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Journal of the Franklin Institute 340 (2003) 43–61

Journal
of The
Franklin Institute

www.elsevier.com/locate/jfranklin

Unknown input observers for singular systems designed by eigenstructure assignment

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Accepted 12 February 2003

Abstract

In this paper, any unknown input observers with orders between minimum and full orders can be established for singular systems by eigenstructure assignment method. The complete and parametric solutions for the observer matrices and for the generalized eigenvectors are obtained. The completeness and parametric forms of the solutions and the flexibility in selecting the observer order allow the designer to choose a suitable observer according to the control purposes; hence, the solutions are quite suitable for advanced applications.

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Keywords: Observer; Singular system; Unknown input; Eigenstructure assignment; Parametric solutions

1. Introduction

Over the past two decades, many researchers have paid attention to the problem of estimating the states of a regular dynamic system subjected to both known and unknown inputs [1–12]. They have developed the reduced and/or full order unknown input observers by different approaches. This problem is of considerable importance, because plant disturbances may exist and some of the inputs to the system are inaccessible. Recently, some researchers have further investigated this problem for the singular systems [13–18]. The state responses of the singular systems may contain differential terms of the input. It is very sensitive even if the input is changed slightly. Hence, the unknown input observer problem is especially meaningful for singular systems.

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In this paper, a new set of solutions of the unknown input observers are established for the singular systems by eigenstructure assignment method. The eigenstructure assignment method has been shown to be useful in the feedback design for regular dynamic systems [19–23], and for singular systems [24–26], since the eigenvalues and the corresponding eigenvectors can be assigned simultaneously. Furthermore, the solutions obtained by eigenstructure assignment are often represented in parametric forms. Hence, more system properties such as robust, sensitivity, minimum gain, etc., can be achieved by choosing the free parameters from optimizing certain objective functions [27–31].

In this paper, the solutions of the unknown input observer with order which is between minimum and full orders are developed. The complete and parametric solutions for the observer matrices and for generalized eigenvector are obtained. The completeness and parametric forms of the solutions and the flexibility in selecting the observer order allow the designer to choose a suitable observer according to the control purposes; hence, the solutions are quite suitable for advanced applications. In most eigenstructure assignment approaches which are applied to feedback design, it is assumed that the system is controllable, so the generalized eigenvectors with any uncontrollable eigenvalue have not been discussed. The transmission zeros in unknown input observer design play the same roles as the uncontrollable eigenvalues in feedback design. If the system contains a transmission zero, it must be an eigenvalue of any possible unknown input observers. In this paper, we also develop the solutions of the generalized eigenvectors with the transmission zeros to complete the problem.

The organization of the paper is as follows. In Section 2, an unknown input observer is introduced. Some preliminary results are presented in Section 3. In Section 4, the problem is formulated and in Section 5 the solutions for the unknown input observers are established. The illustrative examples are provided in Section 6 and Section 7 concludes the paper.

2. Observer design

Consider the following singular system

$$\begin{aligned} E\dot{x} &= Ax + Bu + Dv, \\ y &= Cx + Fu, \end{aligned} \quad (1)$$

where $x \in R^n$ is the state variable, $u \in R^p$ is the input variable, $v \in R^m$ is the unknown disturbance, $y \in R^q$ is the measurable output, $E \in R^{n \times n}$ is a singular matrix and A, B, D, C, F are matrices with appropriate dimensions.

In this paper, the main purpose is to design an r th order observer of the following type

$$\begin{aligned} \dot{z} &= Nz + Ly + Gu, \\ \hat{x} &= Pz + Qy + Zu, \end{aligned} \quad (2)$$

using only the information of the input u and measurable output y and without any knowledge of the unknown disturbance v , where r is an integer in the range $n - q \leq r \leq n$, \hat{x} is the estimation of x , $z \in R^r$ is the state variable of the observer, and N, L, G, P, Q, Z are matrices with appropriate dimensions.

From Eqs. (1) and (2), the following relations can be easily derived for any $T \in R^{r \times n}$:

$$\begin{aligned} (\dot{z} - TE\dot{x}) &= N(z - TE\hat{x}) + (NTE + LC - TA)x - TDv + (G - TB + LF)u, \\ (\hat{x} - x) &= P(z - TE\hat{x}) + (QC - I + PTE)x + (Z + QF)u, \end{aligned} \tag{3}$$

If the following relations hold:

$$NTE - TA + LC = 0, \tag{4}$$

$$TD = 0, \tag{5}$$

$$G - TB + TF = 0, \tag{6}$$

$$PTE + QC = I, \tag{7}$$

$$Z + QF = 0, \tag{8}$$

we can obtain that

$$\begin{aligned} (\dot{z} - TE\dot{x}) &= N(z - TE\hat{x}), \\ (\hat{x} - x) &= P(z - TE\hat{x}). \end{aligned}$$

If N is a stable matrix, it follows that $\hat{x} \rightarrow x$. The behaviors of x and \hat{x} are demonstrated by $u, v, x(0)$ and $z(0)$. Therefore, if one of relations (4)–(8) does not satisfy for any $T \in R^{r \times n}$, there must exist some $u, v, x(0)$ and $z(0)$ which make $(\hat{x} - x)$ not converge to zero. Hence, Eqs. (4)–(8) are the sufficient and necessary condition for the existence of the proposed observer and the problem of the r th order unknown input observer design is to find matrices N, L, G, P, Q, Z and T satisfying Eqs. (4)–(8).

From Eq. (5), it follow that $\text{rank}T \leq n - \text{rank}D$ and from Eq. (7), it should be satisfied that $\text{rank}T \geq \text{rank}TE \geq n - \text{rank}C$ for the existence of P and Q ; Hence, it follows that $n - \text{rank}D \geq n - \text{rank}C$, i.e. $\text{rank}D \leq \text{rank}C$. If the condition is not held, the observer does not exist. From this fact, we assume in the whole paper that C is of full row rank, D is of full column rank and $q \geq m$ without loss of generality.

3. Some preliminary results

In this section, the possible eigenvalues of the matrix N are discussed.

