ACADEMIC PRESS

Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT

Journal of Multivariate Analysis 84 (2003) 351–368

Journal of Multivariate Analysis

http://www.elsevier.com/locate/jmva

Exact tests in simple growth curve models and one-way ANOVA with equicorrelation error structure

Shu-Hui Lin and Jack C. Lee*

Institute of Statistics, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, Taiwan

Received 10 October 2000

Abstract

We consider exact tests with several equicorrelation error structures and combination of equicorrelation covariance structures in simple growth curve model having single or multiple treatments and in one-way ANOVA model. Exact inferences using generalized p-values are obtained. Tests for equal treatment effects under equal equicorrelation error term and for unequal equicorrelation error terms are also developed. Two examples are given to illustrate the importance of our results. According to our findings, we would be better off dropping the assumption of equal variance when the heteroscedasticity is serious. Therefore, tests based on generalized p-values without the assumption of equal variance are much more powerful than tests with this assumption.

AMS 1991 subject classifications: 62H15; 62J10; 62F03

Keywords: Combination of equicorrelation covariances; Generalized p-values; Repeated measure; Testing the fixed treatment effects

1. Introduction

Approximate inferences have been extensively used in applied sciences involving regression analysis. However, the estimation and testing procedures based on inefficient ordinary least squares (OLS) estimates incur inefficient forecasts. On the

*Corresponding author.
E-mail address: jclee@stat.nctu.edu.tw (J.C. Lee).

doi:10.1016/S0047-259X(02)00060-X

other hand, the approximate inference always has the worse size and power problems for the estimation of blockwise heteroscedasticity. If the effects of the variances are sufficiently strong it may lead to an inappropriate standard regression technique. As Krutchkoff [4] pointed out, transformations cannot adjust the heteroscedasticity problem if the data are already normal. If one attempts to solve the problem by performing weighted least squares regression with estimated variances, the required size of the test can become much larger than the intended level. Therefore, exact and size-guaranteed tests as well as intervals for models involving variance components deserved further attention.

Models of treatments or regression coefficients for analyzing Gaussian repeated measures have been studied intensively in the last three decades. More recently, tests and confidence intervals obtained using generalized p-values have been shown, via simulation, to possess great size and power performances (*cf.* [1–3,7,10,12]). Consider repeated observations taken over time for each of the several subjects. The exact test based on sufficient statistics for regression models with an intraclass correlation structure is possible by using generalized p-values, although it is unavailable by conventional methods. Chi and Weerahandi [2] and Weerahandi and Berger [11] developed exact tests for simple growth curve models with usual independent residual error structure by generalized p-values. It is very valuable to examine the exact tests using generalized p-values on regression coefficients and comparing a number of treatments for simple growth curve models with equicorrelation error structure. This covariance structure could be useful for growth curve data when the observations are a mixture of several populations [5]. However, this area needs further investigation. In particular, these results could be important for the extension to the situations in which the variance varies over time.

In this paper, we will show that such an exact inference is possible using generalized p-values for a simple growth curve model with equicorrelation error terms or any finite combination of equicorrelation covariance structures, which is widely used in biomedical and pharmaceutical research areas for which only approximate methods are available. Tests for equal treatment effects under unequal equicorrelation error terms are also developed. Our approach is based on generalized p-values. Section 2 presents the simple growth curve model of complete repeated observations for a single treatment. A test for equality of one or several regression coefficients to some prespecified values using generalized p-values is derived. Section 3 extends the model to the unbalanced multiple treatments. Two tests of equal treatment effects using generalized p-values under the equal equicorrelation error variance assumption as well as unequal equicorrelation error variances are developed. Section 4 is devoted to one-way layout model. Two illustrative examples are given in Section 5, and some conclusions are provided in Section 6.

2. Single treatment group in a simple growth curve model

We consider a single treatment group in a simple linear growth curve model. Let Y_{jt} denote the measurement on unit j at time t, α_j be the random effects associated with unit j, $\boldsymbol{X}_t$ be a set of covariates, $\boldsymbol{\beta}$ be the fixed effects of dimension K, and ε_{jt} be the error term with equicorrelation structure. Then, the model can be formulated as

$$Y_{jt} = \boldsymbol{X}_t'\boldsymbol{\beta} + \alpha_j + \varepsilon_{jt}, \quad j = 1, \ldots, J; \;\; t = 1, \ldots, T. \tag{2.1}$$

In matrix notations

$$\boldsymbol{Y}_j = \boldsymbol{X\beta} + \alpha_j \mathbf{1}_T + \boldsymbol{\varepsilon}_j, \tag{2.2}$$

where $\boldsymbol{X} = (X_1, X_2, \ldots, X_T)'$ is the $T \times K$ design matrix whose first column is 1_T so that a fixed intercept is included and $\boldsymbol{Y}_j = (Y_{j1}, Y_{j2}, \ldots, Y_{jT})'$, $\varepsilon_j \sim N(\mathbf{0}, \boldsymbol{\Sigma}_e)$, $\alpha_j \sim N(0, \sigma_\alpha^2)$ vary independently and $\boldsymbol{\Sigma}_e = \sigma_e^2[(1-\rho)\boldsymbol{I}_T + \rho \mathbf{1}_T \mathbf{1}_T']$, with $\frac{-1}{T-1} < \rho < 1$ which is unknown.

Hence, the covariance matrix of $\boldsymbol{Y}_j$ is

$$\begin{aligned} Cov(\boldsymbol{Y}_j) &= \boldsymbol{\Sigma} = \sigma_\alpha^2 \mathbf{1}_T \mathbf{1}_T' + \boldsymbol{\Sigma}_e = \sigma_e^2(1-\rho)\boldsymbol{I}_T + (\rho\sigma_e^2 + \sigma_\alpha^2)\mathbf{1}_T\mathbf{1}_T' \\ &= \begin{pmatrix} \sigma_e^2 + \sigma_\alpha^2 & \rho\sigma_e^2 + \sigma_\alpha^2 & \cdots & \rho\sigma_e^2 + \sigma_\alpha^2 \\ \rho\sigma_e^2 + \sigma_\alpha^2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho\sigma_e^2 + \sigma_\alpha^2 \\ \rho\sigma_e^2 + \sigma_\alpha^2 & \cdots & \rho\sigma_e^2 + \sigma_\alpha^2 & \sigma_e^2 + \sigma_\alpha^2 \end{pmatrix}. \end{aligned} \tag{2.3}$$

An exact test for a simpler model with $\rho = 0$ was developed using generalized p-values by Weerahandi and Berger [11]. Furthermore, we will derive an exact test for the more challenging model with any finite combination of equicorrelation covariance structures with $\frac{-1}{T-1} < \rho < 1$ by using generalized p-values.

In testing the fixed treatment effects $\mathrm{H}_0 : \beta_k \leqslant \beta_k^*$, where β_k^* is a pre-specified value and $k = 1, \ldots, K$, let $\phi^2 = \sigma_e^2(1-\rho) + T(\rho\sigma_e^2 + \sigma_\alpha^2)$, then

$$\boldsymbol{\Sigma}^{-1} = [\sigma_e^2(1-\rho)]^{-1}\left[\boldsymbol{I}_T - \frac{\phi^2 - \sigma_e^2(1-\rho)}{T\phi^2}\mathbf{1}_T\mathbf{1}_T'\right] \tag{2.4}$$

is a function of $\sigma_e^2(1-\rho)$ and ϕ^2. The residual sum of squares

$$SSE = \sum_{t=1}^{T}\sum_{j=1}^{J}(Y_{jt} - \boldsymbol{X}_t'\hat{\boldsymbol{\beta}})^2 = S_{e,\rho}^2 + S_{\rho,\alpha}^2,$$

where $S_{e,\rho}^2 = \sum_{t=1}^T \sum_{j=1}^J [Y_{jt} - \boldsymbol{X}_t'\hat{\boldsymbol{\beta}} - (\bar{Y}_{j.} - \bar{Y}_{..})]^2$ and $S_{\rho,\alpha}^2 = T\sum_{j=1}^J (\bar{Y}_{j.} - \bar{Y}_{..})^2$ are distributed as

$$W_1 = \frac{S_{e,\rho}^2}{\sigma_e^2(1-\rho)} \sim \chi^2_{J(T-1)-K+1} \quad \text{and} \quad W_2 = \frac{S_{\rho,\alpha}^2}{\phi^2} \sim \chi^2_{J-1}$$

with $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\bar{\boldsymbol{Y}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\bar{\boldsymbol{Y}} \sim N(\boldsymbol{\beta}, \frac{(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}}{J})$, $\bar{\boldsymbol{Y}} = \frac{1}{J}\sum_{j=1}^J \boldsymbol{Y}_j$, $\bar{Y}_{j.} = \frac{1}{T}\sum_{t=1}^T Y_{jt}$, and $\bar{Y}_{..} = \frac{1}{J}\sum_{j=1}^J \bar{Y}_{j.}$.

