Conclusion: In this note, the regulator problem of linear continuous-time systems with nonsymmetrical constrained control is studied. Necessary and sufficient conditions for domain $\mathscr{D}(F, q_1, q_2)$, which generates admissible control by feedback law, to be a positively invariant set w.r.t. system (6), are given. These conditions guarantee that system (1)–(7) is asymptotically stable for every motion emanating from domain $\mathscr{D}(F, q_1, q_2)$. A spectral analysis of equation $FA + FBF = HF$ is also given together with conditions on the existence of matrix H. The necessary condition of the main result is established by using an important property of the $\mathscr{K}\text{er}\, F$: when domain $\mathscr{D}(F, q_1, q_2)$ is positively invariant w.r.t. system (6), $\mathscr{K}\text{er}\, F$ is also positively invariant w.r.t. the system. Finally, the case of symmetrical constrained control is obtained easily by taking $q_1 = q_2 = \rho$.

References

[1] A. Benzaouia, "The regulator problem for linear discrete-time systems with nonsymmetrical constrained control," in *Proc. 30th CDC IEEE-Brighton*, 1991.

[2] A. Benzaouia and A. Hmamed, "Regulator problem for linear continuous systems with nonsymmetrical constrained control using nonsymmetrical Lyapunov functions," in *Proc. 31th CDC IEEE-Arizona*, 1992.

[3] A. Benzaouia and C. Burgat, (a) "Regulator problem for linear discrete-time systems with nonsymmetrical constrained control," *Int. J. Contr.*, vol. 48, no. 6, pp. 2441–2451, 1988; (b) "Existence of nonsymmetrical Lyapunov functions for linear systems," *Int. J. Syst. Sci.*, vol. 20, no. 4, pp. 597–607, 1989; (c) "Existence of nonsymmetrical stability domains for linear systems," *Linear Algebra Appl.*, vol. 121, pp. 217–231, 1989.

[4] G. Bitsoris, "Positively invariant polyhedral sets of discrete-time linear systems," *Int. J. Contr.*, vol. 47, no. 6, pp. 1713–1726, 1988.

[5] J. Cheganças, Sur le concept d'invariance positive appliqué à l'etude de la command avec contrainte des systèmes dynamiques. Thesis, D. I. Res. Rep. 85325. LAAS, Toulouse, 1985.

[6] P. O. Gutman and P. Hagander, "A new design of constrained controllers for linear systems," *IEEE Trans. Automat. Contr.*, vol. 30, pp. 22–33, 1985.

[7] W. Hahn, *Stability of Motion.* New York: Springer-Verlag, 1967.

[8] B. Porter, "Eigenvalue assignment in linear multivariable systems by output feedback," *Int. J. Contr.*, vol. 25, pp. 483–490, 1977.

[9] M. Vassilaki and G. Bitsoris, "Constrained regulation of linear continuous-time dynamical systems," *Syst. Contr. Lett.*, vol. 13, pp. 247–252, 1989.

Optimal Periodic Control Implemented as a Generalized Sampled-Data Hold Output Feedback Control

Nie-Zen Yen and Yung-Chun Wu

***Abstract*—In this note, a conversion method to convert the analog linear quadratic regulation control to a generalized sampled-data hold output feedback control for a linear periodic system or a linear time-invariant system is presented. It is shown that by using such a conversion, one can implement the optimal periodic control scheme in the presence of incomplete and delayed state measurements.**

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I. Introduction

Consider the optimal control problem of a linear periodic system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.a}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{1.b}$$

to minimize the following quadratic performance index:

$$J = \int_0^\infty x^\tau(t)Q(t)x(t) + u^\tau(t)R(t)u(t)\,dt \tag{2}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control, $y \in R^r$ is the output measurement, and $A(t)$, $B(t)$, $C(t)$, $D(t)$, $Q(t)$, and $R(t)$ are continuous matrix functions which satisfy the periodical property that $A(t) = A(t+T)$, $B(t) = B(t+T)$, $C(t) = C(t+T)$, $D(t) = D(t+T)$, $Q(t) = Q(t+T)$ and $R(t) = R(t+T)$, where T is the periodic time, $Q(t) \in R^{n\times n}$ is positive semidefinite, and $R(t) \in R^{m\times m}$ is positive definite.