Definition 1. A complex number λ_i is a transmission zero of the quartet (E, A, D, C) if and only if

$$\text{rank} \begin{bmatrix} (A - \lambda_i E) & D \\ C & 0 \end{bmatrix} < n + m.$$

Definition 2. If there exist a set of linear independent vectors

$$\begin{bmatrix} v_{ij}^k \\ w_{ij}^k \end{bmatrix}, j = 1, \dots, \bar{\theta}_i; \quad k = 1, \dots, \bar{\rho}_{ij},$$

satisfying that

$$\begin{bmatrix} (A - \lambda_i E) & D \\ C & 0 \end{bmatrix} \begin{bmatrix} v_{ij}^k \\ w_{ij}^k \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{ij}^{k-1} \\ w_{ij}^{k-1} \end{bmatrix}, \quad v_{ij}^0 = 0, \tag{9}$$

then we say that v_{ij}^k is a *right generalized transmission vector* of order k with the transmission zero λ_i . Furthermore, if rank

$$\begin{bmatrix} (A - \lambda_i E) & D \\ C & 0 \end{bmatrix} = n + m - \bar{\theta}_i$$

and any nonzero linear combination of

$$\begin{bmatrix} v_{i1}^{\bar{\rho}_{i1}} \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} v_{i\bar{\theta}_i}^{\bar{\rho}_{i\bar{\theta}_i}} \\ 0 \end{bmatrix}$$

is not in the column space of

$$\begin{bmatrix} (A - \lambda_i E) & D \\ C & 0 \end{bmatrix},$$

then we say that $v_{ij}^k, j = 1, \dots, \bar{\theta}_i; k = 1, \dots, \bar{\rho}_{ij}$, form a *complete set* of right transmission vectors with the transmission zero λ_i .

The following lemma discusses the role of transmission zero in an unknown input observer design.

Lemma 3. Assume that observer (2) exists. The eigenvalues of N must contain the transmission zeros of the quartet (E, A, D, C) counting multiplicity and the rest of the eigenvalues can be assigned arbitrarily.

Proof. Let $F \in C^{n \times s}$ be a matrix whose columns are all generalized transmission vectors and A_{TZ} be a Jordan form matrix with all transmission zeros as its eigenvalues. Since $CF=0$, TEF is of full column rank; otherwise, from Eq. (7), the observer does not exist. We can, therefore, find a matrix $Q \in C^{n \times (r-s)}$ such that $[F \ Q]$ and $TE [F \ Q]$ are of full column rank. From right-multiplying by $[F \ Q]$ and left-multiplying by $TE [F \ Q]^{-1}$ to (4), we have

$$\begin{bmatrix} A_{TZ} - A_{TZ} & N_1 - A_1 + L_1 C_2 \\ 0 & N_2 - A_2 + L_2 C_2 \end{bmatrix} = 0, \tag{10}$$

where

$$N[TEF \quad TEQ] = [TEF \quad TEQ] \begin{bmatrix} A_{TZ} & N_1 \\ 0 & N_2 \end{bmatrix}, \tag{11}$$

$$TA[F \quad Q] = [TEF \quad TEQ] \begin{bmatrix} A_{TZ} & A_1 \\ 0 & A_2 \end{bmatrix}, \tag{12}$$

$$C[F \quad Q] = [0 \quad C_2] \quad \text{and} \quad L = [TEF \quad TEQ] \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \tag{13}$$

and the last three Eqs. (11)–(13) can be derived from Eqs. (4), (5) and (9). It can be seen that if the matrices pair (A_2, C_2) is unobservable, then its unobservable eigenvalue is a transmission zero of the original system. So, if A_{TZ} contains all transmission zeros, (A_2, C_2) is an observable pair. Hence, from Eqs. (10) and (11), the result follows. \square

The following notations are given for gathering the chains of generalized transmission vectors with the same lengths to the same group, and then the lengths of generalized transmission vectors in different group would have different lengths.

For a transmission zero λ_i , assume that the elements of the set $\{\bar{\rho}_{i1}, \dots, \bar{\rho}_{i\phi_i}\}$ are arranged as follows: $\bar{\rho}_{i1} \leq \bar{\rho}_{i2} \leq \dots \leq \bar{\rho}_{i\phi_i}$. Denote ϕ_i as the number of distinct elements in the set $\{\bar{\rho}_{i1}, \dots, \bar{\rho}_{i\phi_i}\}$. And the notations $\sigma_{i1}, \sigma_{i2}, \dots, \sigma_{i\phi_i}$, satisfying $\sigma_{i1} < \sigma_{i2} < \dots < \sigma_{i\phi_i}$, represent all distinct elements of the set $\{\bar{\rho}_{i1}, \dots, \bar{\rho}_{i\phi_i}\}$. Assume that there are η_{il} elements with value $\sigma_{il}, l = 1, \dots, \phi_i$, within the set $\{\bar{\rho}_{i1}, \dots, \bar{\rho}_{i\phi_i}\}$. Let

$$\begin{aligned} V_{il}^k &= \left[v_{i(\eta_{i0}+\eta_{i1}+\dots+\eta_{i(l-1)}+1)}^k \quad v_{i(\eta_{i0}+\eta_{i1}+\dots+\eta_{i(l-1)}+2)}^k \quad \dots \quad v_{i(\eta_{i0}+\eta_{i1}+\dots+\eta_{i(l-1)}+\eta_{il})}^k \right], \\ W_{il}^k &= \left[w_{i(\eta_{i0}+\eta_{i1}+\dots+\eta_{i(l-1)}+1)}^k \quad w_{i(\eta_{i0}+\eta_{i1}+\dots+\eta_{i(l-1)}+2)}^k \quad \dots \quad w_{i(\eta_{i0}+\eta_{i1}+\dots+\eta_{i(l-1)}+\eta_{il})}^k \right], \\ k &= 1, \dots, \sigma_{il}, \quad l = 1, \dots, \phi_i \quad \text{and} \quad \eta_{i0} = 0, \end{aligned}$$

where the column vectors of V_{il}^k are the right generalized transmission vectors of grade k , in all chains with length σ_{il} . Then by (9), we have

$$\begin{bmatrix} (A - \lambda_i E) & D \\ C & 0 \end{bmatrix} \begin{bmatrix} V_{il}^k \\ W_{il}^k \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{il}^{k-1} \\ W_{il}^{k-1} \end{bmatrix}, \quad k = 1, \dots, \sigma_{il}, \quad V_{il}^0 = 0. \tag{14}$$

A series of matrices $V_{il}^1, V_{il}^2, \dots, V_{il}^{\sigma_{il}}$, group those chains of right generalized transmission vectors with the same length $\sigma_{il}, l = 1, \dots, \phi_i$.