It is noted that $\hat{\boldsymbol{\beta}}$ is also the maximum likelihood estimator (MLE) and the generalized least squares estimator (GLSE) [6].

Define

$$\boldsymbol{S}_k = S(\sigma_e^2(1-\rho), \phi^2) = \left(\frac{(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}_{kk}}{J} \right)^{\frac{1}{2}}, \quad k = 1, \ldots, K. \tag{2.5}$$

Since (2.4) depends on $\sigma_e^2(1-\rho)$ and ϕ^2, but not on ρ by itself, the generalized p-value for testing $\mathrm{H}_0 : \beta_k \leqslant \beta_k^*$ can be deduced in a similar manner from the $\rho = 0$ case with S_e^2 replaced by $S_{e,\rho}^2$ and $S_{e;\tau}^2$ replaced by $S_{\rho,\alpha}^2$. The generalized p-value can be expressed as

$$\begin{aligned} p &= P\left\{ \frac{\hat{\beta}_k - \beta_k}{S(\sigma_e^2(1-\rho), \phi^2)} \geqslant \frac{b_k - \beta_k^*}{S\left(\sigma_e^2(1-\rho)\frac{s_{e,\rho}^2}{S_{e,\rho}^2}, \phi^2 \frac{s_{\rho,\alpha}^2}{S_{\rho,\alpha}^2} \right)} \right\} \\ &= 1 - E_B\left\{ F_{t_\upsilon}\left[\frac{b_k - \beta_k^*}{S\left(\frac{s_{e,\rho}^2}{B}, \frac{s_{\rho,\alpha}^2}{1-B} \right)} \sqrt{\upsilon} \right] \right\}, \end{aligned} \tag{2.6}$$

where $b_k, s_{e,\rho}^2, s_{\rho,\alpha}^2$ are the observed values of $\hat{\beta}_k, S_{e,\rho}^2, S_{\rho,\alpha}^2$, respectively, $Z \sim N(0,1)$, $B = \frac{W_1}{W} \sim Beta(\frac{J(T-1)-K+1}{2}, \frac{J-1}{2})$, $W = W_1 + W_2 \sim \chi^2_{JT-K}$, and F_{t_υ} is the cumulative distribution function (cdf) of the t-distribution with $\upsilon = JT - K$ degrees of freedom.

Generalized confidence intervals for the parameters can be deduced from (2.6) as well. For example, the generalized $100\gamma\%$ confidence interval of β_k that is symmetric (and shortest) about the point estimate b_k is $[b_k - c_{(1+\gamma)/2}(s_{e,\rho}^2, s_{\rho,\alpha}^2), b_k + c_{(1+\gamma)/2}(s_{e,\rho}^2, s_{\rho,\alpha}^2)]$, with $c_{(1+\gamma)/2} = c_{(1+\gamma)/2}(s_{e,\rho}^2, s_{\rho,\alpha}^2)$ satisfying the equation

$$\begin{aligned} \frac{1+\gamma}{2} &= P\left\{ \frac{\hat{\beta}_k - \beta_k}{S(\sigma_e^2(1-\rho), \phi^2)} \leqslant \frac{c_{(1+\gamma)/2}}{S\left(\frac{s_{e,\rho}^2}{B}, \frac{s_{\rho,\alpha}^2}{1-B} \right)} \right\} \\ &= E_B\left\{ F_{t_\upsilon}\left[\frac{c_{(1+\gamma)/2}}{S\left(\frac{s_{e,\rho}^2}{B}, \frac{s_{\rho,\alpha}^2}{1-B} \right)} \sqrt{\upsilon} \right] \right\} \end{aligned} \tag{2.7}$$

Note that the model can be extended to any finite combination of equicorrelation covariance structures. Let

$$\boldsymbol{Y}_j = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\alpha}_j + \boldsymbol{\varepsilon}_j, \tag{2.8}$$

where $\boldsymbol{\alpha}_j \sim N(\boldsymbol{0}, \boldsymbol{\Sigma}_\alpha)$ and $\boldsymbol{\varepsilon}_j \sim N(\boldsymbol{0}, \boldsymbol{\Sigma}_e)$ vary independently, $\boldsymbol{\Sigma}_\alpha = \sigma_\alpha^2[(1-\rho_1)\boldsymbol{I}_T + \rho_1 \boldsymbol{1}_T\boldsymbol{1}_T']$ and $\boldsymbol{\Sigma}_e = \sigma_e^2[(1-\rho_2)\boldsymbol{I}_T + \rho_2\boldsymbol{1}_T\boldsymbol{1}_T']$. Then, $Cov(\boldsymbol{Y}_j) = \boldsymbol{\Sigma} = [(1-\rho_1)\sigma_\alpha^2 + (1-\rho_2)\sigma_e^2]\boldsymbol{I}_T + (\rho_1\sigma_\alpha^2 + \rho_2\sigma_e^2)\boldsymbol{1}_T\boldsymbol{1}_T'$ and $\boldsymbol{\Sigma}^{-1} = \frac{1}{\phi_2^2}[\boldsymbol{I}_T - \frac{\phi_1^2 - \phi_2^2}{T\phi_1^2}\boldsymbol{1}_T\boldsymbol{1}_T']$ where $\phi_1^2 = T(\rho_1\sigma_\alpha^2 + \rho_2\sigma_e^2) + \phi_2^2$, $\phi_2^2 = (1-\rho_1)\sigma_\alpha^2 + (1-\rho_2)\sigma_e^2$ with $\frac{-1}{T-1} < \rho_1 < 1$ and $\frac{-1}{T-1} < \rho_2 < 1$. With ϕ^2

replaced by ϕ_1^2 and $\sigma_e^2(1-\rho)$ by ϕ_2^2 in (2.4), we can obtain the same equation as (2.6) to express the generalized p-value for the extended model (2.8).

It is noted that the generalized F-test for testing the hypothesis of the form $\mathrm{H}_0 : \boldsymbol{\beta} = \boldsymbol{\beta}^*$, where $\boldsymbol{\beta}^*$ is pre-specified, can be carried out in a similar manner. Define

$$\tilde{S}(\sigma_e^2(1-\rho), \phi^2) = \frac{(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}}{J},$$

the generalized p-values appropriate for testing the null hypothesis is given by

$$p = 1 - E_B\left\{F_{K,JT-K}\left[\frac{JT-K}{K}\left\{(\boldsymbol{b}-\boldsymbol{\beta}^*)'\left[\tilde{S}\left(\frac{s_{e,\rho}^2}{B}, \frac{s_{\rho,\alpha}^2}{1-B}\right)\right]^{-1}(\boldsymbol{b}-\boldsymbol{\beta}^*)\right\}\right]\right\},$$

where $\boldsymbol{b}$ is the observed vector of $\hat{\boldsymbol{\beta}}$ and the expectation is with respect to the beta random variable defined by (2.6).

3. Multiple treatments group in the simple growth curve models

We next consider the test for the fixed treatment effects $\mathrm{H}_0 : \boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_I$ in the following model. Let Y_{ijt} denote the measurement at the tth time point on the jth subject for the ith treatment, α_{ij} be the random effects, $\boldsymbol{\beta}_i$ be the fixed effects of dimension K, and ε_{ijt} be the error term. Then, the model can be formulated as

$$Y_{ijt} = \boldsymbol{X}_{it}'\boldsymbol{\beta}_i + \alpha_{ij} + \varepsilon_{ijt}, \quad i = 1, \ldots, I;\ j = 1, \ldots, n_i;\ t = 1, \ldots, T. \tag{3.1}$$

In matrix notations

$$\boldsymbol{Y}_{ij} = \boldsymbol{X}_i\boldsymbol{\beta}_i + \alpha_{ij}\boldsymbol{1}_T + \boldsymbol{\varepsilon}_{ij}, \tag{3.2}$$

where $\boldsymbol{Y}_{ij} = (Y_{ij1}, Y_{ij2}, \ldots, Y_{ijT})'$, $\boldsymbol{X}_i = (\boldsymbol{X}_{i1}, \boldsymbol{X}_{i2}, \ldots, \boldsymbol{X}_{iT})'$, $\boldsymbol{\varepsilon}_{ij} \sim N(\boldsymbol{0}, \boldsymbol{\Sigma}_{ei})$, $\alpha_{ij} \sim N(0, \sigma_\alpha^2)$ vary independently, and $\boldsymbol{\Sigma}_{ei} = \sigma_i^2[(1-\rho)\boldsymbol{I}_T + \rho\boldsymbol{1}_T\boldsymbol{1}_T']$.