It is known (e.g., [2]–[5]) that the above linear quadratic regulation (LQR) control problem can be solved by the following periodic state feedback:

$$u(t) = G(t)x(t) \tag{3.a}$$

where

$$G(t) = -R(t)^{-1}B^\tau(t)P(t) \tag{3.b}$$

and $P(t) \in R^{n\times n}$ is a periodic positive semidefinite matrix function solved from the following periodic Riccati equation (if the solution exists):

$$-\dot{P}(t) = A^\tau(t)P(t) + P(t)A(t) - P(t)B(t)R(t)^{-1}B^\tau(t)P(t) + Q(t). \tag{3.c}$$

In general, to implement the analog optimal periodic control scheme (3.a) needs complete state measurement. In this note, one converts the control scheme into a generalized-sampled-data hold control using the output measurement (1.b) only.

II. Development

A. Generalized Sampled-Data Hold Control

The periodic system (1) with the optimal periodic control (3.a) yields the following closed-loop system:

$$\dot{x}(t) = (A(t) + B(t)G(t))x(t). \tag{4}$$

Associated to the closed-loop system, one defines $\phi(t,\theta)$, $\Psi(t,\theta)$, and $\Psi_P(t,\theta)$ as the state transition matrices satisfying the following three differential equations, respectively:

$$\frac{d}{dt}\phi(t,\theta) = A(t)\phi(t,\theta); \qquad \phi(\theta,\theta) = I_n \tag{5}$$

$$\frac{d}{dt}\Psi(t,\theta) = (A(t) + B(t)G(t))\Psi(t,\theta); \qquad \Psi(\theta,\theta) = I_n \tag{6}$$

$$\frac{d}{dt}\Psi_P(t,\theta) = \begin{pmatrix} A(t) & B(t)G(t) \\ O_n & A(t)+B(t)G(t) \end{pmatrix}\Psi_P(t,\theta); \qquad \Psi_P(\theta,\theta) = I_{2n}. \tag{7}$$

Where I_n denotes the n dimensional identity matrix, O_n denotes the n dimensional zero matrix. It is easy to check the following equality (see Lemma 1 in the Appendix):

$$\Psi_P(t,\theta) = \begin{pmatrix} \phi(t,\theta) & B_\dagger(t,\theta) \\ O_n & \Psi(t,\theta) \end{pmatrix} \tag{8}$$

where

$$B_{\dagger}(t,\theta) = \int_{\theta}^{t} \phi(t,s)B(s)G(s)\Psi(s,\theta)\,ds = \Psi(t,\theta) - \phi(t,\theta). \quad (9)$$

Based on the optimal periodic control (3.a) and the closed-loop system (4), one conjures a generalized sampled-data hold control (Kabamba [6]) as follows:

$$u(kT+\theta) = G(\theta)\hat{x}(kT+\theta) = G(\theta)\Psi(\theta,0)\hat{x}(kT) \quad (10)$$

where $k = 0,1,2,\cdots$, $\theta \in [0,T)$, $\hat{x}(kT+\theta) = \Psi(\theta,0)\hat{x}(kT)$, and $\hat{x}(kT) \in R^n$. Notice that if $\hat{x}(kT) = x(kT)$, then $\hat{x}(kT+\theta) = x(kT+\theta)$ for all $\theta \in [0,T)$ (see Lemma 2 in the Appendix), so that (10) is just equivalent to (3.a). Now, one defines

$$\hat{\lambda}(kT) = \lim_{\theta \to T^-} \hat{x}((k-1)T+\theta) = \Psi(T,0)\hat{x}((k-1)T) \quad (11)$$

then one has

$$\begin{aligned} u(kT-\sigma) &= G(T-\sigma)\Psi(T-\sigma,0)\hat{x}((k-1)T) \\ &= G(T-\sigma)\Psi(T-\sigma,T)\hat{\lambda}(kT) \\ &= [O_{m\times n} \quad G(T-\sigma)]\Psi_P(T-\sigma,T)\begin{pmatrix} x(kT) \\ \hat{\lambda}(kT)\end{pmatrix} \end{aligned} \quad (12.a)$$