Let

$$U_{il} = [V_{il}^{\sigma_{i1}} \quad \dots \quad V_{il}^{\sigma_{i\phi_i}}]$$

which gathers all last generalized transmission vectors in those chains with length less than or equal to σ_{il} . And it can be seen that

$$U_{il} = [U_{i(l-1)} \quad V_{il}^{\sigma_{il}}].$$

4. Problem formulation

The key problem in this chapter is now stated as follows:

Unknown input observer design by eigenstructure assignment: Given a symmetric set of complex numbers $\{\lambda_1, \dots, \lambda_\pi\}$ which contain the transmission zeros, and a set of positive integers $\rho_{i1}, \dots, \rho_{i\theta_i}, i = 1, \dots, \pi$, representing the multiplicities and satisfying that $\sum_{i=1}^\pi \sum_{j=1}^{\theta_i} \rho_{ij} = r$, we want to find the parametric solutions of the matrices N, L, G, P, Q, Z and T over the field of real number satisfying (4)–(7) where the matrix N has eigenvalues $\{\lambda_1, \dots, \lambda_\pi\}$ and left generalized eigenvectors $h_{ij}^1, \dots, h_{ij}^{\rho_{ij}}, j = 1, \dots, \theta_i; i = 1, \dots, \pi$ satisfying the relation:

$$\begin{aligned} h_{ij}^k N &= \lambda_i h_{ij}^k + h_{ij}^{k-1}, \text{ for } i = 1, \dots, \pi, \quad j = 1, \dots, \theta_i, \\ k &= 1, \dots, \rho_{ij}, \text{ and } h_{ij}^0 = 0. \end{aligned} \tag{15}$$

Since $h_{ij}^k, i = 1, \dots, \pi; j = 1, \dots, \theta_i; k = 1, \dots, \rho_{ij}$ are linearly independent, Eqs. (4) and (5) are equivalent to the following conditions:

$$\begin{aligned} -h_{ij}^k T(A - \lambda_i I) + h_{ij}^k LC &= -h_{ij}^{k-1} T, \quad h_{ij}^k TD = 0, \\ i &= 1, \dots, \pi, \quad j = 1, \dots, \theta_i, \quad k = 1, \dots, \rho_{ij}. \end{aligned} \tag{16}$$

Let $\tilde{t}_{ij}^k = -h_{ij}^k T$ and $\tilde{l}_{ij}^k = h_{ij}^k L$; then Eq. (16) is equivalent to

$$\begin{bmatrix} \tilde{t}_{ij}^k & \tilde{l}_{ij}^k \end{bmatrix} \begin{bmatrix} (A - \lambda_i E) & D \\ C & 0 \end{bmatrix} = \begin{bmatrix} \tilde{t}_{ij}^{k-1} E & 0 \end{bmatrix}. \tag{17}$$

Define H as

$$H = \begin{bmatrix} H_1 \\ \vdots \\ H_\pi \end{bmatrix}, \text{ where } H_i = \begin{bmatrix} H_{i1} \\ \vdots \\ H_{i\theta_i} \end{bmatrix}, \text{ and } H_{ij} = \begin{bmatrix} h_{ij}^1 \\ \vdots \\ h_{ij}^{\rho_{ij}} \end{bmatrix} \tag{18}$$

and the matrices \tilde{T} and \tilde{L} are defined in the same way.

If the solutions of $\tilde{t}_{ij}^1, \dots, \tilde{t}_{ij}^{\rho_{ij}}$ and $\tilde{l}_{ij}^1, \dots, \tilde{l}_{ij}^{\rho_{ij}}, j = 1, \dots, \theta_i$ corresponding to eigenvalues $\lambda_i, i = 1, \dots, \pi$ have been found from Eq. (17), then by choosing a nonsingular H , the solutions of T, L, N and G can be obtained as follows:

$$T = -H^{-1} \tilde{T}, \quad L = H^{-1} \tilde{L}, \quad N = H^{-1} A H \quad \text{and} \quad G = TB, \tag{19}$$

where A is a Jordan form matrix in lower case with eigenvalues $\{\lambda_1, \dots, \lambda_\mu\}$ and multiplicity $\{\rho_{i1}, \dots, \rho_{i\theta_i}; i = 1, \dots, \mu\}$.

Furthermore, if the matrix \tilde{T} or T satisfies the following relation:

$$\text{rank} \begin{bmatrix} TE \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} \tilde{T}E \\ C \end{bmatrix} = n,$$

we can obtain a pseudo-inverse of $\begin{bmatrix} TE \\ C \end{bmatrix}$ as follows:

$$M = \left(\begin{bmatrix} TE \\ C \end{bmatrix}^H \begin{bmatrix} TE \\ C \end{bmatrix} \right)^{-1} \begin{bmatrix} TE \\ C \end{bmatrix}^H,$$

where X^H means the Hermitian adjoint of the matrix X . The solutions of P and Q are as follows:

$$\begin{bmatrix} P & Q \end{bmatrix} = M + K \left(I_{r+q} - \begin{bmatrix} TE \\ C \end{bmatrix} M \right),$$

where $K \in R^{n \times (r+q)}$ represents free parameters. Then the matrix Z can be obtained by $Z = -QF$.

From the above discussion, the following proposition, similar to that given by Moore [19] and Klein and Moore [20], characterizing all possible solutions of the observer is given.

Lemma 4. Let $\{\lambda_1, \dots, \lambda_\mu\}$ be a symmetric set of complex numbers and let $\{\rho_{i1}, \dots, \rho_{i\theta_i}; i = 1, \dots, \mu\}$ be a set of positive integers satisfying that $\sum_{i=1}^\mu \sum_{j=1}^{\theta_i} \rho_{ij} = r$. There exist N, G, P, Q, T of real number and $h_{ij}^1, \dots, h_{ij}^{\rho_{ij}}; j = 1, \dots, \theta_i; i = 1, \dots, \pi$, satisfying (4)–(7) and (15) if and only if

- (a) The matrix H is nonsingular.
- (b) For each $i \in \{1, \dots, \pi\}$ there exists $i' \in \{1, \dots, \pi\}$ such that $\lambda_i = (\lambda_{i'})^*, \theta_i = \theta_{i'}, \rho_{ij} = \rho_{i'j}, j = 1, \dots, \theta_i$, and $h_{ij}^k = (h_{ij}^k)^*, i = 1, \dots, \pi; j = 1, \dots, \theta_i; k = 1, \dots, \rho_{ij}$, where $(h_{ij}^k)^*$ means the component-wise conjugate of the vector h_{ij}^k .
- (c) For each $i \in \{1, \dots, \mu\}$, there exists a set of vectors $\{\tilde{t}_{ij}^k, \bar{t}_{ij}^k, i = 1, \dots, \pi; j = 1, \dots, \theta_i; k = 1, \dots, \rho_{ij}\}$ satisfying Eq. (17) and

$$\text{rank} \begin{bmatrix} \tilde{T}E \\ C \end{bmatrix} = n. \tag{20}$$

Proof. The sufficiency has been stated above. Assume that the solutions exist. Since the matrix H represents the left generalized eigenvectors, it is nonsingular. If Eq. (20) is not satisfied, P and Q do not exist. Hence, the necessity follows. \square

In the above deriving process, it can be seen that the matrix H can be arbitrary chosen to satisfy conditions (a) and (b) in Lemma 4. Hence, the existing condition of the observer can be stated briefly as that the observer exists if and only if a matrix \tilde{T}

satisfying Eqs. (17) and (20) exists. The following lemma discusses the possible orders of the observer.