The covariance matrix of $\boldsymbol{Y}_{ij}$ is

$$\begin{aligned} Cov(\boldsymbol{Y}_{ij}) &= \boldsymbol{\Sigma}_i = \sigma_\alpha^2\boldsymbol{1}_T\boldsymbol{1}_T' + \boldsymbol{\Sigma}_{ei} \\ &= \sigma_i^2(1-\rho)\boldsymbol{I}_T + (\rho\sigma_i^2 + \sigma_\alpha^2)\boldsymbol{1}_T\boldsymbol{1}_T', \quad i = 1, \ldots, I \end{aligned} \tag{3.3}$$

and

$$\boldsymbol{\Sigma}_i^{-1} = [\sigma_i^2(1-\rho)]^{-1}\left[I_T - \frac{\phi_i^2 - \sigma_i^2(1-\rho)}{T\phi_i^2}\boldsymbol{1}_T\boldsymbol{1}_T'\right], \tag{3.4}$$

with $\phi_i^2 = \sigma_i^2(1-\rho) + T(\rho\sigma_i^2 + \sigma_\alpha^2)$.

The test was considered by Chi and Weerahandi [2] by generalized p-values with $\rho = 0$, $\boldsymbol{X}_i = \boldsymbol{X}$, $n_i = n$ for $i = 1, \ldots, I$. We will extend the test to the unbalanced data and unequal design matrices for both equal equicorrelation error term and heteroscedastic error variances.

3.1. Equal equicorrelation error term

In this case, $\sigma_1^2 = \cdots = \sigma_I^2 = \sigma_e^2$, then

$$Cov(\boldsymbol{Y}_{ij}) = \boldsymbol{\Sigma} = \sigma_\alpha^2 \mathbf{1}_T \mathbf{1}_T' + \boldsymbol{\Sigma}_e = \sigma_e^2(1-\rho)\mathbf{I}_T + (\rho\sigma_e^2 + \sigma_\alpha^2)\mathbf{1}_T\mathbf{1}_T', \quad i = 1, \dots, I \tag{3.5}$$

and

$$\boldsymbol{\Sigma}^{-1} = [\sigma_e^2(1-\rho)]^{-1}\left[\boldsymbol{I}_T - \frac{\phi^2 - \sigma_e^2(1-\rho)}{T\phi^2}\mathbf{1}_T\mathbf{1}_T'\right], \tag{3.6}$$

with $\phi^2 = \sigma_e^2(1-\rho) + T(\rho\sigma_e^2 + \sigma_\alpha^2)$.

The residual sum of squares is

$$SSE = \sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{t=1}^{T}(Y_{ijt} - \boldsymbol{X}_{it}'\hat{\boldsymbol{\beta}}_i)^2 = S_{e,\rho}^2 + S_{\rho,\alpha}^2,$$

where $S_{e,\rho}^2 = \sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{t=1}^{T}[Y_{ijt} - \boldsymbol{X}_{it}'\hat{\boldsymbol{\beta}}_i - (\bar{Y}_{ij.} - \bar{Y}_{i..})]^2$ and $S_{\rho,\alpha}^2 = T\sum_{i=1}^{I}\sum_{j=1}^{n_i}(\bar{Y}_{ij.} - \bar{Y}_{i..})^2$ are distributed as

$$V = \frac{S_{e,\rho}^2}{\sigma_e^2(1-\rho)} \sim \chi_{\upsilon_1}^2 \quad \text{and} \quad U = \frac{S_{\rho,\alpha}^2}{\phi^2} \sim \chi_{\upsilon_2}^2, \tag{3.7}$$

with $\upsilon_1 = T\sum_{i=1}^{I} n_i - IK - \sum_{i=1}^{I} n_i + I$, $\upsilon_2 = \sum_{i=1}^{I} n_i - I$. A proof of Eq. (3.7) is given in the appendix. The proof of the independence of U and V is also provided.

Furthermore,

$$\begin{aligned}\hat{\boldsymbol{\beta}}_i &= (\boldsymbol{X}_i'\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i)^{-1}\boldsymbol{X}_i'\boldsymbol{\Sigma}^{-1}\bar{\boldsymbol{Y}}_{i.} \\ &= (\boldsymbol{X}_i'\boldsymbol{X}_i)^{-1}\boldsymbol{X}_i'\bar{\boldsymbol{Y}}_{i.} \sim N\left(\boldsymbol{\beta}_i, \frac{(\boldsymbol{X}_i'\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i)^{-1}}{n_i}\right),\end{aligned} \tag{3.8}$$

where $\bar{\boldsymbol{Y}}_{i.} = \frac{1}{n_i}\sum_{j=1}^{n_i}\boldsymbol{Y}_{ij}$, $\bar{\boldsymbol{Y}}_{ij.} = \frac{1}{T}\sum_{t=1}^{T} Y_{ijt}$, and $\bar{Y}_{i..} = \frac{1}{n_i}\sum_{j=1}^{n_i}\bar{Y}_{ij.}$, $i = 1, \dots, I$. Letting $\Sigma^{-1/2}$ denote a positive definite square root matrix of Σ^{-1} and pre-multiplying both sides of Eq. (3.2), we can rewrite the model as

$$\tilde{\boldsymbol{Y}}_{ij} = \tilde{\boldsymbol{X}}_i\boldsymbol{\beta}_i + \boldsymbol{e}_{ij},$$

where $e_{ij} \sim N(0, I_T)$. Let $\tilde{S}_{12}^2(\sigma_e^2(1-\rho), \phi^2)$ be the standardized residual sum of squares under null hypothesis and $\tilde{S}_{1,2}^2(\sigma_e^2(1-\rho), \phi^2) = \tilde{S}_1^2(\sigma_e^2(1-\rho),^2) + \cdots + \tilde{S}_I^2(\sigma_e^2(1-\rho), \phi^2)$ be the standardized residual sum of squares under the alternative based on the standardized model, then $\tilde{S}_{12}^2(\sigma_e^2(1-\rho), \phi^2) \sim \chi^2_{T(\sum_{i=1}^{I} n_i)-K}$ and $\tilde{S}_{1,2}^2(\sigma_e^2(1-\rho), \phi^2) \sim \chi^2_{T(\sum_{i=1}^{I} n_i)-IK}$. The potential extreme region defined by the inequality

$$\left\{\tilde{S}_{12}^2(\sigma_e^2(1-\rho), \phi^2) \geqslant \tilde{s}_{12}^2\left(\frac{s_{e,\rho}^2}{S_{e,\rho}^2/\sigma_e^2(1-\rho)}, \frac{s_{\rho,\alpha}^2}{S_{\rho,\alpha}^2/\phi^2}\right)\right\} \tag{3.9}$$

is a well-defined subset of the sample space, where $s^2_{e,\rho}, s^2_{\rho,\alpha}$ are the observed values of $S^2_{e,\rho}, S^2_{\rho,\alpha}$, respectively, and the observed sample point falls on the boundary of the extreme region. Therefore, the generalized p-value for testing $\mathrm{H}_0 : \beta_1 = \cdots = \beta_I$ can be calculated as

$$p = Pr\left\{\frac{\tilde{S}^2_{12}(\sigma^2_e(1-\rho), \phi^2)}{U+V} \geqslant \tilde{s}^2_{12}\left(\frac{s^2_{e,\rho}}{V/(U+V)}, \frac{s^2_{\rho,\alpha}}{U/(U+V)}\right)\right\}$$
$$= 1 - E_B\left\{F_{r_1,r_2}\left[\frac{r_2}{r_1}\left\{\tilde{s}^2_{12}\left(\frac{s^2_{e,\rho}}{B}, \frac{s^2_{\rho,\alpha}}{1-B}\right) - 1\right\}\right]\right\}, \quad (3.10)$$

where F_{r_1,r_2} is the cdf of the F distribution with degrees of freedom $r_1 = (I-1)K$ and $r_2 = T\sum_{i=1}^{I} n_i - IK$. The expectation is with respect to the beta random variable

$$B = \frac{V}{U+V} \sim Beta\left(\frac{\upsilon_1}{2}, \frac{\upsilon_2}{2}\right),$$

where $\upsilon_1 = T\sum_{i=1}^{I} n_i - IK - \sum_{i=1}^{I} n_i + I$, $\upsilon_2 = \sum_{i=1}^{I} n_i - I$.