and

$$\begin{aligned} x(kT-\sigma) &= \phi(kT-\sigma,kT)x(kT) + \int_{kT}^{kT-\sigma} \phi(kT-\sigma,s)B(s)u(s)\,ds \\ &= \phi(T-\sigma,T)x(kT) + \int_{T}^{T-\sigma} \phi(T-\sigma,s)B(s)G(s)\Psi \cdot(s,0)\hat{x}((k-1)T)\,ds \\ &= \phi(T-\sigma,T)x(kT) + \int_{T}^{T-\sigma} \phi(T-\sigma,s)B(s)G(s)\Psi \cdot(s,T)\hat{\lambda}(kT)\,ds \\ &= [I_n \quad O_n]\Psi_P(T-\sigma,T)\begin{pmatrix} x(kT) \\ \hat{\lambda}(kT)\end{pmatrix} \end{aligned} \quad (12.b)$$

for all $\sigma \in (0,T]$. Thus by (12.a) and (12.b), it is concluded that

$$\begin{aligned} y(kT-\sigma) &= C(kT-\sigma)x(kT-\sigma) + D(kT-\sigma)u(kT-\sigma) \\ &= C_h(-\sigma)\begin{pmatrix} x(kT) \\ \hat{\lambda}(kT)\end{pmatrix} \end{aligned} \quad (13)$$

where

$$\begin{aligned} C_h(-\sigma) &= [C(kT-\sigma) \quad D(kT-\sigma)G(T-\sigma)]\Psi_P \cdot(T-\sigma,T) \\ &= [C(-\sigma) \quad D(-\sigma)G(-\sigma)]\Psi_P(-\sigma,0). \end{aligned} \quad (14)$$

B. Conversion Algorithm

Assume that $\sigma_1, \sigma_2, \cdots, \sigma_f$ are positive real numbers which satisfy $0 < \sigma_1 < \cdots < \sigma_f \le T$ and

$$\operatorname{rank}\begin{pmatrix} C_h(-\sigma_1) \\ C_h(-\sigma_2) \\ \vdots \\ C_h(-\sigma_f)\end{pmatrix} = 2n. \quad (15)$$

By (13) and (15), one can obtain

$$x(kT) = [I_n \quad O_n]\begin{pmatrix} x(kT) \\ \hat{\lambda}(kT)\end{pmatrix} = L\begin{pmatrix} y(kT-\sigma_1) \\ y(kT-\sigma_2) \\ \vdots \\ y(kT-\sigma_f)\end{pmatrix} \quad (16)$$

where $L \in R^{n\times fr}$ is given by

$$L = [I_n \quad O_n]\left(\begin{pmatrix} C_h(-\sigma_1) \\ C_h(-\sigma_2) \\ \vdots \\ C_h(-\sigma_f)\end{pmatrix}^{\tau}\begin{pmatrix} C_h(-\sigma_1) \\ C_h(-\sigma_2) \\ \vdots \\ C_h(-\sigma_f)\end{pmatrix}\right)^{-1}\begin{pmatrix} C_h(-\sigma_1) \\ C_h(-\sigma_2) \\ \vdots \\ C_h(-\sigma_f)\end{pmatrix}^{\tau}. \quad (17)$$

This means that $x(kT)$ can be exactly predicted by $y(kT-\sigma_1), y(kT-\sigma_2), \cdots, y(kT-\sigma_f)$ if the generalized sampled-data hold control (10) is valid in the interval $[(k-1)T, T)$, so that

$$u(kT+\theta) = G(\theta)\Psi(\theta,0)L\begin{pmatrix} y(kT-\sigma_1) \\ y(kT-\sigma_2) \\ \vdots \\ y(kT-\sigma_f)\end{pmatrix} \quad (18)$$

is just an equivalent control of (3.a) for all $k \ge 1$ and $\theta \in [0,T)$.