Lemma 5. *If the observer exists, its order can be chosen within $n - q \leq r \leq n$.*

Proof. If an observer of order r exists, a matrix $\tilde{T} \in R^{r \times n}$ satisfying Eqs. (17) and (20) can be found. It follows from Eq. (20) that $\text{rank } \tilde{T}E \geq n - q$. Hence, we have $r \geq n - q$. This means that the order of the observers must be at least $n - q$. If an observer of order $n - q$ is required, a new matrix $\tilde{T}' \in R^{(n-q) \times n}$ satisfying Eq. (20) can be obtained by carefully deleting $r - (n - q)$ rows of \tilde{T} and an observer of order $n - q$ can be obtained by this new matrix \tilde{T}' from the process discussed above. If an observer of order n is required, a new matrix $\tilde{T}'' \in R^{n \times n}$ satisfying (20) can be obtained by adding $n - r$ additional solutions of Eq. (17) to the row vectors of \tilde{T} and an observer of order n can be obtained from this new matrix \tilde{T}'' . Other observers of order within $n - q \leq r \leq n$ can be obtained in a similar way. \square

5. Main results

From Lemma 4, the problem now is to find the solutions of Eq. (17). In this section, we shall establish the complete parametric solutions of $\tilde{l}_{ij}^1, \dots, \tilde{l}_{ij}^{p_{ij}}$ and $\tilde{l}_{ij}^1, \dots, \tilde{l}_{ij}^{p_{ij}}$ with an eigenvalue λ_i from Eq. (17). The properties of Eq. (17) depend on whether the eigenvalue λ_i is a transmission zero or not; hence, we shall first establish the parametric solutions of Eq. (17) for eigenvalues which are not transmission zeros and then the parametric solutions of Eq. (17) for eigenvalues which are transmission zeros.

5.1. The solutions for eigenvalues which are not transmission zeros

If λ_i is not a transmission zero, then

$$\text{rank} \begin{bmatrix} (A - \lambda_i E) & D \\ C & 0 \end{bmatrix} = n + m.$$

We can obtain a nonsingular transformation as follows:

$$\begin{bmatrix} \Upsilon_i^{11} & \Upsilon_i^{12} \\ \Upsilon_i^{21} & \Upsilon_i^{22} \end{bmatrix} \begin{bmatrix} (A - \lambda_i E) & D \\ C & 0 \end{bmatrix} \begin{bmatrix} \Psi_i^1 \\ \Psi_i^2 \end{bmatrix} = \begin{bmatrix} I_{(n+m) \times (n+m)} \\ 0 \end{bmatrix}, \quad (21)$$

where $\Upsilon_i^{11} \in C^{(n+m) \times n}$, $\Upsilon_i^{12} \in C^{(n+m) \times q}$, $\Upsilon_i^{21} \in C^{(q-m) \times n}$, $\Upsilon_i^{22} \in C^{(q-m) \times q}$, $\Psi_i^1 \in C^{n \times (n+m)}$, $\Psi_i^2 \in C^{m \times (n+m)}$ and $I_{(n+m) \times (n+m)}$ is an $(n + m) \times (n + m)$ identity matrix. We can obtain the following theorem.

Theorem 6. If λ_i is not a transmission zero, the complete solution of Eq. (17) are as follows:

$$\begin{bmatrix} \tilde{t}_{ij}^k & \tilde{l}_{ij}^k \end{bmatrix} = \begin{bmatrix} \tilde{t}_{ij}^{k-1} E \Psi_i^1 & f_{ij}^{k-1} \end{bmatrix} \begin{bmatrix} \Upsilon_i^{11} & \Upsilon_i^{12} \\ \Upsilon_i^{21} & \Upsilon_i^{22} \end{bmatrix}, \quad k = 1, \dots, \rho_{ij}, \quad \tilde{t}_{ij}^0 = 0, \quad (22)$$

where f_{ij}^{k-1} are vectors representing the free parameters.

Proof. *Necessity:* Introduce the following variable transformation:

$$\begin{bmatrix} \tilde{t}_{ij}^k & \tilde{l}_{ij}^k \end{bmatrix} = \begin{bmatrix} a_{ij}^k & f_{ik}^{k-1} \end{bmatrix} \begin{bmatrix} \Upsilon_i^{11} & \Upsilon_i^{12} \\ \Upsilon_i^{21} & \Upsilon_i^{22} \end{bmatrix}, \quad k = 1, \dots, \rho_{ij}. \quad (23)$$

From Eqs. (17) and (21), we have

$$\begin{bmatrix} \tilde{t}_{ij}^{k-1} & E & 0 \end{bmatrix} \begin{bmatrix} \Psi_i^1 \\ \Psi_i^2 \end{bmatrix} = \begin{bmatrix} a_{ij}^k & f_{ij}^k \end{bmatrix} \begin{bmatrix} I_{(n+m) \times (n+m)} \\ 0 \end{bmatrix}, \quad k = 1, \dots, \rho_{ij},$$

hence

$$a_{ij}^k = \tilde{t}_{ij}^{k-1} E \Psi_i^1, \quad k = 1, \dots, \rho_{ij}.$$

By applying this relation into Eq. (23), Eq. (22) is obtained.

Sufficiency: Using Eqs. (22) and (21), we have

$$\begin{aligned} \begin{bmatrix} \tilde{t}_{ij}^k & \tilde{l}_{ij}^k \end{bmatrix} \begin{bmatrix} (A - \lambda_i E) & D \\ C & 0 \end{bmatrix} &= \begin{bmatrix} \tilde{t}_{ij}^{k-1} E \Psi_i^1 & \tilde{z}_{ij}^k \end{bmatrix} \begin{bmatrix} \Upsilon_i^{11} & \Upsilon_i^{12} \\ \Upsilon_i^{21} & \Upsilon_i^{22} \end{bmatrix} \begin{bmatrix} (A - \lambda_i E) & D \\ C & 0 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{t}_{ij}^{k-1} E \Psi_i^1 & \tilde{z}_{ij}^k \end{bmatrix} \begin{bmatrix} I_{(n+m) \times (n+m)} \\ 0 \end{bmatrix} \begin{bmatrix} \Psi_i^1 \\ \Psi_i^2 \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{t}_{ij}^{k-1} E & 0 \end{bmatrix} \begin{bmatrix} \Psi_i^1 \\ \Psi_i^2 \end{bmatrix} \begin{bmatrix} \Psi_i^1 \\ \Psi_i^2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \tilde{t}_{ij}^{k-1} E & 0 \end{bmatrix}. \end{aligned}$$

Therefore the vectors given by Eq. (22) satisfy Eq. (17). \square

5.2. The solutions for eigenvalues which are transmission zeros

If λ_i is a transmission zero, then

$$\text{rank} \begin{bmatrix} (A - \lambda_i E) & D \\ C & 0 \end{bmatrix} = r_i < n + m.$$