3.2. Heteroscedastic error variances

In the case of the heteroscedastic error variances, we have

$$Cov(\boldsymbol{Y}_{ij}) = \boldsymbol{\Sigma}_i = \sigma^2_\alpha \mathbf{1}_T\mathbf{1}'_T + \boldsymbol{\Sigma}_{ei}$$
$$= \sigma^2_i(1-\rho)\mathbf{I}_T + (\rho\sigma^2_i + \sigma^2_\alpha)\mathbf{1}_T\mathbf{1}'_T, \quad i = 1, \ldots, I \quad (3.11)$$

and

$$\boldsymbol{\Sigma}_i^{-1} = [\sigma^2_i(1-\rho)]^{-1}\left[\boldsymbol{I}_T - \frac{\phi^2_i - \sigma^2_i(1-\rho)}{T\phi^2_i}\mathbf{1}_T\mathbf{1}'_T\right], \quad (3.12)$$

with $\phi^2_i = \sigma^2_i(1-\rho) + T(\rho\sigma^2_i + \sigma^2_\alpha)$. The residual sum of squares is

$$SSE = \sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{t=1}^{T}(Y_{ijt} - \boldsymbol{X}'_{it}\hat{\boldsymbol{\beta}}_i)^2 = \sum_{i=1}^{I} S^2_i + \sum_{i=1}^{I} \Lambda^2_i,$$

where $S^2_i = \sum_{j=1}^{n_i}\sum_{t=1}^{T}[Y_{ijt} - \boldsymbol{X}'_{it}\hat{\boldsymbol{\beta}}_i - (\bar{Y}_{ij.} - \bar{Y}_{i..})]^2$ and $\Lambda^2_i = T\sum_{j=1}^{n_i}(\bar{Y}_{ij.} - \bar{Y}_{i..})^2$, $i = 1, \ldots, I$ are distributed as $V_i = \frac{S^2_i}{\sigma^2_i(1-\rho)} \sim \chi^2_{Tn_i-K-n_i+1}$ and $U_i = \frac{\Lambda^2_i}{\phi^2_i} \sim \chi^2_{n_i-1}$. It is easy to see that

$$U = \sum_{i=1}^{I} U_i \sim \chi^2_\upsilon \quad \text{and} \quad H = \sum_{i=1}^{I}(U_i + V_i) \sim \chi^2_\omega,$$

where

$$\upsilon = \sum_{i=1}^{I} n_i - I, \quad \omega = T\sum_{i=1}^{I} n_i - IK. \quad (3.13)$$

Furthermore,

$$\hat{\boldsymbol{\beta}}_i = (\boldsymbol{X}_i'\boldsymbol{X}_i)^{-1}\boldsymbol{X}_i'\bar{\boldsymbol{Y}}_{i.} \sim N\left(\boldsymbol{\beta}_i, \frac{(\boldsymbol{X}_i'\boldsymbol{\Sigma}_i^{-1}\boldsymbol{X}_i)^{-1}}{n_i}\right), \tag{3.14}$$

with $\bar{\boldsymbol{Y}}_{i.} = \frac{1}{n_i}\sum_{j=1}^{n_i}\boldsymbol{Y}_{ij}$, $\bar{Y}_{ij.} = \frac{1}{T}\sum_{t=1}^{T} Y_{ijt}$, and $\bar{Y}_{i..} = \frac{1}{n_i}\sum_{j=1}^{n_i}\bar{Y}_{ij.}$ $i = 1, \ldots, I$.

Let $\tilde{S}_{12}^2(\sigma_1^2(1-\rho), \ldots, \sigma_I^2(1-\rho), \phi_1^2, \ldots, \phi_I^2)$ be the standardized residual sum of squares under null hypothesis and $\tilde{S}_{1,2}^2(\sigma_1^2(1-\rho), \ldots, \sigma_I^2(1-\rho), \phi_1^2, \ldots, \phi_I^2) = \tilde{S}_1^2(\sigma_1^2(1-\rho), \phi_1^2) + \cdots + \tilde{S}_I^2(\sigma_I^2(1-\rho), \phi_I^2)$ be the standardized residual sum of squares under the alternative. The potential extreme region for testing $\mathrm{H}_0 : \boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_I$ is

$$\begin{aligned}&\big\{\tilde{S}_{12}^2(\sigma_1^2(1-\rho), \ldots, \sigma_I^2(1-\rho), \phi_1^2, \ldots, \phi_I^2)\\ &\quad \geqslant \tilde{s}_{12}^2\left(\frac{s_1^2}{S_1^2/\sigma_1^2(1-\rho)}, \ldots, \frac{s_I^2}{S_I^2/\sigma_I^2(1-\rho)}, \frac{\lambda_1^2}{\Lambda_1^2/\phi_1^2}, \ldots, \frac{\lambda_I^2}{\Lambda_I^2/\phi_I^2}\right)\right\}.\end{aligned} \tag{3.15}$$

The observed sample point $(s_1^2, \ldots, s_I^2, \lambda_1^2, \ldots, \lambda_I^2)$ falls on the boundary of this set. The generalized p-value for testing $\mathrm{H}_0 : \beta_1 = \cdots = \beta_I$ can be expressed as

$$\begin{aligned}p &= Pr\left\{\frac{\tilde{S}_{12}^2(\sigma_1^2(1-\rho), \ldots, \sigma_I^2(1-\rho), \phi_1^2, \ldots, \phi_I^2)}{H}\right.\\ &\quad \left.\geqslant \tilde{s}_{12}^2\left(\frac{s_1^2}{V_1/H}, \ldots, \frac{s_I^2}{V_I/H}, \frac{\lambda_1^2}{U_1/H}, \ldots, \frac{\lambda_I^2}{U_I/H}\right)\right\}\\ &= Pr\left\{\frac{\tilde{S}_{12}^2 - \tilde{S}_{1,2}^2}{\tilde{S}_{1,2}^2} \geqslant \tilde{s}_{12}^2\left(\frac{s_1^2}{V_1/H}, \ldots, \frac{s_I^2}{V_I/H}, \frac{\lambda_1^2}{U_1/H}, \ldots, \frac{\lambda_I^2}{U_I/H}\right) - 1\right\}\\ &= 1 - E_{B_1,\ldots,B_{2I-1}}\left\{F_{\upsilon_1,\upsilon_2}\left[\frac{\upsilon_2}{\upsilon_1}\left\{\tilde{s}_{12}^2\left(\frac{s_1^2}{B_1B_2\cdots B_{2I-1}}, \ldots, \frac{s_I^2}{(1-B_{I-1})B_I\cdots B_{2I-1}},\right.\right.\right.\right.\\ &\quad \left.\left.\left.\left.\frac{\lambda_1^2}{(1-B_I)B_{I+1}\cdots B_{2I-1}}, \ldots, \frac{\lambda_I^2}{(1-B_{2I-1})}\right) - 1\right\}\right]\right\},\end{aligned} \tag{3.16}$$

where $s_1^2, \ldots, s_I^2, \lambda_1^2, \ldots, \lambda_I^2$ are the observed values of $S_1^2, \ldots, S_I^2, \Lambda_1^2, \ldots, \Lambda_I^2$, $W_t = \begin{cases} V_t, & \mathrm{t} \leqslant I \\ U_{t-I}, & \mathrm{t} > I \end{cases}$, $q_t = \begin{cases} Tn_t - n_t - K + 1, & \mathrm{t} \leqslant I \\ n_{t-I} - 1, & \mathrm{t} > I \end{cases}$, $W_t \sim \chi_{q_t}^2, t = 1, \ldots, 2I$, $F_{\upsilon_1,\upsilon_2}$ is the cdf of the F distribution with degrees of freedom $\upsilon_1 = (I-1)K$ and $\upsilon_2 = T\sum_{i=1}^{I} n_i - IK$, and the expectation is taken with respect to the independent beta random variables

$$B_t = \frac{\sum_{i=1}^{t} W_i}{\sum_{i=1}^{t+1} W_i} \sim Beta\left(\frac{\sum_{i=1}^{t} q_i}{2}, \frac{q_{t+1}}{2}\right), \quad t = 1, \ldots, 2I-1.$$

Moreover, the model can be extended to the following:

$$\boldsymbol{Y}_{ij} = \boldsymbol{X}_i\boldsymbol{\beta}_i + \boldsymbol{\alpha}_{ij} + \boldsymbol{\varepsilon}_{ij},$$

where $\boldsymbol{\alpha}_{ij} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\alpha)$ and $\boldsymbol{\varepsilon}_{ij} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{ei})$ vary independently with $\boldsymbol{\Sigma}_\alpha = \sigma_\alpha^2[(1-\rho_1)\boldsymbol{I}_T + \rho_1 \mathbf{1}_T \mathbf{1}_T']$, $\boldsymbol{\Sigma}_{ei} = \sigma_i^2[(1-\rho_2)\boldsymbol{I}_T + \rho_2 \mathbf{1}_T \mathbf{1}_T']$, and $\frac{-1}{T-1} < \rho_1 < 1$, $\frac{-1}{T-1} < \rho_2 < 1$. Let $\phi_{1i}^2 = T(\rho_1\sigma_\alpha^2 + \rho_2\sigma_i^2) + \phi_{2i}^2$, $\phi_{2i}^2 = (1-\rho_1)\sigma_\alpha^2 + (1-\rho_2)\sigma_i^2$ and $\phi_1^2 = T(\rho_1\sigma_\alpha^2 + \rho_2\sigma_e^2) + \phi_2^2$, $\phi_2^2 = (1-\rho_1)\sigma_\alpha^2 + (1-\rho_2)\sigma_e^2$, then

$$Cov(\boldsymbol{Y}_{ij}) = \boldsymbol{\Sigma}_i = [(1-\rho_1)\sigma_\alpha^2 + (1-\rho_2)\sigma_i^2]\boldsymbol{I}_T + (\rho_1\sigma_\alpha^2 + \rho_2\sigma_i^2)\mathbf{1}_T\mathbf{1}_T',$$

and $\boldsymbol{\Sigma}_i^{-1} = \frac{1}{\phi_{2i}^2}[\boldsymbol{I}_T - \frac{\phi_{1i}^2 - \phi_{2i}^2}{T\phi_{1i}^2}\mathbf{1}_T\mathbf{1}_T']$ under unequal covariances.

