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A Bayesian-like estimator of the process capability index C_{pmk}

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Abstract. Pearn et al. (1992) proposed the capability index C_{pmk}, and investigated the statistical properties of its natural estimator $\hat{C}_{pmk}$ for stable normal processes with constant mean μ. Chen and Hsu (1995) showed that under general conditions the asymptotic distribution of $\hat{C}_{pmk}$ is normal if $\mu \neq m$, and is a linear combination of the normal and the folded-normal distributions if $\mu = m$, where m is the mid-point between the upper and the lower specification limits. In this paper, we consider a new estimator $\tilde{C}_{pmk}$ for stable processes under a different (more realistic) condition on process mean, namely, $P(\mu \geq m) = p$, $0 \leq p \leq 1$. We obtain the exact distribution, the expected value, and the variance of $\tilde{C}_{pmk}$ under normality assumption. We show that for $P(\mu \geq m) = 0$, or 1, the new estimator $\tilde{C}_{pmk}$ is the MLE of C_{pmk}, which is asymptotically efficient. In addition, we show that under general conditions $\tilde{C}_{pmk}$ is consistent and is asymptotically unbiased. We also show that the asymptotic distribution of $\tilde{C}_{pmk}$ is a mixture of two normal distributions.

Keywords and Phrases: process capability index; Bayesian-like estimator; consistent; mixture distribution

1. Introduction

Pearn et al. (1992) proposed the process capability index C_{pmk}, which combines the merits of two earlier indices C_{pk} (Kane (1986)) and C_{pm} (Chan et al. (1988)). The index C_{pmk} alerts the user if the process variance increases and/or the process mean deviates from its target value, and is designed to monitor the normal and the near-normal processes. The index C_{pmk} is considered arguably the most useful index to date for processes with two-sided specification

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limits (Boyles (1994), Wright (1995)). The index C_{pmk}, referred to as the third-generation capability index, has been defined as the following:

$$C_{pmk} = \min\left\{\frac{USL-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}}, \frac{\mu-LSL}{3\sqrt{\sigma^2+(\mu-T)^2}}\right\}, \tag{1}$$

where USL and LSL are the upper and the lower specification limits, respectively, μ is the process mean, σ is the process standard deviation, and T is the target value. We note that C_{pmk} can be rewritten as:

$$C_{pmk} = \frac{d-|\mu-m|}{3\sqrt{\sigma^2+(\mu-T)^2}}, \tag{2}$$

where m is the mid-point between the upper and the lower specification limits, and d is the half length of the specification interval $[LSL, USL]$. That is, $m = (USL+LSL)/2$, and $d = (USL-LSL)/2$. For stable processes where the process mean μ is assumed to be a constant (unknown), Pearn et al. (1992) considered the natural estimator of C_{pmk} which is defined as:

$$\hat{C}_{pmk} = \frac{d-|\overline{X}-m|}{3\sqrt{S_n^2+(\overline{X}-T)^2}}, \tag{3}$$

where $\overline{X} = (\sum_{i=1}^{n} X_i)/n$ and $S_n = \{n^{-1}\sum_{i=1}^{n}(X_i-\overline{X})^2\}^{1/2}$ are conventional estimators of the process mean and the process standard deviation, μ and σ, respectively. If the process characteristic follows the normal distribution, Pearn et al. (1992) showed that for the case with $T = m$ (symmetric tolerance) the distribution of the natural estimator $\hat{C}_{pmk}$ is a mixture of the chi-square distribution and the non-central chi-square distribution, as expressed in the following:

$$\hat{C}_{pmk} \sim \frac{\frac{d\sqrt{n}}{\sigma}-\chi'_1(\lambda)}{3\sqrt{\chi^2_{n-1}+\chi'^2_{n-1}(\lambda)}}, \tag{4}$$

where χ^2_{n-1} is the chi-square distribution with $n-1$ degrees of freedom, $\chi'_1(\lambda)$ is the non-central chi distribution with one degree of freedom and non-centrality parameter λ, and $\chi'^2_{n-1}(\lambda)$ is the non-central chi-square distribution with $n-1$ degrees of freedom and non-centrality parameter λ, where $\lambda = n(\mu-T)^2/\sigma^2$. Chen and Hsu (1995) showed that the natural estimator $\hat{C}_{pmk}$ is asymptotically unbiased. Chen and Hsu (1995) also showed that under general conditions the natural estimator $\hat{C}_{pmk}$ converges to the normal distribution $N(0, \sigma^2_{pmk})$, where