We can obtain the following nonsingular transformation:

$$\begin{bmatrix} \Upsilon_i^{11} & \Upsilon_i^{12} \\ \Upsilon_i^{21} & \Upsilon_i^{22} \end{bmatrix} \begin{bmatrix} (A - \lambda_i E) & D \\ C & 0 \end{bmatrix} \begin{bmatrix} \Psi_i^1 & \Psi_i^1 \\ \Psi_i^2 & \Psi_i^2 \end{bmatrix} = \begin{bmatrix} I_{r_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad (24)$$

where $\Upsilon_i^{11} \in C^{r_i \times n}$, $\Upsilon_i^{12} \in C^{r_i \times q}$, $\Upsilon_i^{21} \in C^{(n+q-r_i) \times n}$, $\Upsilon_i^{22} \in C^{(n+q-r_i) \times q}$, $\Psi_i^1 \in C^{n \times r_i}$, $\Psi_i^1 \in C^{n \times (n+m-r_i)}$, $\Psi_i^2 \in C^{m \times r_i}$, $\Psi_i^2 \in C^{m \times (n+m-r_i)}$, and I_{r_i} is an $r_i \times r_i$ identity matrix. We can obtain the following lemma.

Lemma 7. $\Upsilon_i^{21}EU_{il}$ is of full column rank for $l = 1, \dots, \phi_i$.

Proof. If $\Upsilon_i^{21}EU_{il}$ is not of full column rank, then we can find a nonzero vector f such that $\Upsilon_i^{21}EU_{il}f = 0$. Since the matrix

$$\begin{bmatrix} I_{r_i} & 0 \end{bmatrix} \begin{bmatrix} \Psi_i^1 & \Psi_i^1 \\ \Psi_i^2 & \Psi_i^2 \end{bmatrix}^{-1}$$

is of full row rank, we can find a vector $\begin{bmatrix} \bar{v} \\ \bar{w} \end{bmatrix}$ such that

$$\begin{bmatrix} I_{r_i} & 0 \end{bmatrix} \begin{bmatrix} \Psi_i^1 & \Psi_i^1 \\ \Psi_i^2 & \Psi_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{v} \\ \bar{w} \end{bmatrix} = \begin{bmatrix} \Upsilon_i^{11} & \Upsilon_i^{12} \end{bmatrix} \begin{bmatrix} EU_{il}f \\ 0 \end{bmatrix}.$$

Then we can obtain that

$$\begin{bmatrix} A - \lambda_i E & D \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{v} \\ \bar{w} \end{bmatrix} = \begin{bmatrix} EU_{il}f \\ 0 \end{bmatrix}.$$

So, $\begin{bmatrix} EU_{il}f \\ 0 \end{bmatrix}$ is in the column space of

$$\begin{bmatrix} A - \lambda_i E & D \\ C & 0 \end{bmatrix}.$$

It is a contradiction to the definition of U_{il} . \square

From Lemma 7, for each $U_{il}, l = 1, \dots, \phi_i$, we can obtain the following nonsingular transformation:

$$\begin{bmatrix} S_{il}^1 \\ S_{il}^2 \end{bmatrix} \Upsilon_i^{21}EU_{il} = \begin{bmatrix} I_{\eta_{i1} + \dots + \eta_{il}} \\ 0 \end{bmatrix}, \tag{25}$$

where $S_{il}^1 \in R^{(\eta_{i1} + \dots + \eta_{il}) \times (n+q-r_i)}$, $S_{il}^2 \in R^{(n+q-r_i-\eta_{i1}-\dots-\eta_{il}) \times (n+q-r_i)}$ and the matrix $\begin{bmatrix} S_{il}^1 \\ S_{il}^2 \end{bmatrix}$ is nonsingular. To find S_{il}^1 and S_{il}^2 for each $l \in \{1, \dots, \phi_i\}$, we only need to calculate $S_{i\phi_i}^1$ and $S_{i\phi_i}^2$. Let

$$\begin{bmatrix} S_{i\phi_i}^1 \\ S_{i\phi_i}^2 \end{bmatrix} = \begin{bmatrix} s_{i1} \\ \vdots \\ s_{i\phi_i} \\ s_{i(\phi_i+1)} \end{bmatrix},$$

where $s_{il} \in \mathbb{R}^{\eta_{il} \times (n+q-r_i)}$, $l = 1, \dots, \phi_i$ and $s_{i(\phi_i+1)} \in \mathbb{R}^{(n+q-r_i-\eta_{i1}-\dots-\eta_{i\phi_i}) \times (n+q-r_i)}$. Then S_{il}^1 and S_{il}^2 can be selected as

$$S_{il}^1 = \begin{bmatrix} s_{i1} \\ \vdots \\ s_{il} \end{bmatrix} \quad \text{and} \quad S_{i2}^1 = \begin{bmatrix} s_{i(l+1)} \\ \vdots \\ s_{i(\phi_i+1)} \end{bmatrix}.$$

The following theorem can now be given.

Theorem 8. Assume that λ_i is a transmission zero.

(a) If $\rho_{ij} \leq \sigma_{i1}$, the complete solutions of Eq. (17) can be represented as follows:

$$\begin{bmatrix} \tilde{t}_{ij}^k & \tilde{t}_{ij}^k \end{bmatrix} = \begin{bmatrix} \tilde{t}_{ij}^{k-1} E \Psi_i^1 & f_{ij}^{k-1} \end{bmatrix} \begin{bmatrix} \gamma_i^{11} & \gamma_i^{12} \\ \gamma_i^{21} & \gamma_i^{22} \end{bmatrix} \tilde{t}_{ij}^0 = 0, \quad k = 1, \dots, \rho_{ij}, \quad (26)$$

where f_{ij}^{k-1} , $k = 1, \dots, \rho_{ij}$ are vectors representing the free parameters.