C. Another Conversion

An alternative conversion using less output measurements is also possible. To do so, one assumes

$$\begin{pmatrix} C_h(-\sigma_1) \\ C_h(-\sigma_2) \\ \vdots \\ C_h(-\sigma_g)\end{pmatrix} = [M_1 \quad M_2] \quad (19)$$

where $g \le f$, $M_1 \in R^{gr\times n}$, $M_2 \in R^{gr\times n}$, and $\operatorname{rank}[M_1] = n$. By (13) and (19), one obtains

$$x(kT) = (M_1^{\tau}M_1)^{-1}M_1^{\tau}\begin{pmatrix} y(kT-\sigma_1) \\ y(kT-\sigma_2) \\ \vdots \\ y(kT-\sigma_g)\end{pmatrix} - (M_1^{\tau}M_1)^{-1}M_1^{\tau}M_2\hat{\lambda}(kT). \quad (20)$$

Thus, an alternative conversion can be taken as follows:

$$u(kT+\theta) = G(\theta)\Psi(\theta,0)\hat{x}(kT) \quad (21.a)$$

$$\hat{x}(kT) = L_1\begin{pmatrix} y(kT-\sigma_1) \\ y(kT-\sigma_2) \\ \vdots \\ y(kT-\sigma_g)\end{pmatrix} + L_2\hat{x}((k-1)T) \quad (21.b)$$

where $L_1 = (M_1^{\tau}M_1)^{-1}M_1^{\tau} \in R^{n\times gr}$, $L_2 = -(M_1^{\tau}M_1)^{-1}M_1^{\tau}M_2\Psi(T,0) \in R^{n\times n}$, $k = 1,2,\cdots$, and $\theta \in [0,T)$.

Remark 1: Since $u(\theta)$ for $\theta \in [-T,0)$ may not obey (10), a practical control scheme in the first periodic time can be taken as follows:

$$u(\theta) = G(\theta)\Psi(\theta,0)E(x(0)) \quad (22)$$

for $\theta \in [0, T)$, where the expectation $E(x(0))$ can be substituted by an estimated vector using any other approaches (e.g., the exact reconstruction method developed in [7]).

Remark 2: A necessary and sufficient condition to exist real numbers $\sigma_1, \cdots, \sigma_f$ $(0 < \sigma_1 < \cdots < \sigma_f \leq T)$ for satisfying (15) is that the following Gramian matrix has full rank (see Lemma 3 in the Appendix).

$$\int_0^T \Psi_p^\tau(-\sigma, 0)[C(-\sigma) \quad D(-\sigma)G(-\sigma)]^\tau \cdot [C(-\sigma) \quad D(-\sigma)G(-\sigma)]\Psi_p(-\sigma, 0)\, d\sigma. \quad (23)$$

Similarly, a necessary and sufficient condition to exist real numbers $\sigma_1, \cdots, \sigma_g$ $(0 < \sigma_1 < \cdots < \sigma_g \leq T)$ for satisfying $\text{rank}[M_1] = n$ is that the following Gramian matrix has full rank

$$\int_0^T \phi^\tau(-\sigma, 0) C^\tau(-\sigma) C(-\sigma) \phi(-\sigma, 0)\, d\sigma. \quad (24)$$

Remark 3: It is interesting to compare the presented approach with a multirate output feedback control given as follows (e.g., [8]):

$$u(kT + iT/\bar{f} + \theta) = L_i y(kT) \quad (25)$$

where $\bar{f}$ is a positive integer, $\theta \in [0, T/\bar{f})$, and $L_i \in R^{m \times r}$, $i = 0, 1, 2, \cdots, \bar{f} - 1$ are the piecewise output feedback gains. From the theoretical viewpoint, the converted generalized sampled-data hold control (18) [or (21)] is really an optimal solution of the continuous-time LQR control problem, it produces the least cost and arises no intersample ripple. On the other hand, a multirate output feedback control can be considered as a suboptimal approach using convenient structure. A significant advantage of this approach is that the minimization problem of the index (2) subject to the multirate structure can be converted into a discrete-time LQR control problem, in particular, if the system has complete state information, then the optimal solution can simply be solved from a discrete-time algebraic Riccati equation (see [1]). Besides, a multirate output feedback control scheme is easier for practical implementation because it only uses a zero-order hold and needs less output measurements.