It is important to point out that replacing ϕ^2 by ϕ_1^2 and $\sigma_e^2(1-\rho)$ by ϕ_2^2 in (3.6) for the equal equicorrelation covariance model, then we can get the same equation as (3.10) to express the generalized p-value for testing $\mathrm{H}_0 : \boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_I$. Similarly, replacing ϕ_i^2 by ϕ_{1i}^2 and $\sigma_i^2(1-\rho)$ by ϕ_{2i}^2 in (3.12) for the unequal equicorrelation covariance model, then we can get the same equation as (3.16) for testing $\mathrm{H}_0 : \boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_I$.

3.3. Multiple comparisons and generalized confidence region

In this subsection, we demonstrate multiple comparisons in the situation when the hypothesis of equal treatment effect has been rejected at a certain nominal level. Based on the Scheffe's methods, the multiple comparisons as well as pre-planned comparisons can be extended to the cases when the variances are unequal.

Consider the null hypothesis $\mathrm{H}_0 : \sum_{i=1}^I c_i\boldsymbol{\beta}_i = \mathbf{0}$ for all $c_i \in R$ such that $\sum_{i=1}^I c_i = 0$. It is evident that

$$\sum_{i=1}^I c_i\hat{\boldsymbol{\beta}}_i \sim N\left(\sum_{i=1}^I c_i\boldsymbol{\beta}_i, \sum_{i=1}^I c_i^2 \frac{(\boldsymbol{X}_i'\boldsymbol{\Sigma}_i^{-1}\boldsymbol{X}_i)^{-1}}{n_i}\right). \tag{3.17}$$

Define $S_i(\sigma_i^2(1-\rho), \phi_i^2) = \frac{(\boldsymbol{X}_i'\boldsymbol{\Sigma}_i^{-1}\boldsymbol{X}_i)^{-1}}{n_i}$, $i = 1, \ldots, I$, then the generalized p-value for testing the null hypothesis can be obtained by

$$\begin{aligned} p = 1 - E_{B_1,\ldots,B_{2I-1}}\Bigg\{ F_{\upsilon_1,\upsilon_2}\Bigg[\frac{\upsilon_2}{\upsilon_1}\Bigg\{\left(\sum_{i=1}^I c_i\boldsymbol{b}_i\right)'\left[\sum_{i=1}^I c_i^2 S_i\left(\frac{s_i^2}{R_i}, \frac{\lambda_i^2}{R_{I+i}}\right)\right]^{-1} \\ \times \left(\sum_{i=1}^I c_i\boldsymbol{b}_i\right)\Bigg\}\Bigg]\Bigg\}, \end{aligned} \tag{3.18}$$

where s_i^2, λ_i^2, $\boldsymbol{b}_i$ are the observed values of S_i^2, Λ_i^2, $\hat{\boldsymbol{\beta}}_i$, respectively, $F_{\upsilon_1,\upsilon_2}$ is the cdf of the F distribution with degrees of freedom $\upsilon_1 = (I-1)K$ and $\upsilon_2 = T\sum_{i=1}^I n_i - IK$, and the expectation is taken with respect to the random variables

$$\begin{aligned} &R_1 = B_1 B_2 \cdots B_{2I-1} \quad \text{and} \quad R_{2I} = 1 - B_{2I-1}, \\ &R_i = (1 - B_{i-1})B_i \cdots B_{2I-1}, \quad i = 2, \ldots, 2I-1, \end{aligned} \tag{3.19}$$

where B_i, $i = 1, \ldots, 2I-1$ are the independent beta random variables defined by (3.16). It is noted that the solution to the problem of pre-planned comparisons can be deduced from (3.18) by simply replacing υ_1 by the appropriate degrees of freedom ($\upsilon_1 = K$ if just one comparison is planned).

Moreover, a set of simultaneous generalized confidence region with confidence coefficient $1-\alpha$ for the linear contrasts, $\sum_{i=1}^{I} c_i\boldsymbol{\beta}_i$ with $\sum_{i=1}^{I} c_i = 0$, can be obtained by solving the equation

$$1-\alpha = E_{B_1,\ldots,B_{2I-1}}\left\{F_{\upsilon_1,\upsilon_2}\left[\frac{\upsilon_2}{\upsilon_1}\left(\sum_{i=1}^{I} c_i\boldsymbol{b}_i - \sum_{i=1}^{I} c_i\boldsymbol{\beta}_i\right)'\right.\right.$$
$$\left.\left.\times\left[\sum_{i=1}^{I} c_i^2 S_i\left(\frac{s_i^2}{R_i},\frac{\lambda_i^2}{R_{I+i}}\right)\right]^{-1}\left(\sum_{i=1}^{I} c_i\boldsymbol{b}_i - \sum_{i=1}^{I} c_i\boldsymbol{\beta}_i\right)\right\}, \tag{3.20}$$

where s_i^2, λ_i^2, $\boldsymbol{b}_i$ are the observed values of S_i^2, Λ_i^2, $\hat{\boldsymbol{\beta}}_i$, respectively, $F_{\upsilon_1,\upsilon_2}(\alpha)$ is the $(1-\alpha)$th quantile of the F distribution with degrees of freedom $\upsilon_1 = (I-1)K$ and $\upsilon_2 = T\sum_{i=1}^{I} n_i - IK$, and $R_i,\ i = 1,\ldots,2I$ are defined in (3.19).

4. One-way ANOVA under heteroscedastic error variances

For the special case $\boldsymbol{X} = \mathbf{1}_T,\ \beta_i = \mu_i, i = 1,\ldots,I,$ then the model $\boldsymbol{Y}_{ij} = \boldsymbol{X}_i\boldsymbol{\beta}_i + \alpha_{ij}\mathbf{1}_T + \boldsymbol{\varepsilon}_{ij}$ becomes

$$\boldsymbol{Y}_{ij} = \mu_i\mathbf{1}_T + \alpha_i\mathbf{1}_T + \boldsymbol{\varepsilon}_{ij} = \begin{pmatrix}\mu_i\\ \vdots\\ \mu_i\end{pmatrix} + \begin{pmatrix}\alpha_i\\ \vdots\\ \alpha_i\end{pmatrix} + \begin{pmatrix}\varepsilon_{ij1}\\ \vdots\\ \varepsilon_{ij1}\end{pmatrix},$$
$$i = 1,\ldots,I;\quad j = 1,\ldots,n_i;\ \ t = 1,\ldots,T, \tag{4.1}$$

where $\boldsymbol{\varepsilon}_{ij} \sim N(\mathbf{0},\boldsymbol{\Sigma}_{ei})$, $\boldsymbol{\Sigma}_{ei} = \sigma_i^2[(1-\rho)\boldsymbol{I}_T + \rho\mathbf{1}_T\mathbf{1}_T']$, and $Cov(\boldsymbol{Y}_{ij}) = \boldsymbol{\Sigma}_i$ with

$$\boldsymbol{\Sigma}_i^{-1} = [\sigma_i^2(1-\rho)]^{-1}\left[\boldsymbol{I}_T - \frac{\phi_i^2 - \sigma_i^2(1-\rho)}{T\phi_i^2}\mathbf{1}_T\mathbf{1}_T'\right] \tag{4.2}$$

and $\phi_i^2 = \sigma_i^2(1-\rho) + T(\rho\sigma_i^2 + \sigma_\alpha^2)$. Since $\hat{\mu}_i = \bar{Y}_{i..} \sim N(\mu_i, \frac{\phi_i^2}{Tn_i})$, the test of the fixed treatment effects $\mathrm{H}_0: \mu_1 = \cdots = \mu_I$ can be reduced to the one-way ANOVA under the heteroscedastic error variance case. As discussed in [9], ANOVA under heteroscedasticity (in which $\rho = 0$ and $\alpha_i = 0,\ i = 1,\ldots,I$) can be solved by the generalized p-values. In this paper, the treatment of model (4.1) is an extension of the one-way ANOVA under heteroscedastic error variances.