$$\sigma^2_{pmk} = \frac{\sigma^2}{9[\sigma^2+(\mu-T)^2]} + \left\{\frac{12(\mu-T)\sigma^2-6\mu_3}{18[\sigma^2+(\mu-T)^2]^{3/2}}\right\}C_{pmk} + \left\{\frac{144(\mu-T)^2\sigma^2-144(\mu-T)\mu_3+36(\mu_4-\sigma^4)}{144[\sigma^2+(\mu-T)^2]^{3/2}}\right\}C^2_{pmk}, \tag{5}$$

μ_3, μ_4 are the third and fourth central moment of the process, respectively.

2. A Bayesian-like estimator

In real-world applications, the production may require multiple supplies with different quality characteristics on each single shipment of the raw materials, multiple manufacturing lines with inconsistent precision in machine settings and engineering effort for each manufacturing line, or multiple workmanship shifts with unequal performance level on each shift. Therefore, the basic and common assumption that the process mean stay as a constant may not be satisfied in real situations. Consequently, using the natural estimator $\hat{C}_{pmk}$ to measure the potential and performance of such a process is inappropriate as the resulting capability measure would not be accurate. For stable processes under those conditions, if the knowledge on the process mean, $P(\mu \geq m) = p$, $0 \leq p \leq 1$, is available, then we can consider the following new estimator $\tilde{C}_{pmk}$. In general, the probability $P(\mu \geq m) = p$, $0 \leq p \leq 1$, can be obtained from historical information of a stable process.

$$\tilde{C}_{pmk} = \frac{b_{n-1}[d - (\overline{X} - m)I_A(\mu)]}{3\sqrt{S_n^2 + (\overline{X} - T)^2}}, \tag{6}$$

where $b_{n-1} = \sqrt{2/(n-1)}\{\Gamma[(n-1)/2]/\Gamma[(n-2)/2]\}$ is the correction factor, $I_A(\cdot)$ is the indicator function defined as $I_A(\mu) = 1$ if $\mu \in A$, and $I_A(\mu) = -1$ if $\mu \notin A$, where $A = \{\mu \,|\, \mu \geq m\}$. We note that the new estimator $\tilde{C}_{pmk}$ can be rewritten as the following:

$$\tilde{C}_{pmk} = \frac{b_{n-1}[d - (\overline{X} - m)I_A(\mu)]}{3\sqrt{S_n^2 + (\overline{X} - T)^2}} = \frac{\dfrac{d - (\overline{X} - m)I_A(\mu)}{3b_{n-1}^{-1}S_n}}{\sqrt{1 + \dfrac{(\overline{X} - T)^2}{S_n^2}}} = \frac{\tilde{C}_{pk}}{\sqrt{1 + \dfrac{(\overline{X} - T)^2}{S_n^2}}}, \tag{7}$$

where $\tilde{C}_{pk} = b_{n-1}[d - (\overline{X} - m)I_A(\mu)]/(3S_n)$ as defined by Pearn and Chen (1996). If the process characteristic follows the normal distribution, $N(\mu, \sigma^2)$, then we can show the following Theorem.

Theorem 1. *If the process characteristic follows the normal distribution, then* $\tilde{C}_{pmk} \sim \dfrac{b_{n-1}\left[\sqrt{n}C_p - \dfrac{N(\eta, 1)}{3}\right]}{\sqrt{\chi_n'^2}}$, *where* $N(\eta, 1)$ *is the normal distribution with mean* $\eta = 3\sqrt{n}(C_p - C_{pk})$, $\chi_n'^2$ *is the non-central chi-square distribution with n degrees of freedom and non-centrality parameter* $\lambda = n(\mu - T)^2/\sigma^2$.