III. Example

Consider the optimal control problem of the following linear periodic system

$$\dot{x}(t) = \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix} u(t) \quad (26.a)$$

$$y(t) = [\cos(2\pi t) \quad \sin(2\pi t)] x(t) \quad (26.b)$$

to minimize a quadratic performance index as follows:

$$J = \int_0^T x^\tau(t) x(t) + u^\tau(t) u(t)\, dt. \quad (27)$$

By solving the periodic Riccati equation (3.c), one obtains the optimal periodic control as follows:

$$u(t) = \begin{pmatrix} -\cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & -\cos(2\pi t) \end{pmatrix} x(t). \quad (28)$$

This control scheme yields the following closed-loop system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x(t). \quad (29)$$

Now, let $T = 1$, $\sigma_1 = 0.4$, $\sigma_2 = 0.6$, $\sigma_3 = 0.8$, and $\sigma_4 = 1.0$. By (14), one obtains

$$\begin{pmatrix} C_h(-0.4) \\ C_h(-0.6) \\ C_h(-0.8) \\ C_h(-1.0) \end{pmatrix} = \begin{pmatrix} -0.8090 & -0.5878 & -0.3979 & -0.2891 \\ -0.8090 & 0.5878 & -0.6651 & 0.4832 \\ 0.3090 & 0.9511 & 0.3787 & 1.1656 \\ 1.0000 & 0.0000 & 1.7183 & -0.0000 \end{pmatrix} \quad (30)$$

and by (17), one obtains

$$L = \begin{pmatrix} -0.7196 & -1.3087 & 0.3641 & -0.7535 \\ -1.7058 & 2.0658 & -1.2796 & 0.6866 \end{pmatrix}. \quad (31)$$

thus, the converted generalized sampled-data hold control (18) is given by

$$u(kT + \theta) = \begin{pmatrix} -\cos(2\pi\theta) & \sin(2\pi\theta) \\ -\sin(2\pi\theta) & -\cos(2\pi\theta) \end{pmatrix} \cdot \left(e^{-\theta} \begin{pmatrix} -0.719 & -1.308 & 0.364 & -0.753 \\ -1.705 & 2.065 & -1.279 & 0.686 \end{pmatrix} \cdot \begin{pmatrix} y(kT - 0.4) \\ y(kT - 0.6) \\ y(kT - 0.8) \\ y(kT - 1.0) \end{pmatrix} \right). \quad (32)$$

This converted control scheme is checked by simulation as shown in Fig. 1.

IV. Conclusions

In this note, a conversion method to convert the analog optimal periodic control to a generalized sampled-data hold output feedback control for a linear periodic system is developed. Such a conversion enables us to implement the optimal periodic control scheme in the presence of incomplete state measurement. Besides, the converted control scheme can use the delayed output feedback to offer a leisure time for on line computation, so that it can provide ability to tolerate the time delay (such as: measurement delay, computation time lag, etc). Such a conversion algorithm is also applicable to a linear time-invariant system just by considering the system as a periodic model with an arbitrary periodic time.

Appendix

Lemma 1: We only have to show that

$$B_\dagger(t, \theta) = \int_\theta^t \phi(t, s) B(s) G(s) \Psi(s, \theta)\, ds = \Psi(t, \theta) - \phi(t, \theta).$$

Proof: One has

$$\frac{d}{dt}\left\{\int_\theta^t \phi(t, s) B(s) G(s) \Psi(s, \theta)\, ds\right\} = A(t)\left\{\int_\theta^t \phi(t, s) B(s) G(s) \Psi(s, \theta)\, ds\right\} + B(t) G(t) \Psi(t, \theta) \quad (A.1)$$