4.1. The generalized F-test for one-way ANOVA

The residual sum of squares

$$\Lambda_i^2 = T\sum_{j=1}^{n_i}(\bar{Y}_{ij.} - \bar{Y}_{i..})]^2,\quad i = 1,\ldots,I$$

is distributed as

$$\boldsymbol{U}_i = \frac{\Lambda_i^2}{\phi_i^2} \sim \chi^2_{n_i - 1}.$$

Now $\boldsymbol{U} = \sum_{i=1}^{I} \boldsymbol{U}_i \sim \chi^2_{\sum_{i=1}^{I} n_i - I}$ and define the standardized between-group sum of squares

$$\tilde{S}_B^2 = \tilde{S}_B^2(\phi_1^2, \ldots, \phi_I^2) = \sum_{i=1}^{I} \frac{Tn_i}{\phi_i^2} \bar{Y}_{i..}^2 - \frac{\left(\sum_{i=1}^{I} \frac{Tn_i}{\phi_i^2} \bar{Y}_{i..}\right)^2}{\sum_{i=1}^{I} \left(\frac{Tn_i}{\phi_i^2}\right)}, \tag{4.3}$$

where $\phi_i^2 = \sigma_i^2(1-\rho) + T(\rho\sigma_i^2 + \sigma_\alpha^2)$ and $\tilde{s}_B^2$ is the observed value of $\tilde{S}_B^2$.

The potential extreme region for $\mathrm{H}_0 : \mu_1 = \cdots = \mu_I$ is

$$\left\{ \tilde{S}_B^2(\phi_1^2, \ldots, \phi_I^2) \geqslant \tilde{s}_B^2 \left(\frac{\lambda_1^2}{\Lambda_1^2/\phi_1^2}, \ldots, \frac{\lambda_I^2}{\Lambda_I^2/\phi_I^2} \right) \right\}. \tag{4.4}$$

The observed sample point $(\lambda_1^2, \ldots, \lambda_I^2)$ of $(\Lambda_1^2, \ldots, \Lambda_I^2)$ falls on the boundary of this set. The generalized p-value can be expressed as

$$\begin{aligned} p &= Pr\left\{ \tilde{S}_B^2(\phi_1^2, \ldots, \phi_I^2) \geqslant \tilde{s}_B^2 \left(\frac{\lambda_1^2}{\Lambda_1^2/\phi_1^2}, \ldots, \frac{\lambda_I^2}{\Lambda_I^2/\phi_I^2} \right) \right\} \\ &= Pr\left\{ \frac{\tilde{S}_B^2(\phi_1^2, \ldots, \phi_I^2)}{U} \geqslant \tilde{s}_B^2 \left(\frac{\lambda_1^2}{U_1/U}, \ldots, \frac{\lambda_I^2}{U_I/U} \right) \right\} \\ &= 1 - E_{B_1, \ldots, B_{I-1}} \left\{ F_{\upsilon_1, \upsilon_2} \left[\frac{\upsilon_2}{\upsilon_1} \left\{ \tilde{s}_B^2 \left(\frac{\lambda_1^2}{B_1 B_2 \cdots B_{I-1}}, \ldots, \right.\right.\right.\right. \\ &\qquad \left.\left.\left.\left. \frac{\lambda_k^2}{(1 - B_{k-1}) B_k \cdots B_{I-1}}, \ldots, \frac{\lambda_I^2}{(1 - B_{I-1})} \right) \right\} \right] \right\}, \end{aligned} \tag{4.5}$$

where $F_{\upsilon_1, \upsilon_2}$ is the cdf of the F distribution with degrees of freedom $\upsilon_1 = I - 1$ and $\upsilon_2 = \sum_{i=1}^{I} n_i - I$. The expectation is taken with respect to the independent beta random variables

$$B_t = \frac{\sum_{i=1}^{t} U_i}{\sum_{i=1}^{t+1} U_i} \sim Beta\left(\frac{\sum_{i=1}^{t} n_i - t}{2}, \frac{n_{t+1} - 1}{2} \right), \quad k = 1, \ldots, I. \tag{4.6}$$

Note that the model can also be extended to the following: $\boldsymbol{Y}_{ij} = \mu_i \mathbf{1}_T + \boldsymbol{\alpha}_{ij} + \boldsymbol{\varepsilon}_{ij}$, with $\boldsymbol{\alpha}_{ij} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\alpha)$ and $\boldsymbol{\varepsilon}_{ij} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{ei})$ vary independently where $\boldsymbol{\Sigma}_\alpha = \sigma_\alpha^2[(1-\rho_1)\boldsymbol{I}_T + \rho_1 \mathbf{1}_T \mathbf{1}_T']$, $\boldsymbol{\Sigma}_{ei} = \sigma_i^2[(1-\rho_2)\boldsymbol{I}_T + \rho_2 \mathbf{1}_T \mathbf{1}_T']$, with $\frac{-1}{T-1} < \rho_1 < 1$ and $\frac{-1}{T-1} < \rho_2 < 1$.

Let $\phi_{1i}^2 = T(\rho_1 \sigma_\alpha^2 + \rho_2 \sigma_i^2) + \phi_{2i}^2$ and $\phi_{2i}^2 = (1-\rho_1)\sigma_\alpha^2 + (1-\rho_2)\sigma_i^2$. Then, replacing ϕ_i^2 by ϕ_{1i}^2 and $\sigma_i^2(1-\rho)$ by ϕ_{2i}^2 in (4.2), we can get the same result as (4.5) for testing $\mathrm{H}_0 : \mu_1 = \cdots = \mu_I$.

4.2. Behrens–Fisher problem

In case $I = 2$, the test for $\mathrm{H}_0 : \mu_1 = \mu_2$ can be treated as an extended Behrens–Fisher problem.

The generalized p-value for testing $\mathrm{H}_0 : \mu_1 = \mu_2$ is given by

$$p = 1 - E_B\left\{F_{1,n_1+n_2-2}\left[(n_1 + n_2 - 2)\tilde{s}_B^2\left(\frac{\lambda_1^2}{B}, \frac{\lambda_2^2}{(1-B)}\right)\right]\right\}, \tag{4.7}$$

where

$$\tilde{s}_B^2\left(\frac{\lambda_1^2}{B}, \frac{\lambda_2^2}{(1-B)}\right) = (\bar{Y}_{1..} - \bar{Y}_{2..})^2\left(\frac{\lambda_1^2/B}{Tn_1} + \frac{\lambda_2^2/(1-B)}{Tn_2}\right)^{-1}, \tag{4.8}$$

F_{1,n_1+n_2-2} is the cdf of the F distribution with 1 and $n_1 + n_2 - 2$ degrees of freedom and the expectation is with respect to the beta random variable

$$B = \frac{U_1}{U} \sim Beta\left(\frac{n_1 - 1}{2}, \frac{n_2 - 1}{2}\right).$$

Note that the test can be reduced to Tsui and Weerahandi [8] by generalized p-values with $\rho = 0$, and $T = 1$.

5. Illustrative examples

Two numerical examples are given to illustrate the advantages of the proposed tests when the assumption of equal variance is violated. In these examples, data are generated from normal distributions under the assumed models (4.1) and (3.1), respectively. In the first example, we consider the performance of the F-test with respect to Type I error in the one-way ANOVA with unequal error variances. In the second example, we consider performance of the F-test with respect to Type II error in the growth curve model involving three treatment groups with heteroscedastic error variances.

5.1. Example 1

First, the data are generated assuming model (4.1) $\boldsymbol{Y}_{ij} = \mu_i \mathbf{1}_T + \alpha_i \mathbf{1}_T + \boldsymbol{\varepsilon}_{ij}$ with $i = 1, \ldots, 5$; $t = 1, 2$; $\sigma_\alpha = 2$; $\rho = 0.2$. We will consider the performance of the F-test with respect to Type I error in the one-way ANOVA with unequal error variances; that is, we will consider the rejection probability when the null hypothesis is actually true. The larger the p-value, the stronger the evidence to support the null hypothesis. Since the test has the exact specified size when the variances are equal, its performance should be studied when the variances are quite different. We shall demonstrate the test with a set of simulated data from normal distributions. The problem of comparing three means is considered. The mean of each distribution is taken to be twenty so that the null hypothesis $\mathrm{H}_0 : \mu_1 = \mu_2 = \mu_3 = 20$ is true. Table 1 shows the results of a simulated experiment in which data are generated from normal

Table 1
Summary statistics of simulated data

Treatments	n_i	σ_i	ϕ_i	$\bar{X}_{i..}$	λ_i^2
A	12	1	2.28	19.5	6.01
B	10	2	2.97	19.72	10.15
C	8	3	3.85	19.10	39.15
D	6	4	4.82	22.43	77.22
E	4	4	4.82	18.79	37.04

Table 2
P-values with and without assumptions of equal variance for one-way ANOVA

Treatments compared	p_e	p_u
A, B, and C	0.4401	0.5699(0.003)
A, B, and D	0.0003	0.0952(0.004)
A, B, and E	0.2928	0.6246(0.003)
A, C, and D	0.0014	0.0980(0.005)
A, C, and E	0.5950	0.7150(0.003)
A, D, and E	0.0055	0.1055(0.005)
B, C, and D	0.0040	0.0920(0.004)
B, C, and E	0.4821	0.5365(0.003)
B, D, and E	0.0150	0.1186(0.005)
C, D, and E	0.0270	0.0960(0.005)

distributions with mean 20 and various values of n_i, σ_i, ϕ_i, $i = 1, \ldots, 5$, as indicated. Meanwhile, Table 1 provides the summary statistics, namely treatment means $\bar{X}_{i..}$ and residual sum of squares λ_i^2 corresponding to each value of ϕ_i.