(b) If $\sigma_{i1} < \rho_{ij} \leq \sigma_{i\phi_i}$ there exists b such that $\sigma_{ib} < \rho_{ij} \leq \sigma_{i(b+1)}$, and if $\rho_{ij} > \sigma_{i\phi_i}$, then $b = \phi_i$. Under the above conditions, the complete solutions of Eq. (17) can be represented as follows:

$$\begin{aligned} \begin{bmatrix} \tilde{t}_{ij}^k & \tilde{t}_{ij}^k \end{bmatrix} &= \begin{bmatrix} \tilde{t}_{ij}^{k-1} E \Psi_i^1 & f_{ij}^{k-1} \end{bmatrix} \begin{bmatrix} I & -\gamma_i^{11} E U_{ib} S_{ib}^1 \\ 0 & S_{ib}^2 \end{bmatrix} \begin{bmatrix} \gamma_i^{11} & \gamma_i^{12} \\ \gamma_i^{21} & \gamma_i^{22} \end{bmatrix}, \\ \tilde{t}_{ij}^0 &= 0, k = 1, \dots, (\rho_{ij} - \sigma_{ib}), \\ \begin{bmatrix} \tilde{t}_{ij}^k & \tilde{t}_{ij}^k \end{bmatrix} &= \begin{bmatrix} \tilde{t}_{ij}^{k-1} E \Psi_i^1 & f_{ij}^{k-1} \end{bmatrix} \begin{bmatrix} I & -\gamma_i^{11} E U_{il} S_{il}^1 \\ 0 & S_{il}^2 \end{bmatrix} \begin{bmatrix} \gamma_i^{11} & \gamma_i^{12} \\ \gamma_i^{21} & \gamma_i^{22} \end{bmatrix}, \\ k &= (\rho_{ij} - \sigma_{i(l+1)}) + 1, \dots, (\rho_{ij} - \sigma_{il}), \quad l = (b - 1), \dots, 1, \\ \begin{bmatrix} \tilde{t}_{ij}^k & \tilde{t}_{ij}^k \end{bmatrix} &= \begin{bmatrix} \tilde{t}_{ij}^{k-1} E \gamma_i^1 & f_{ij}^{k-1} \end{bmatrix} \begin{bmatrix} \gamma_i^{11} & \gamma_i^{12} \\ \gamma_i^{21} & \gamma_i^{22} \end{bmatrix}, \quad k = (\rho_{ij} - \sigma_{i1}) + 1, \dots, \rho_{ij}, \quad (27) \end{aligned}$$

where f_{ij}^{k-1} , $k = 1, \dots, \rho_{ij}$ are vectors representing the free parameters.

Proof. Here, we will show that Eq. (26) is equivalent to Eqs. (17) and (27) is equivalent to Eq. (17).

Necessity: A variable transformation is adopted as follows:

$$\begin{bmatrix} \tilde{t}_{ij}^k & \tilde{t}_{ij}^k \end{bmatrix} = \begin{bmatrix} a_{ij}^k & c_{ij}^k \end{bmatrix} \begin{bmatrix} \gamma_i^{11} & \gamma_i^{12} \\ \gamma_i^{21} & \gamma_i^{22} \end{bmatrix}, \quad k = 1, \dots, \rho_{ij}, \quad (28)$$

where $a_{ij}^k \in C^{1 \times r}$ and $c_{ij}^k \in C^{1 \times (m+n-r)}$. It can be seen that Eq. (17) is equivalent to the following relations:

$$a_{ij}^k = \tilde{t}_{ij}^{k-1} E \Psi_i^1, \tag{29}$$

$$0 = \tilde{t}_{ij}^{k-1} E \Psi_i^1, \quad k = 1, \dots, \rho_{ij}. \tag{30}$$

Observe that the column space of Ψ_i^1 is the same as the column space of $V_{il}^1, l = 1, \dots, \phi_i$. Hence, Eq. (30) is equivalent to

$$\tilde{t}_{ij}^{k-1} E V_{il}^1 = 0, \quad l = 1, \dots, \phi_i, \quad k = 1, \dots, \rho_{ij}.$$

From Eqs. (14) and (17), it can be further shown that

$$\tilde{t}_{ij}^{k-1} E V_{il}^1 = \begin{cases} \tilde{t}_{ij}^{(k-\sigma_{il})} E V_{il}^{\sigma_{il}} & \text{if } k > \sigma_{il}, \\ \tilde{t}_{ij}^0 E V_{il}^k \equiv 0 & \text{if } k \leq \sigma_{il}. \end{cases} \tag{31}$$

We discuss the problem in the following two cases:

- (a) It can be seen from Eq. (31) that if $\rho_{ij} \leq \sigma_{il}$, Eq. (30) would be satisfied for all possible \tilde{t}_{ij}^{k-1} . Hence, Eq. (17) is equivalent to Eq. (29). Let $f_{ij}^{k-1} = c_{ij}^k, k = 1, \dots, \rho_{ij}$. By applying Eqs. (29) to (28), Eq. (26) is obtained.
- (b) If $\sigma_{i1} < \rho_{ij} \leq \sigma_{i\phi_i}$, there exists a number b such that $\sigma_{ib} < \rho_{ij} \leq \sigma_{i(b+1)}$, and if $\rho_{ij} > \sigma_{i\phi_i}$, then $b = \phi_i$. Under the above conditions, from Eq. (31), Eq. (30) is equivalent to the following relations:

$$\begin{aligned} \tilde{t}_{ij}^{k-1} E U_{ib} &= 0, \text{ where } k = 2, \dots, (\rho_{ij} - \sigma_{ib}^b) + 1, \\ \tilde{t}_{ij}^{k-1} E U_{i(b-1)} &= 0, \text{ where } k = (\rho_{ij} - \sigma_{ib}) + 2, \dots, (\rho_{ij} - \sigma_{i(b-1)}) + 1, \\ &\vdots \\ \tilde{t}_{ij}^{k-1} E U_{i1} &= 0, \text{ where } k = (\rho_{ij} - \sigma_{i2}) + 2, \dots, (\rho_{ij} - \sigma_{i1}) + 1. \end{aligned}$$

From Eq. (28), we have

$$\begin{aligned} (a_{ij}^{k-1} \Upsilon_i^{11} + c_{ik}^{k-1} \Upsilon_i^{21}) E U_{ib} &= 0, \quad k = 2, \dots, (\rho_{ij} - \sigma_{ib}) + 1, \\ (a_{ij}^{k-1} \Upsilon_i^{11} + c_{ij}^{k-1} \Upsilon_i^{21}) E U_{il} &= 0, \quad k = (\rho_{ij} - \sigma_{i(l+1)}) + 2, \dots, (\rho_{ij} - \sigma_{il}) + 1, \\ l &= 1, \dots, (b - 1). \end{aligned}$$

From Eqs. (25) and (29), it can be seen that the above equations are equivalent to

$$\begin{aligned} c_{ij}^k &= -\tilde{t}_{ij}^{k-1} E \Psi_i^1 \Upsilon_i^{11} E U_{1b} S_{ib}^1 + f_{ij}^{k-1} S_{ib}^2, \quad k = 1, \dots, (\rho_{ij} - \sigma_{ib}), \\ c_{ij}^k &= -\tilde{t}_{ij}^{k-1} E \Psi_i^1 \Upsilon_i^{11} E U_{1l} S_{il}^1 + f_{ij}^{k-1} S_{il}^2, \quad k = (\rho_{ij} - \sigma_{i(l+1)}) + 1, \dots, (\rho_{ij} - \sigma_{il}), \\ l &= 1, \dots, (b - 1), \end{aligned} \tag{32}$$

where f_{ij}^k are column vectors with appropriate dimension and represent the free parameters. For $k = (\rho_{ij} - \sigma_{il}), \dots, \rho_{ij}$, c_{ij}^k is free; hence, define

$$c_{ij}^k = f_{ij}^{k-1}, k = (\rho_{ij} - \sigma_{il}) + 1, \dots, \rho_{ij}. \tag{33}$$

By applying Eqs. (29), (32) and Eqs. (33) to (28), Eq. (27) is obtained.