Proof: We note that $3b_{n-1}^{-1}S_n\tilde{C}_{pk} = d - (\overline{X} - m)I_A(\mu)$ is distributed as the normal distribution $N(3\sigma C_{pk}, \sigma^2/n)$. Therefore, $b_{n-1}[d - (\overline{X} - m)I_A(\mu)]/(3\sigma) = b_{n-1}\{[d/(3\sigma)] - [(\overline{X} - m)I_A(\mu)/(3\sigma)]$ is distributed as $b_{n-1}\{C_p - [N(\eta, 1)/(3\sqrt{n})]\}$, where $N(\eta, 1)$ is the normal distribution with mean $\eta = 3\sqrt{n}(C_p - C_{pk})$. We also note that $[nS_n^2 + n(\overline{X} - T)^2]/\sigma^2 = \sum_{i=1}^{n}(X_i - T)^2/\sigma^2$ is distributed as $\chi_n'^2$, the non-central chi-square distribution with n degrees of freedom and non-centrality parameter $\lambda = n(\mu - T)^2/\sigma^2$. Therefore, $\tilde{C}_{pmk}$ is distributed as $b_{n-1}\{\sqrt{n}C_p - [N(\eta, 1)/3]\}/\sqrt{\chi_n'^2}$.

The r-th moment (about zero) of $\tilde{C}_{pmk}$, therefore, can be obtained as:

$$E(\tilde{C}^r_{pmk})^r = E\left\{\frac{b_{n-1}\left[\sqrt{n}C_p - \frac{N(\eta,1)}{3}\right]}{\sqrt{\chi'^2_n}}\right\}^r$$

$$= \sum_{i=0}^{r} b^r_{n-1}\binom{r}{i}E\left\{\left[\frac{-N(\eta,1)}{3\sqrt{n}C_p}\right]^i\left[\frac{\sqrt{n}C_p}{\sqrt{\chi'^2_n}}\right]^r\right\}, \quad (8)$$

By setting $r = 1$, and $r = 2$, we may obtain the first two moments and the variance as:

$$E(\tilde{C}_{pmk}) = \sum_{i=0}^{1} b_{n-1}\binom{1}{i}E\left\{\left[\frac{-N(\eta,1)}{3\sqrt{n}C_p}\right]^i\left[\frac{\sqrt{n}C_p}{\sqrt{\chi'^2_n}}\right]\right\}, \quad (9)$$

$$E(\tilde{C}^2_{pmk}) = \sum_{i=0}^{2} b^2_{n-1}\binom{2}{i}E\left\{\left[\frac{-N(\eta,1)}{3\sqrt{n}C_p}\right]^i\left[\frac{\sqrt{n}C_p}{\sqrt{\chi'^2_n}}\right]^2\right\}, \quad (10)$$

$$\mathrm{Var}(\tilde{C}_{pmk}) = E(\tilde{C}^2_{pmk}) - [E(\tilde{C}_{pmk})]^2. \quad (11)$$

We note that for the case with $p(\mu \geq m) = 1$, $\tilde{C}_{pmk} < \hat{C}_{pmk}$ for $\overline{X} \geq m$ and $\tilde{C}_{pmk} > \hat{C}_{pmk}$ for $\overline{X} < m - d[(1 - b_{n-1})/(1 + b_{n-1})]$. If the process distribution is normal, then the probability $P(\overline{X} \geq m) = \Phi\{\sqrt{n}[(\mu - m)/\sigma]\}$ converges to 1. Thus, for large values of n, we expect to have $\tilde{C}_{pmk} < \hat{C}_{pmk}$. On the other hand, if $P(\mu \geq m) = 0$, then we have $\tilde{C}_{pmk} < \hat{C}_{pmk}$ for $\overline{X} \leq m$ and $\tilde{C}_{pmk} > \hat{C}_{pmk}$ for $\overline{X} > m + d[(1 - b_{n-1})/(1 + b_{n-1})]$. If the process distribution is normal, then the probability $P(\overline{X} \leq m) = \Phi\{\sqrt{n}[(m - \mu)/\sigma]\}$ converges to 1. Thus, for large values of n, we also expect to have $\tilde{C}_{pmk} < \hat{C}_{pmk}$. Explicit forms of the expected value and the variance of $\tilde{C}_{pmk}$ are analytically intractable. But, for the cases with $P(\mu \geq m) = 1$ or 0, the probability density function may be obtained (the proof is omitted for the simplicity of the presentation).

3. Asymptotic distribution of $\tilde{C}_{pmk}$

In the following, we show that if the knowledge on the process mean, the probabilities $P(\mu \geq m) = p$, and $P(\mu < m) = 1 - p$, with $0 \leq p \leq 1$ is given, then the asymptotic distribution of the proposed new estimator $\tilde{C}_{pmk}$ is a mixture of two normal distributions. We first present some Lemmas. The proofs for these Lemmas can be found in the reference Serfling (1980). A direct consequence of our result is that for the cases with either $P(\mu \geq m) = 1$, or $P(\mu \geq m) = 0$, the asymptotic distribution will then be an ordinary normal distribution.

