overline{CD} . Therefore no part-specific sorting can get us out of \overline{CD} .

To extend this example to general k , another $k-3$ parts each of which contains only one point in it are added between the arc $(1, 3)$ in Fig. 9(a), arc $(4, 1)$ in Fig. 9(b) and arc $(1, 4)$ in Fig. 9(c). Then, similar to the $k=3$ case, sort part $(1, 2, 4, 5, \dots, k+1)$ in (a) we may obtain partition (b); while sort part $(1, 3, 4, 5, \dots, k+1)$ we may obtain partition (c). \square

Corollary 5.9. *CD is not (strong, k , shape) sortable for all $k \geq 3$.*

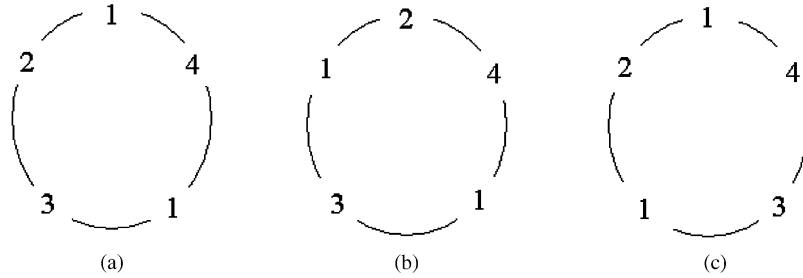


Fig. 9. Cycling among partition not satisfying CD.

Theorem 5.10. *M is (strong, k, open) sortable for all $k \geq 2$.*

Proof. Set $s(\pi)$ to be the number of parts containing more than one element. It is easily seen that $s(\pi)$ is decreased in an M -sorting. \square

Since D and NP reduce to consecutiveness in \mathbb{R}^1 ; NC and SN coincide in \mathbb{R}^1 , then by known results in \mathbb{R}^1 [7], we have:

D and NP are not (strong, k , shape) sortable for all $k \geq 3$.

NC and SN are not (strong, k , shape) sortable for all $k \geq 2$.

Furthermore, since AC is not consistent, it is not (weak, k , open) sortable for all $k \geq 2$.

We present the sortability results of Q by the following tables where the set of k such that (l, k, t) is sortable is given in the (l, t) cell, while \bar{k} denotes the set of k not sortable. For example, $k \geq 2$ means k -sortable for all $k \geq 2$, while $\bar{k} \geq 2$ means not k -sortable for all $k \geq 2$.

For D :

Open	Strong		Sort-specific		
	Size	Shape	Shape	Size	Open
$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$
?	?	?	$k \geq 2$	$k \geq 2$	$k \geq 2$
	Part-specific			Weak	

For NC :

Open	Strong		Sort-specific		
	Size	Shape	Shape	Size	Open
$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$
?	?	?	$k \geq 2$	$k \geq 2$	$k \geq 2$
	Part-specific			Weak	

For NP:

Strong			Sort-specific		
Open	Size	Shape	Shape	Size	Open
$2? \bar{k} \geq 3$	$2? \bar{k} \geq 3$	$2? \bar{k} \geq 3$	$k \geq 2$	$k \geq 2$	$k \geq 2$
?	?	?	$k \geq 2$	$k \geq 2$	$k \geq 2$
Part-specific			Weak		

For SN:

Strong			Sort-specific		
Open	Size	Shape	Shape	Size	Open
$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$
?	?	?	$k \geq 2$	$k \geq 2$	$k \geq 2$
Part-specific			Weak		

For CD:

Strong			Sort-specific		
Open	Size	Shape	Shape	Size	Open
$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$
$2? \bar{k} \geq 3$	$2? \bar{k} \geq 3$	$2? \bar{k} \geq 3$	$k \geq 2$	$k \geq 2$	$k \geq 2$
Part-specific			Weak		

For M:

Strong			Sort-specific		
Open	Size	Shape	Shape	Size	Open
$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$
$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$
Part-specific			Weak		

For AC:

Strong			Sort-specific		
Open	Size	Shape	Shape	Size	Open
$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$
$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$
Part-specific			Weak		

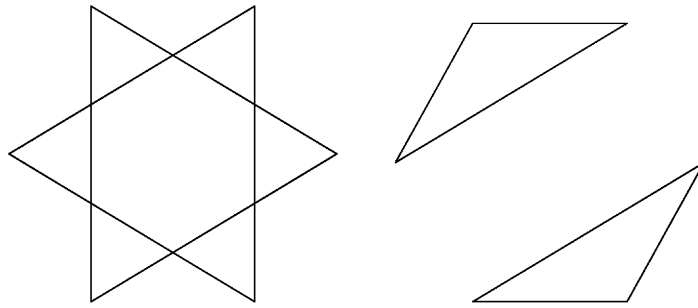


Fig. 10. Invariant diameters in sorting 6 points.

6. An example

Capoyleas et al. [6] considered the problem of partitioning points in \mathbb{R}^d into two parts π_1 and π_2 to minimize some monotone function of the diameters (or radius) of π_1 and π_2 . To be definite, let it be the maximum of the two diameters.

Let the *diagonal sorting* for disjointness mean the following:

(i) If $\text{Conv}(\pi_1) \subset \text{Conv}(\pi_2)$, then $\text{Conv}(\pi'_1)$ is a single (arbitrary) vertex of $\text{Conv}(\pi_2)$, and $\text{Conv}(\pi'_2)$ consists of all other points.

(ii) If $\text{Conv}(\pi_1) \cap \text{Conv}(\pi_2)$ is a line or a point, then $\pi'_1 = \pi_1 \setminus \text{Conv}(\pi_2)$ and $\pi'_2 = \{p: p \in \text{Conv}(\pi_2)\}$.

(iii) If $P = \text{Conv}(\pi_1) \cap \text{Conv}(\pi_2)$ is a polygon, then π'_1 and π'_2 are separated by an (arbitrary) diagonal of P .

Note that a diagonal sorting does not guarantee a decrease of the maximum diameter. In Fig. 10, the maximum diameter remains the same after the sorting:

Nevertheless, Capoyleas, Rote and Woeginger proved that disjointness is (diagonal sorting, 2, size)-sortable (not stated in these terms) by setting $s(\pi)$ to be the sum of perimeters of $\text{Conv}(\pi_1)$ and $\text{Conv}(\pi_2)$. Therefore there exists a disjoint optimal partition.

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