Three treatments out of five in Table 1 are compared at a time. When the classical ANOVA F-test is used under the assumption of equal variance, the p-values are denoted by p_e. On the other hand, when (4.5) is used without the assumption of equal variance, the p-values are denoted by p_u. It is noted that p_u are computed by Monte Carlo integration with 5000 sets of beta random numbers. The results are displayed in Table 2. The absolute error bounds, shown within parentheses, are computed by $3 \times \frac{\sigma_p}{\sqrt{N}}$ where N is the number of replications and σ_p is the simulated standard deviation. With probability 0.999, the estimated p-values are accurate up to the error bound. It is noted that the classical F-test tends to reject the null hypothesis if the ϕ^2 of the treatments are substantially different. Even in comparing treatments B, C and D, this test suggests that we have strong evidence to reject the null hypothesis although the data are generated with the hypothesis being true. It is important to point out that p_u is much bigger than the corresponding p_e. Thus, compared with the classical F-test, the procedure of generalized p-values provides a more efficient way to detect the significance of mean differences.

Table 3
P-values with and without assumptions of equal variance for testing equality of fixed treatment effects

$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$	$\boldsymbol{\beta}_3$	σ_1	σ_2	σ_3	σ_α	p_e	p_u
$\binom{10}{1}$	$\binom{12}{1}$	$\binom{14}{1}$	1	2	3	3	0.08378(0.00178)	0.00340(0.00032)
$\binom{10}{1}$	$\binom{12}{1}$	$\binom{14}{1}$	1	2	3	2	0.28063(0.00301)	0.01255(0.00272)
$\binom{10}{1}$	$\binom{12}{1}$	$\binom{14}{1}$	1	2	3	1	0.31505(0.00191)	0.03415(0.00121)
$\binom{10}{1}$	$\binom{12}{1}$	$\binom{14}{1}$	2	3	4	1	0.10687(0.00255)	0.03124(0.00186)
$\binom{10}{1}$	$\binom{12}{1}$	$\binom{14}{1}$	2	3	4	2	0.07623(0.00269)	0.01367(0.00092)
$\binom{10}{1}$	$\binom{12}{1}$	$\binom{14}{1}$	2	3	4	4	0.06596(0.00075)	0.03330(0.00262)

5.2. Example 2

The data are generated assuming the model $\boldsymbol{Y}_{ij} = \boldsymbol{X}_i\boldsymbol{\beta}_i + \alpha_{ij}\mathbf{1}_T + \boldsymbol{\varepsilon}_{ij}$ with $i = 1, 2, 3$; $n_1 = 10$; $n_2 = 8$; $n_3 = 6$; $T = 3$; $\rho = 0.2$; and $\boldsymbol{X}_i' = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \end{pmatrix}$ for $i = 1, 2, 3$. We will consider the performance of the F-test with respect to Type II error in the growth curve model involving the fixed effects $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, and $\boldsymbol{\beta}_3$ of the three treatment groups and with heteroscedastic error variances; that is, we will consider the rejection probability when the alternative hypothesis is true. The smaller the p-valves, the stronger the evidence to reject the null hypothesis. The p-valves for testing the fixed treatment effects, using formula (3.10), with the assumption of equal variance are denoted by p_e. The p-values, computed by Monte Carlo integration based on 5000 sets of beta random numbers by using formula (3.16) without the assumption of equal variance, are denoted by p_u. The results are displayed in Table 3. The absolute error bounds calculated as above are shown within parentheses. The p-values suggest that when the heteroscedasticity is serious, the test without the assumption of equal variance is much more powerful than the test with the assumption of equal variance. Especially, when σ_α^2 does not dominate the other variances and $\sigma_1^2, \sigma_2^2, \sigma_3^2$ are significantly different, then the p-values without the assumption of equal variances are quite efficient to test the fixed treatment effects.

6. Conclusions

In this paper, we consider several equicorrelation error structures and combination of equicorrelation covariance structures in simple growth curve model with single or multiple treatments and in one-way ANOVA model, which are widely used in many research areas. Unfortunately, so far only approximate methods are available. We show that exact inferences based on generalized p-values can be obtained. According to our findings, the assumption of equal variance is not reasonable in many applications, and in such situation, we would be better off dropping this assumption when the heteroscedasticity is serious.

Acknowledgments

We would thank two referees and the associate editor for their constructive comments and suggestions which lead to a substantial improvement of the paper and R.S. Wang and Y.L. Hsu for some helpful discussions regarding the appendix.

Appendix. Proof of Eq. (3.7)

We will provide the proof of (3.7) and show the independence of U and V in this appendix. Consider the model in (3.1). Recall that $\hat{\boldsymbol{\beta}}_i$ is the maximum likelihood estimator (MLE) of $\boldsymbol{\beta}_i$ with $\hat{\boldsymbol{\beta}}_i = (\boldsymbol{X}_i'\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i)^{-1}\boldsymbol{X}_i'\boldsymbol{\Sigma}^{-1}\bar{\boldsymbol{Y}}_{i.} = (\boldsymbol{X}_i'\boldsymbol{X}_i)^{-1}\boldsymbol{X}_i'\bar{\boldsymbol{Y}}_{i.}$. In view of the point estimates, we have the following decomposition:

$$Y_{ijt} - \boldsymbol{X}_{it}'\hat{\boldsymbol{\beta}}_i = (Y_{ijt} - \boldsymbol{X}_{it}'\hat{\boldsymbol{\beta}}_i - (\bar{Y}_{ij.} - \bar{Y}_{i..})) + (\bar{Y}_{ij.} - \bar{Y}_{i..}), \tag{A.1}$$

with the sum of cross product

$$\begin{aligned}
&\sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{t=1}^{T}(Y_{ijt} - \boldsymbol{X}_{it}'\hat{\boldsymbol{\beta}}_i - (\bar{Y}_{ij.} - \bar{Y}_{i..}))(\bar{Y}_{ij.} - \bar{Y}_{i..})\\
&\quad = \sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{t=1}^{T}(Y_{ijt} - \boldsymbol{X}_{it}'\hat{\boldsymbol{\beta}}_i)(\bar{Y}_{ij.} - \bar{Y}_{i..}) - \sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{t=1}^{T}(\bar{Y}_{ij.} - \bar{Y}_{i..})^2\\
&\quad = \sum_{i=1}^{I}\sum_{j=1}^{n_i}(T\bar{Y}_{ij.} - \mathbf{1}_T'\boldsymbol{X}_i\hat{\boldsymbol{\beta}}_i)(\bar{Y}_{ij.} - \bar{Y}_{i..}) - \sum_{i=1}^{I}\sum_{j=1}^{n_i}T(\bar{Y}_{ij.} - \bar{Y}_{i..})^2\\
&\quad = \sum_{i=1}^{I}(T\bar{Y}_{i..} - \mathbf{1}_T'\boldsymbol{X}_i\hat{\boldsymbol{\beta}}_i)\sum_{j=1}^{n_i}(\bar{Y}_{ij.} - \bar{Y}_{i..}) = 0.
\end{aligned} \tag{A.2}$$

Hence, on summation of squared terms in (A.1) yields the orthogonal decomposition of sums of squares,

$$SSE = \sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{t=1}^{T}(Y_{ijt} - \boldsymbol{X}_{it}'\hat{\boldsymbol{\beta}}_i)^2 = S_{e,\rho}^2 + S_{\rho,\alpha}^2, \tag{A.3}$$

with $S^2_{e,\rho} = \sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{t=1}^{T}[Y_{ijt} - \boldsymbol{X}'_{it}\hat{\boldsymbol{\beta}}_i - (\bar{Y}_{ij.} - \bar{Y}_{i..})]^2$ and $S^2_{\rho,\alpha} = T\sum_{i=1}^{I}\sum_{j=1}^{n_i}(\bar{Y}_{ij.} - \bar{Y}_{i..})^2$.