Lemma 1: *If $\mu_4 = E(X - \mu)^4$ exists, then $\sqrt{n}(\overline{X} - \mu, S^2_n - \sigma^2)$ converges to $N((0,0),\Sigma)$ in distribution, where* $\Sigma = \begin{bmatrix} \sigma^4 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}$.

Lemma 2: *If $g(x, y)$ is a real-valued differentiable function, then $\sqrt{n}[g(\overline{X}, S_n^2) - g(\mu, \sigma^2)]$ converges to $N(0, D\Sigma D')$ in distribution, if* $D = \left(\left.\frac{\partial g}{\partial x}\right|_{(\mu,\sigma^2)}, \left.\frac{\partial g}{\partial y}\right|_{(\mu,\sigma^2)} \right) \neq (0, 0)$.

Lemma 3: *If the random vector $(w_{1n}, w_{2n}, \ldots, w_{kn})$ converges to the random vector $(w_1, w_2, \ldots, w_k)$ in distribution, and the random vector $(v_{1n}, v_{2n}, \ldots, v_{kn})$ converges to the random vector $(v_1, v_2, \ldots, v_k)$ in probability, then the random vector $(v_{1n}w_{1n}, v_{2n}w_{2n}, \ldots, v_{kn}w_{kn})$ converges to the random vector $(v_1w_1, v_2w_2, \ldots, v_kw_k)$ in distribution.*

Lemma 4: *If the random vector $(w_{1n}, w_{2n}, \ldots, w_{kn})$ converges to the random vector $(w_1, w_2, \ldots, w_k)$ in distribution, and the function g is continuous with probability one, then $g(w_{1n}, w_{2n}, \ldots, w_{kn})$ converges to $g(w_1, w_2, \ldots, w_k)$ in distribution.*

Lemma 5: *If the random vector $(v_{1n}, v_{2n}, \ldots, v_{kn})$ converges to the random vector $(v_1, v_2, \ldots, v_k)$ in probability, and the function g is continuous with probability one, then $g(v_{1n}, v_{2n}, \ldots, v_{kn})$ converges to $g(v_1, v_2, \ldots, v_k)$ in probability.*

Lemma 6: *If $\mu_4 = E(X - \mu)^4$ exists, then $\sqrt{n}(\overline{X} - \mu, \overline{X} - \mu, S_n^2 - \sigma^2)$ converges to $N((0,0,0), \Sigma^*)$ in distribution, where* $\Sigma^* = \begin{bmatrix} \sigma^2 & \sigma^2 & \mu_3 \\ \sigma^2 & \sigma^2 & \mu_3 \\ \mu_3 & \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}$.

Proof: See Chen and Hsu (1995).

Theorem 2: *The estimator $\tilde{C}_{pmk}$* is consistent.

Proof: We first note that $(\overline{X}, S_n^2)$ converges to (μ, σ^2) in probability and b_{n-1} converges to 1 as $n \to \infty$. Since $\tilde{C}_{pmk}$ is a continuous function of $(\overline{X}, S_n^2)$, then it follows directly from Lemma 5 that $\tilde{C}_{pmk}$ converges to C_{pmk} in probability. Hence, $\tilde{C}_{pmk}$ must be consistent.

Theorem 3: *Under general conditions, if the fourth central moment $\mu_4 = E(X - \mu)^4$ exists, then $\sqrt{n}(\tilde{C}_{pmk} - C_{pmk})$ converges to $p \cdot N(0, \sigma_{pmk1}^2) + (1 - p) \cdot N(0, \sigma_{pmk2}^2)$ in distribution, where*

$$\sigma_{pmk1}^2 = \frac{\Delta_1^2}{9}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-1} + \frac{\Delta_1}{3}\frac{\mu_3}{\sigma^3}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-3/2} C_{pmk1} + \frac{1}{4}\frac{\mu_4 - \sigma^4}{\sigma^4}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-2} C_{pmk1}^2$$