Sufficiency: Define the variable transformation (28). Since the matrix

$$\begin{bmatrix} \Upsilon_i^{11} & \Upsilon_i^{12} \\ \Upsilon_i^{21} & \Upsilon_i^{22} \end{bmatrix}$$

is nonsingular, Eq. (28) is a nonsingular transformation. It can also be seen that Eq. (31) is a equivalent relation; Hence, from Eqs. (26) or (27), we can derive Eq. (17) by reversing the procedure in the necessity. \square

5.3. The realness of the solutions

The realness of the solutions is guarantee by the following lemma:

Lemma 9. *Under the conditions proposed by Lemma 4, if the free parameters in Theorems 6 and 8 are chosen as $f_{ij}^k = (f_{ij}^k)^*$ when $\lambda_i = (\lambda_i)^*$, then the solutions N, L, G, P, Q and W are real.*

Proof. Observe that h_{ij}^k , the row vector of H , is the left generalized eigenvector of N . Let $V = H^{-1}$, and v_{ij}^k be the corresponding column vector of V , representing the right generalized eigenvector of V . Then, under the conditions (2) of Lemma 4, we have $v_{ij}^k = (v_{ij}^k)^*$ when $\lambda_i = (\lambda_i)^*$. Then, we have $(N)^* = (H^{-1}AH)^* = \sum_{i=1}^{\pi} \sum_{j=1}^{\theta_i} \sum_{k=1}^{\rho_{ij}} (\lambda_i)^* (v_{ij}^k)^* (h_{ij}^k)^* = \sum_{i=1}^{\pi} \sum_{j=1}^{\theta_i} \sum_{k=1}^{\rho_{ij}} \lambda_i v_{ij}^k h_{ij}^k = H^{-1}AH = N$. Hence, the matrix N is real.

If $f_{ij}^k = (f_{ij}^k)^*$, when $\lambda_i = (\lambda_i)^*$, then from Lemma 4 and $*$, we have $\tilde{t}_{ij}^k = (\tilde{t}_{ij}^k)^*$, and $\tilde{l}_{ij}^k = (\tilde{l}_{ij}^k)^*$ when $\lambda_i = (\lambda_i)^*$. Then $(T)^* = (-H^{-1}\tilde{T})^* = -\sum_{i=1}^{\pi} \sum_{j=1}^{\theta_i} \sum_{k=1}^{\rho_{ij}} (\lambda_i)^* (v_{ij}^k)^* (\tilde{t}_{ij}^k)^* = -\sum_{i=1}^{\pi} \sum_{j=1}^{\theta_i} \sum_{k=1}^{\rho_{ij}} \lambda_i v_{ij}^k \tilde{t}_{ij}^k = -H^{-1}\tilde{T} = T$ and $(L)^* = (-H^{-1}\tilde{L})^* = -\sum_{i=1}^{\pi} \sum_{j=1}^{\theta_i} \sum_{k=1}^{\rho_{ij}} (\lambda_i)^* (v_{ij}^k)^* (\tilde{l}_{ij}^k)^* = -\sum_{i=1}^{\pi} \sum_{j=1}^{\theta_i} \sum_{k=1}^{\rho_{ij}} \lambda_i v_{ij}^k \tilde{l}_{ij}^k = H^{-1}\tilde{L} = L$. Hence, the matrices T and L are real.

Since T is real. It follows that P, Q, G and W are real. \square

6. Example

Example 1. Consider system (1) with the following matrices [18]

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This system contain no transmission zero, so eigenvalues of the observer can be arbitrarily assigned. We illustrate the design of the proposed method for unknown input observer from minimum order to full order.

(a) *First(minimum)-order observer*: Let the eigenvalue of the observer be λ , then

$$T = -h^{-1}f_{11}^0 \begin{bmatrix} 0 & -1 & -\lambda \end{bmatrix}, L = h^{-1}f_{11}^0 \begin{bmatrix} -\lambda^2 & 1 \end{bmatrix}, [P \quad Q] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{h}{f_{11}^0} & -\lambda & 0 \end{bmatrix}.$$

Let $a = h^{-1}f_{11}^0$, then the parametric solutions of the observers are

$$\begin{aligned} \dot{z} &= \lambda z + [-a\lambda^2 \quad a]y, \\ \hat{x} &= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{a} \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -\lambda & 0 \end{bmatrix} y. \end{aligned} \quad (\text{b})$$

(b) *Second-order observer*: Let the eigenvalues of the observers be two complex conjugate numbers $-1+i$ and $-1-i$, then

$$\begin{aligned} \tilde{t}_1 &= f_{11}^0 \begin{bmatrix} 0 & -1 & 1-i \end{bmatrix}, \tilde{t}_2 = f_{21}^0 \begin{bmatrix} 0 & -1 & 1+i \end{bmatrix}, \\ \tilde{l}_1 &= f_{11}^0 \begin{bmatrix} 2i & 1 \end{bmatrix}, \tilde{l}_2 = f_{21}^0 \begin{bmatrix} -2i & 1 \end{bmatrix}. \end{aligned}$$

For considering the realness of the observers, choose the free parameters to be complex conjugate as $f_{11}^0 = \bar{f}_{21}^0 = 1+i$,

$$H = \begin{bmatrix} 1 & 1+i \\ 1 & 1-i \end{bmatrix} \text{ and } K = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

then the observer can be obtained as

$$\begin{aligned} \dot{z} &= \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} -4 & 0 \\ 2 & 1 \end{bmatrix} y, \\ \hat{x} &= \begin{bmatrix} -1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} -1 & 0 \\ -2 & 1 \\ 0 & 0 \end{bmatrix} y. \end{aligned}$$

For this system, the minimum order of the observer is 1. However, if the convergence of the error vector is required to be sinusoidal, then its order must be at least 2 for the realness of the observer. Thus, if only the method for minimum order observer design is obtained, the observer with sinusoidal convergence cannot be obtained; while by the present method, this requirement can be easily achieved as shown in the example.