Due to the orthogonality of vectors on which they are based, the sums of squares on the right-hand side of (A.3) are independently distributed. The distribution of each sum of squares can be derived by averaging (3.2) appropriately and using (3.5), (3.6) and (3.8). For example, it follows from the fact that

$$\bar{Y}_{ij.} = \frac{1}{T}\mathbf{1}'_T\boldsymbol{Y}_{ij} \sim N\left(\frac{1}{T}\mathbf{1}'_T\boldsymbol{X}_i\boldsymbol{\beta}_i, \frac{1}{T^2}\mathbf{1}'_T\boldsymbol{\Sigma}\mathbf{1}_T\right), \quad i = 1, \ldots, I;$$

$$j = 1, 2, \ldots, n_i, \tag{A.4}$$

with $\frac{1}{T^2}\mathbf{1}'_T\boldsymbol{\Sigma}\mathbf{1}_T = \frac{1}{T}\phi^2$ and $\phi^2 = \sigma^2_e(1-\rho) + T(\rho\sigma^2_e + \sigma^2_\alpha)$,

$$\frac{\sum_{j=1}^{n_i}(\bar{Y}_{ij.} - \bar{Y}_{i..})^2}{\phi^2/T} \sim \chi^2_{n_i-1}. \tag{A.5}$$

Since the I populations are independent,

$$U = \frac{S^2_{\rho,\alpha}}{\phi^2} = \frac{T\sum_{i=1}^{I}\sum_{j=1}^{n_i}(\bar{Y}_{ij.} - \bar{Y}_{i..})^2}{\phi^2} \sim \chi^2_{\sum_{i=1}^{I} n_i - I}. \tag{A.6}$$

On the other hand,

$$\boldsymbol{Y}_{ij} - \boldsymbol{X}_i\hat{\beta}_i = \left[\boldsymbol{I} - \frac{1}{n_i}\boldsymbol{X}_i(\boldsymbol{X}'_i\boldsymbol{X}_i)^{-1}\boldsymbol{X}'_i\right]\boldsymbol{Y}_{ij}$$

$$- \sum_{l=1,\, l\neq j}^{n_i}\frac{1}{n_i}\boldsymbol{X}_i(\boldsymbol{X}'_i\boldsymbol{X}_i)^{-1}\boldsymbol{X}'_i\boldsymbol{Y}_{il} \sim N(\mathbf{0}, \Sigma^*), \tag{A.7}$$

where $\boldsymbol{\Sigma}^* = Cov(\boldsymbol{Y}_{ij} - \boldsymbol{X}_i\hat{\boldsymbol{\beta}}_i) = \boldsymbol{\Sigma} - \frac{1}{n_i}\boldsymbol{\Sigma}\boldsymbol{P} - \frac{1}{n_i}\boldsymbol{P}\boldsymbol{\Sigma} + \frac{1}{n_i}\boldsymbol{P}\,\boldsymbol{\Sigma}\boldsymbol{P}$ and $\boldsymbol{P} = \boldsymbol{X}_i(\boldsymbol{X}'_i\boldsymbol{X}_i)^{-1}\boldsymbol{X}'_i$ is an idempotent matrix of rank K. The residual sum of square, SSE, can be also expressed as

$$SSE = \sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{t=1}^{T}(Y_{ijt} - \boldsymbol{X}'_{it}\hat{\boldsymbol{\beta}}_i)^2 = \sum_{i=1}^{I}\sum_{j=1}^{n_i}(\boldsymbol{Y}_{ij} - \boldsymbol{X}_i\hat{\boldsymbol{\beta}}_i)'(\boldsymbol{Y}_{ij} - \boldsymbol{X}_i\hat{\beta}_i), \tag{A.8}$$

and the expectation of SSE can be obtained by the following steps with the properties of $tr(\boldsymbol{\Sigma}\boldsymbol{P}) = tr(\boldsymbol{P}\boldsymbol{\Sigma}), \boldsymbol{P}^2 = \boldsymbol{P}$ and $\boldsymbol{P}' = \boldsymbol{P}$,

$$\sum_{i=1}^{I}\sum_{j=1}^{n_i} tr[Cov(\boldsymbol{Y}_{ij} - \boldsymbol{X}_i\hat{\boldsymbol{\beta}}_i)] = \sum_{i=1}^{I}\sum_{j=1}^{n_i} tr\left[\left(\boldsymbol{I}_T - \frac{1}{n_i}\boldsymbol{P}\right)\boldsymbol{\Sigma}\right]$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{n_i} tr\left\{\left(\boldsymbol{I}_T - \frac{1}{n_i}\boldsymbol{P}\right)[\sigma^2_e(1-\rho)\boldsymbol{I}_T + (\rho\sigma^2_e + \sigma^2_\alpha)\mathbf{1}_T\mathbf{1}'_T]\right\}$$

$$= \sum_{i=1}^{I} \sum_{j=1}^{n_i} \Big[(\sigma_e^2(1-\rho)T + (\rho\sigma_e^2 + \sigma_\alpha^2)T$$

$$- \frac{1}{n_i}\sigma_e^2(1-\rho)K - \frac{1}{n_i}(\rho\sigma_e^2 + \sigma_\alpha^2)T \Big]$$

$$= \left(\sum_{i=1}^{I} n_i - I \right) \phi^2 + \sigma_e^2(1-\rho)\left[T\sum_{i=1}^{I} n_i - IK - \sum_{i=1}^{I} n_i + I \right], \tag{A.9}$$

where $\phi^2 = \sigma_e^2(1-\rho) + T(\rho\sigma_e^2 + \sigma_\alpha^2)$.

In (A.9) we have utilized the following equality:

$$\mathbf{1}_T'\boldsymbol{X}_i(\boldsymbol{X}_i'\boldsymbol{X}_i)^{-1}\boldsymbol{X}_i'\mathbf{1}_T = T, \tag{A.10}$$

by noting that $\boldsymbol{X}_i = (\mathbf{1}_T, \boldsymbol{Z})$, $\mathbf{0} = [\boldsymbol{I} - \boldsymbol{X}_i(\boldsymbol{X}_i'\boldsymbol{X}_i)^{-1}\boldsymbol{X}_i']\boldsymbol{X}_i = [\boldsymbol{I} - \boldsymbol{X}_i(\boldsymbol{X}_i'\boldsymbol{X}_i)^{-1}\boldsymbol{X}_i']$ $(\mathbf{1}_T, \boldsymbol{Z})$, and $\mathbf{1}_T - \boldsymbol{X}_i(\boldsymbol{X}_i'\boldsymbol{X}_i)^{-1}\boldsymbol{X}_i'1_T = \mathbf{0}$.

The expectation of $S_{e,\rho}^2$, with $S_{e,\rho}^2 = \sum_{i=1}^{I} \sum_{j=1}^{n_i} \sum_{t=1}^{T} [Y_{ijt} - \boldsymbol{X}_{it}'\hat{\boldsymbol{\beta}}_i - (\bar{\boldsymbol{Y}}_{ij.} - \bar{\boldsymbol{Y}}_{i..})]^2$, is $\sigma_e^2(1-\rho)[T\sum_{i=1}^{I} n_i - IK - \sum_{i=1}^{I} n_i + I]$, which can be readily obtained by (A.9) and hence we can get

$$V = \frac{S_{e,\rho}^2}{\sigma_e^2(1-\rho)} \sim \chi^2_{T\sum_{i=1}^{I} n_i - IK - \sum_{i=1}^{I} n_i + I}. \tag{A.11}$$

This completes the proof of (3.7). $\square$

References

[1] M.M.A. Ananda, S. Weerahandi, Testing the difference of two exponential means using generalized *p*-values, Comm. Statist.-Simulation Comput. 25 (2) (1996) 521–532.

[2] E.M. Chi, S. Weerahandi, Comparing treatments under growth curve models: exact tests using generalized *p*-values, J. Statist. Plann. Inference 71 (1998) 179–189.

[3] J. Gamage, S. Weerahandi, Size performance of some tests in one-way ANOVA, Comm. Statist.-Simulation 27 (3) (1998) 625–640.

[4] R.G. Krutchkoff, One-way fixed effects analysis of variance when the error variances may be unequal, J. Statist. Comput. Simulation 30 (1998) 259–271.

[5] J.C. Lee, Prediction and estimation of growth curve with special covariance structures, J. Amer. Statist. Assoc. 83 (1988) 432–440.

[6] C.R. Rao, Least-squares theory using an estimated dispersion matrix and its application to measurement of signals, Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, Vol. 1, 1967, pp. 355–372.

[7] J.G. Thursby, A comparison of several exact tests and approximate tests for structural shift under heteroscedasticity, J. Econometrics 53 (1992) 363–386.

[8] K. Tsui, S. Weerahandi, Generalized *p*-values in significance testing of hypotheses in the presence of nuisance parameters, J. Amer. Statist. Assoc. 84 (1989) 602–607.

[9] S. Weerahandi, ANOVA under unequal error variances, Biometrics 51 (1995) 589–599.

[10] S. Weerahandi, D. Amaratunga, A performance comparison of methods in mixed models, Statistics Med., 2003, to appear.
[11] S. Weerahandi, V.W. Berger, Exact inference for growth curves with intraclass correlation structure, Biometrics 55 (1999) 921–924.
[12] L. Zhou, T. Mathew, Some tests for variance components using generalized *p*-values, Technometrics 36 (4) (1994) 394–402.