$$\sigma_{pmk2}^2 = \frac{\Delta_2^2}{9}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-1} + \frac{\Delta_2}{3}\frac{\mu_3}{\sigma^3}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-3/2} C_{pmk2} + \frac{1}{4}\frac{\mu_4 - \sigma^4}{\sigma^4}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-2} C_{pmk2}^2$$

$$\Delta_1 = \frac{9(\mu - T)C_{pmk1}^2}{d - (\mu - m)} + 1, \quad C_{pmk1} = \frac{d - (\mu - m)}{3\sqrt{\sigma^2 + (\mu - T)^2}},$$

$$\Delta_2 = \frac{9(\mu - T)C_{pmk2}^2}{d + (\mu - m)} - 1, \quad C_{pmk2} = \frac{d + (\mu - m)}{3\sqrt{\sigma^2 + (\mu - T)^2}}.$$

Proof: (CASE I) If $\mu > m$, we define the function $g_1(x, y) = \dfrac{d - (x - m)}{3\sqrt{y + (x - T)^2}}$, where $x > m,\ y > 0$. Since g_1 is differentiable, then we have

$$\left.\frac{\partial g_1}{\partial x}\right|_{(\mu,\sigma^2)} = \frac{-\Delta_1 C_{pmk1}}{d - (\mu - m)}, \quad \left.\frac{\partial g_1}{\partial y}\right|_{(\mu,\sigma^2)} = -\frac{9}{2}\frac{C_{pmk1}^3}{[d - (\mu - m)]^2},$$

where $\Delta_1 = \dfrac{9(\mu - T)C_{pmk1}^2}{d - (\mu - m)} + 1$, and $C_{pmk1} = \dfrac{d - (\mu - m)}{3\sqrt{\sigma^2 + (\mu - T)^2}}$. If we define $D_1 = \left(\left.\dfrac{\partial g_1}{\partial x}\right|_{(\mu,\sigma^2)}, \left.\dfrac{\partial g_1}{\partial y}\right|_{(\mu,\sigma^2)}\right)$, then $D_1 \neq (0, 0)$.

By Lemma 1 and Lemma 2, $\sqrt{n}(b_{n-1}^{-1}\tilde{C}_{pmk} - C_{pmk}) = \sqrt{n}[g_1(\overline{X}, S_n^2) - g_1(\mu, \sigma^2)]$ converges to $N(0, \sigma_{pmk1}^2)$ in distribution, where

$$\sigma_{pmk1}^2 = D_1 \Sigma D_1' = \frac{\Delta_1^2}{9}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-1} + \frac{\Delta_1}{3}\frac{\mu_3}{\sigma^3}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-3/2} C_{pmk1}$$
$$+ \frac{1}{4}\frac{\mu_4 - \sigma^4}{\sigma^4}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-2} C_{pmk1}^2$$

(CASE II) If $\mu < m$, we define the function $g_2(x, y) = \dfrac{d + (x - m)}{3\sqrt{y + (x - T)^2}}$, where $x < m,\ y > 0$. Since g_1 is differentiable, then we have

$$\left.\frac{\partial g_2}{\partial x}\right|_{(\mu,\sigma^2)} = \frac{-\Delta_2 C_{pmk2}}{d + (\mu - m)}, \quad \left.\frac{\partial g_2}{\partial y}\right|_{(\mu,\sigma^2)} = -\frac{9}{2}\frac{C_{pmk2}^3}{[d + (\mu - m)]^2},$$

where $\Delta_2 = \dfrac{9(\mu - T)C_{pmk2}^2}{d + (\mu - m)} - 1$, and $C_{pmk2} = \dfrac{d + (\mu - m)}{3\sqrt{\sigma^2 + (\mu - T)^2}}$. If we define $D_2 = \left(\left.\dfrac{\partial g_2}{\partial x}\right|_{(\mu,\sigma^2)}, \left.\dfrac{\partial g_2}{\partial y}\right|_{(\mu,\sigma^2)}\right)$, then $D_2 \neq (0, 0)$.