(c) *Third(full)-order observer*: Let the eigenvalues of the observers be $-1+i, -1-i$, and -2 then

$$\begin{aligned}\tilde{t}_1 &= f_{11}^0 [0 \quad -1 \quad 2], & \tilde{t}_2 &= f_{21}^0 [0 \quad -1 \quad 1-i], & \tilde{t}_3 &= f_{31}^0 [0 \quad -1 \quad 1+i], \\ \tilde{l}_1 &= f_{11}^0 [-4 \quad 1], & \tilde{l}_2 &= f_{21}^0 [2i \quad 1], & \tilde{l}_3 &= f_{31}^0 [-2i \quad 1].\end{aligned}$$

For considering the realness of the observers, the free parameters are chosen to be complex conjugate. Then arbitrarily the parameters are chosen as $f_{11}^0 = f_{21}^0 = f_{31}^0 = 1$,

$$H \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1+i & 1-i \\ 1 & -1-i & 1+i \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 2 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix},$$

then the observer can be obtained as

$$\begin{aligned}\dot{z} &= \begin{bmatrix} 0 & -2 & 2 \\ 0 & -2 & 0 \\ -1 & 0 & -2 \end{bmatrix} z + \begin{bmatrix} 2 & 1 \\ -4 & 1 \\ -6 & 1 \end{bmatrix} y, \\ \hat{x} &= \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} z + \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} y.\end{aligned}$$

In this example, the unknown input observers for the three possible orders are designed in a unified way. In practice, a suitable order and free parameters can be chosen by practical consideration and specific control requirement.

Example 2. Consider system (1) with the following matrices

$$E = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 3 \\ 0 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

and $D = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

This system has one transmission zero -2 with $\bar{\theta}_1 = 1$ and $\bar{\rho}_{11} = 1$, so it is an eigenvalue of the observer.

(a) *First(minimum)-order observer*: The eigenvalue should be -2 and it follows that

$$\begin{bmatrix} \tilde{t}_{ij}^1 & \tilde{l}_{ij}^1 \end{bmatrix} = f_{ij}^0 \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 1 \end{bmatrix}.$$

Let $f_{11}^0 = [g_1 \quad g_2]$ and

$$T = \left[\frac{1}{h}(g_1 - g_2) - \frac{1}{h}(-g_1 + 2g_2) - \frac{1}{h}f_1 \right],$$

$$L = \left[-\frac{1}{h}g^2 \quad \frac{1}{h}g^2 \right], [P \quad Q] = \begin{bmatrix} 0 & 1 & 1 \\ \frac{h}{g_1 - 2g_2} & 1 & -\frac{1}{g_1 - 2g_2}g^2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $a_1 = h^{-1}g_1$ and $a_2 = h^{-1}g_2$, then the parametric solutions of the observers are

$$\dot{z} = -2z + [-a_2 \quad a_2]y,$$

$$\hat{x} = \begin{bmatrix} 0 \\ 1 \\ a_1 - 2a_2 \\ 0 \end{bmatrix} z + \begin{bmatrix} 1 & 1 \\ 1 & \frac{a_2}{a_1 - 2a_2} \\ 0 & -1 \end{bmatrix} y.$$

(b) *Third(full)-order observer*: Here, we assigned the eigenvalues of the observer to be the transmission zero $\lambda_1 = -2$ with $\theta_1 = 1$ and $\rho_{11} = 3$ to illustrate the assignment of generalized eigenvectors with an eigenvalue which is a transmission zero. By a simple computation, it follows that

$$[\tilde{r}_{1j}^k \quad \tilde{l}_{1j}^k] = [\tilde{r}_{1j}^{k-1} \quad f_{1j}^{k-1}] \begin{bmatrix} 1 & 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -2 & 1 & -2 \\ 1 & 0 & 2 & -1 & 1 \end{bmatrix}, \quad k = 1, 2,$$

$$[\tilde{r}_{1j}^3 \quad \tilde{l}_{1j}^3] = [\tilde{r}_{1j}^2 \quad f_{1j}^2] \begin{bmatrix} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 1 \end{bmatrix}.$$

Choose $f_{11}^0 = 1, f_{11}^1 = 2, f_{11}^2 = [1 \quad -1]$,

$$K = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The obtained observer is

$$\dot{z} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 1 \\ -1 & -1 & -2 \end{bmatrix} z + \begin{bmatrix} 0 & -1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} y,$$

$$\hat{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} y.$$

(c) *Third(full)-order observer:* Let the eigenvalues of the observer be $\lambda_1 = -2$ with $\rho_1 = 1$ and $\lambda_2 = -1$ with $\rho_2 = 2$ to illustrate the assignment of the generalized eigenvectors with an eigenvalue which is not a transmission zero. The solutions of the eigenvalues $\lambda_2 = -1$ are

$$\begin{bmatrix} \tilde{t}_{2j}^k & \tilde{l}_{2j}^k \end{bmatrix} = \begin{bmatrix} \tilde{t}_{2j}^{k-1} & f_{2j}^{k-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}, \quad k = 1, 2.$$

Choose $f_{11}^0 = [-1 \ 1]$, $f_{21}^0 = -1$, $f_{21}^1 = 1$,

$$H = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The obtained observer is

$$\dot{z} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ -2 & 1 & -3 \end{bmatrix} z + \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} y,$$

$$\hat{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} y.$$

Example 3. Consider system (1) with the following matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This system has been studied by Kawaji and Sawada [17] for the design of a first-order observer. It does not contain any transmission zero. Let the eigenvalue of the

observer be λ , then it follows that

$$T = -h^{-1}f_{11}^0 \begin{bmatrix} -\lambda & -1 & -\lambda \end{bmatrix}, \quad L = h^{-1}f_{11}^0 \begin{bmatrix} -\lambda(5 + \lambda) & 1 \end{bmatrix},$$

$$[P \quad Q] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{h}{f_{11}^0} & -\lambda & 0 \end{bmatrix}.$$

Let $a = h^{-1}f_{11}^0$, then the parametric solutions of the observers are

$$\dot{z} = \lambda z + \begin{bmatrix} -a\lambda(5 + \lambda) & a \end{bmatrix} y,$$

$$\hat{x} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{a} \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\lambda & 0 \end{bmatrix} y. \quad (34)$$

In Eq. (34), there is one more free parameter a than the results given by Kawaji and Sawada [14] where only the solutions when $a = 1$ are obtained. It can be seen that a is not a trivial parameter and it provides a more degree of freedom to assign a suitable observer according to the control requirement.

7. Conclusions

In this paper, the complete parametric solutions of the unknown input observers with orders between minimum and full orders for singular systems have been established. The complete and parametric solutions of the observer matrices and of the generalized eigenvector are obtained. In the illustrative examples, it can be seen that the solutions obtained by the proposed method have more free parameters than the previous results. Furthermore, in the proposed method, the order of the observer can be chosen from the range between $n-q$ and n , according to the control purposes. The completeness and parametric form of the solutions and flexibility in selecting the observer order make them more suitable for advanced applications.

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