By Lemma 1 and Lemma 2, $\sqrt{n}(b_{n-1}^{-1}\tilde{C}_{pmk} - C_{pmk}) = \sqrt{n}[g_2(\overline{X}, S_n^2) - g_2(\mu, \sigma^2)]$ converges to $N(0, \sigma_{pmk2}^2)$ in distribution, where

$$\sigma_{pmk2}^2 = D_2 \Sigma D_2' = \frac{\Delta_2^2}{9}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-1} + \frac{\Delta_2}{3}\frac{\mu_3}{\sigma^3}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-3/2} C_{pmk2}$$
$$+ \frac{1}{4}\frac{\mu_4 - \sigma^4}{\sigma^4}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-2} C_{pmk2}^2$$

(CASE III) If $\mu = m$, then

$$\sqrt{n}(b_{n-1}^{-1}\tilde{C}_{pmk} - C_{pmk}) = -\frac{\sqrt{n}(\overline{X}-\mu)}{3\sqrt{S_n^2 + (\overline{X}-T)^2}}$$

$$-\frac{d}{3}\frac{[\sqrt{n}(S_n^2-\sigma^2) + \sqrt{n}(\overline{X}^2-\mu^2) - 2\sqrt{n}(\overline{X}-\mu)T]}{\sqrt{\sigma^2+(\mu-T)^2}\sqrt{S_n^2+(\overline{X}-T)^2}[\sqrt{\sigma^2+(\mu-T)^2}+\sqrt{S_n^2+(\overline{X}-T)^2}]}.$$

We define

$$v_{1n} = -\frac{1}{3\sqrt{S_n^2+(\overline{X}-T)^2}},$$

$$v_{2n} =$$

$$-\frac{d}{3[\sqrt{\sigma^2+(\mu-T)^2}\sqrt{S_n^2+(\overline{X}-T)^2}][\sqrt{\sigma^2+(\mu-T)^2}+\sqrt{S_n^2+(\overline{X}-T)^2}]},$$

$$w_{1n} = \sqrt{n}(\overline{X}-\mu), \quad w_{2n} = \sqrt{n}(S_n^2-\sigma^2) + \sqrt{n}(\overline{X}^2-\mu^2) - 2\sqrt{n}(\overline{X}-\mu)T.$$

Since $(\overline{X}, S_n^2)$ converges to (μ, σ^2) in probability, then (v_{1n}, v_{2n}) converges to (v_1, v_2) in probability, where $v_1 = -\left(\frac{C_{pmk0}}{d}\right)$, $v_2 = -\left(\frac{9C_{pmk0}^3}{2d^2}\right)$, and $C_{pmk0} = \frac{d}{3\sqrt{\sigma^2+(\mu-T)^2}}$. Define the function $G(x, y, z) = (x, z + xy - 2xT)$. Then by Lemma 4 and Lemma 6, $(w_{1n}, w_{2n}) = \sqrt{n}[G(\overline{X}, \overline{X}, S_n^2) - G(\mu, \mu, \sigma^2)]$ converges to (w_1, w_2) which is distributed as $N((0,0), \Sigma_G)$, where

$$\Sigma_G = \begin{bmatrix} 1 & 0 & 0 \\ a & \mu & 1 \end{bmatrix} \begin{bmatrix} \sigma^2 & \sigma^2 & \mu_3 \\ \sigma^2 & \sigma^2 & \mu_3 \\ \mu_3 & \mu_3 & b \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & \mu \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sigma^2 & c \\ c & d \end{bmatrix},$$

with $a = \mu - 2T, b = \mu_4 - \sigma^4, c = 2(\mu - T)\sigma^2 + \mu_3,$

$$d = 4(\mu-T)^2 + 4(\mu-T)\mu_3 + (\mu_4 - \sigma^4).$$

Hence, by Lemma 3,

$$\Bigg[\frac{-\sqrt{n}(\overline{X}-\mu)}{3\sqrt{S_n^2+(\overline{X}-T)^2}},$$

$$-\frac{3}{d}\frac{\sqrt{n}(S_n^2-\sigma^2) + \sqrt{n}(\overline{X}^2-\mu^2) - 2\sqrt{n}(\overline{X}-\mu)T}{\sqrt{\sigma^2+(\mu-T)^2}\sqrt{S_n^2+(\overline{X}-T)^2}[\sqrt{\sigma^2+(\mu-T)^2}+\sqrt{S_n^2+(\overline{X}-T)^2}]}\Bigg]$$

converges to $(v_1 w_1, v_2 w_2) = \left(-\frac{C_{pmk}}{d} w_1, -\frac{9C_{pmk}^3}{2d^2} w_2\right)$ in distribution. Define $H(x, y) = x + y$. Then, by Lemma 4, $\sqrt{n}(b_{n-1}^{-1}\tilde{C}_{pmk} - C_{pmk})$ converges to $Y = -\frac{C_{pmk}}{d} w_1 - \frac{9}{2}\frac{C_{pmk}^3}{d^2} w_2$, which is a normal distribution with $E(Y) = 0$,

$$\mathrm{Var}(Y) = \frac{\Delta_0^2}{9}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-1} + \frac{\Delta_0}{3}\frac{\mu_3}{\sigma^3}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-3/2} C_{pmk0}$$
$$+ \frac{1}{4}\frac{\mu_4 - \sigma^4}{\sigma^4}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-2} C_{pm0}^2$$

where $\Delta_0 = \frac{9(\mu - T)C_{pmk0}^2}{d} + 1$, $C_{pmk0} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}}$.

Since $P\{\sqrt{n}(b_{n-1}^{-1}\tilde{C}_{pmk} - C_{pmk}) \le r\} = P\{\mu \ge m\}P\{\sqrt{n}(b_{n-1}^{-1}\tilde{C}_{pmk} - C_{pmk}) \le r \mid \mu \ge m\} + P\{\mu < m\}P\{\sqrt{n}(b_{n-1}^{-1}\tilde{C}_{pmk} - C_{pmk}) \le r \mid \mu < m\}$ for all real number r, then it follows that $\sqrt{n}(b_{n-1}^{-1}\tilde{C}_{pmk} - C_{pmk})$ converges to $p \cdot N(0, \sigma_{pmk1}^2) + (1 - p) \cdot N(0, \sigma_{pmk2}^2)$ in distribution. Since $\sqrt{n}(\tilde{C}_{pmk} - C_{pmk}) = \sqrt{n}(b_{n-1}^{-1}\tilde{C}_{pmk} - C_{pmk}) + \sqrt{n}(\tilde{C}_{pmk} - b_{n-1}^{-1}\tilde{C}_{pmk})$ and b_{n-1} converges to 1 as $n \to \infty$, thus by Slutsky's theory, the theorem proved.

Corollary 3.1: *The estimator $\tilde{C}_{pmk}$ is asymptotically unbiased.*

Proof: From Theorem 3, $\sqrt{n}(\tilde{C}_{pmk} - C_{pmk})$ converges to the following $p \cdot N(0, \sigma_{pmk1}^2) + (1 - p) \cdot N(0, \sigma_{pmk2}^2)$ in distribution. Therefore, $E\{\sqrt{n}(\tilde{C}_{pmk} - C_{pmk})\}$ converges to zero, and so $\tilde{C}_{pmk}$ must be asymptotically unbiased.

Corollary 3.2: *If the process characteristic follows the normal distribution, then $\sqrt{n}(\tilde{C}_{pmk} - C_{pmk})$ converges to $p \cdot N(0, \sigma_{pmk1'}^2) + (1 - p) \cdot N(0, \sigma_{pmk2'}^2)$, a mixture of two normal distributions, where*

$$\sigma_{pmk1'}^2 = \frac{\Delta_1^2}{9}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-1} + \frac{1}{2}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-2} C_{pmk1}^2,$$

$$\sigma_{pmk2'}^2 = \frac{\Delta_2^2}{9}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-1} + \frac{1}{2}\left[1 + \frac{(\mu - T)^2}{\sigma^2}\right]^{-2} C_{pmk2}^2,$$

$$\Delta_1 = \frac{9(\mu - T)C_{pmk1}^2}{d - (\mu - m)} + 1, \quad C_{pmk1} = \frac{d - (\mu - m)}{3\sqrt{\sigma^2 + (\mu - T)^2}},$$

$$\Delta_2 = \frac{9(\mu - T)C_{pmk2}^2}{d + (\mu - m)} - 1, \quad C_{pmk2} = \frac{d + (\mu - m)}{3\sqrt{\sigma^2 + (\mu - T)^2}}.$$

Theorem 4: *If the process characteristic follows the normal distribution, then* *for the case with* $P(\mu \geq m) = 0$, *or* 1, *(i)* $\tilde{C}_{pmk}$ *is the MLE of* C_{pmk}, *(ii)* $\tilde{C}_{pmk}$ *is asymptotically efficient.*

Proof: (i) For normal distributions, $(\bar{X}, S_n^2)$ is the MLE of (μ, σ^2). By the invariance property, $\tilde{C}_{pmk}$ is the MLE of C_{pmk}.

(ii) The Fisher information matrix can be calculated as:

$$I(\theta) = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} \sigma^{-2} & 0 \\ 0 & (2\sigma^2)^{-2} \end{bmatrix},$$

where $\theta = (\mu, \sigma^2), a' = E\left[\frac{\partial}{\partial \mu} \ln f(x, \theta)\right]^2$,

$$b' = c' = E\left[\frac{\partial}{\partial \mu} \ln f(x, \theta) \frac{\partial}{\partial \sigma^2} \ln f(x, \theta)\right], \text{ and } d' = E\left[\frac{\partial}{\partial \sigma^2} \ln f(x, \theta)\right]^2.$$

If $P(\mu \geq m) = 1$, then the information lower bound reduces to

$$\left[\frac{\partial}{\partial \mu} C_{pmk}, \frac{\partial}{\partial \sigma^2} C_{pmk}\right] \frac{I^{-1}(\theta)}{n} \begin{bmatrix} \frac{\partial}{\partial \mu} C_{pmk} \\ \frac{\partial}{\partial \sigma^2} C_{pmk} \end{bmatrix}$$

$$= \frac{\Delta_1^2}{9n} \left\{1 + \frac{(\mu - T)^2}{\sigma^2}\right\}^{-1} + \frac{C_{pmk1}^2}{2} \left\{1 + \frac{(\mu - T)^2}{\sigma^2}\right\}^{-2} = \frac{\sigma_{pmk1'}^2}{n}.$$

On the other hand, if $P(\mu \geq m) = 0$, then the information lower bound reduces to

$$\left[\frac{\partial}{\partial \mu} C_{pmk}, \frac{\partial}{\partial \sigma^2} C_{pmk}\right] \frac{I^{-1}(\theta)}{n} \begin{bmatrix} \frac{\partial}{\partial \mu} C_{pmk} \\ \frac{\partial}{\partial \sigma^2} C_{pmk} \end{bmatrix}$$

$$= \frac{\Delta_2^2}{9n} \left\{1 + \frac{(\mu - T)^2}{\sigma^2}\right\}^{-1} + \frac{C_{pmk2}^2}{2} \left\{1 + \frac{(\mu - T)^2}{\sigma^2}\right\}^{-2} = \frac{\sigma_{pmk2'}^2}{n}.$$

Since the information lower bound is achieved (Corollary 3.2), then for the case with $P(\mu \geq m) = 0$, or 1, $\tilde{C}_{pmk}$ is asymptotically efficient.

In practice, to evaluate the estimator $\tilde{C}_{pmk}$ we need to determine the value of the indicator which requires additionally the knowledge of $P(\mu \geq m)$, or $P(\mu < m)$. If historical information of the process shows $P(\mu \geq m) = p$, then we may determine the value $I_A(\mu) = 1$, or -1 using available random number tables. For example, assume $p = 0.375$ is given, then $I_A(\mu) = 1$ if the generated 3-digit random number is no greater than 375, and $I_A(\mu) = -1$ otherwise.

4. Conclusions

Pearn et al. (1992) proposed the capability index C_{pmk}, which is designed to monitor the normal and the near-normal processes. The index C_{pmk} is considered to be the most useful index to date for processes with two-sided specification limits. Pearn et al. (1992) investigated the statistical properties of the natural estimator $\hat{C}_{pmk}$ for stable normal processes with constant mean μ. In this paper, we considered stable processes under a different condition (more realistic) where the process mean may not be a constant. For stable processes under such conditions with given knowledge of $P(\mu \geq m) = p$, $0 \leq p \leq 1$, we investigated a new estimator $\tilde{C}_{pmk}$ using the given information.

We obtained the exact distribution of the new estimator, and derived its expected value and variance under normality assumption. For cases with $P(\mu \geq m) = 0$, or 1, we showed that the new estimator $\tilde{C}_{pmk}$ is the MLE of C_{pmk}. In addition, we showed that under general conditions $\tilde{C}_{pmk}$ is consistent and is asymptotically unbiased. We also showed that the asymptotic distribution of $\tilde{C}_{pmk}$ is a mixture of two normal distributions. The results obtained in this paper allow us to perform a more accurate capability measure for processes under more realistic conditions in which using existing method (estimator) is inappropriate.

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