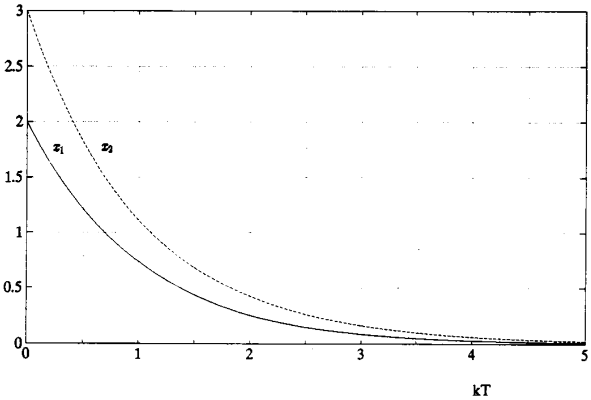

Fig. 1. The response of the periodic system (26) with the converted control scheme (32), where one assumes $x(0) = [x_1(0)x_2(0)]^\tau = [2\ 3]^\tau$.

and

$$\frac{d}{dt}\{\Psi(t,\theta) - \phi(t,\theta)\} = A(t)\{\Psi(t,\theta) - \phi(t,\theta)\} + B(t)G(t)\Psi(t,\theta). \quad \text{(A.2)}$$

Thus, the equality can be obtained by checking the differential equation (7) directly. □

Lemma 2: If $\hat{x}(kT) = x(kT)$, then the control (10) equals to (3.a) on $[kT, (k+1)T)$.

Proof: With the control (10), the state of the periodic system (1) becomes

$$\begin{aligned} x(kT+\theta) &= \phi(\theta,0)x(kT) + \int_0^\theta \phi(\theta,s)B(s)u(kT+s)\,ds \\ &= \phi(\theta,0)x(kT) \\ &\quad + \int_0^\theta \phi(\theta,s)B(s)G(s)\Psi(s,0)\hat{x}(kT)\,ds \\ &= \phi(\theta,0)x(kT) + B_\dagger(\theta,0)\hat{x}(kT) \\ &= \phi(\theta,0)(x(kT) - \hat{x}(kT)) + \Psi(\theta,0)\hat{x}(kT) \quad \text{(A.3)} \end{aligned}$$

Thus, if $\hat{x}(kT) = x(kT)$, then $x(kT+\theta) = \hat{x}(kT+\theta)$ for all $\theta \in [0, T)$. □

Lemma 3: Assume $H(t)$ is a continuous matrix function from $[0, T]$ into $R^{n_1 \times n_2}$, then the following Gramian matrix

$$\Omega_T = \int_0^T H(t)H^\tau(t)\,dt \quad \text{(A.4)}$$

is positive definite, if and only if there exist finite points $\sigma_1, \sigma_2, \cdots, \sigma_f$ of $(0, T]$, such that

$$[H(\sigma_1) \colon H(\sigma_2) \colon \cdots \colon H(\sigma_f)] \quad \text{(A.5)}$$

has full row rank.

Proof: **(If)** If there exist finite points $\sigma_1, \sigma_2, \cdots, \sigma_f$ of $(0, T]$, such that the given matrix (A.5) has full row rank, then for any nonzero vector $\xi \in R^{n_1}$, one can find at least a point $\sigma_i \in \{\sigma_1, \sigma_2, \cdots, \sigma_f\}$, such that $\xi^\tau H(\sigma_i) \neq 0$. Since $H(t)$ is continuous, this implies

$$\xi^\tau \Omega_T \xi = \int_0^T \xi^\tau H(t)H^\tau(t)\xi\,dt > 0 \quad \text{(A.6)}$$

so that Ω_T is positive definite.

(Only if) Assume Ω_T is positive definite, and consider arbitrary finite points $\sigma_1, \sigma_2, \cdots, \sigma_f$ of $(0, T]$, if the given matrix (A.5) has not full row rank, then one can find $\xi \in R^{n_1}$ and $\sigma_{f+1} \in (0, T]$, such that

$$\xi^\tau H(\sigma_{f+1}) \neq 0 \quad \text{(A.7)}$$

and

$$\xi^\tau [H(\sigma_1) \colon H(\sigma_2) \colon \cdots \colon H(\sigma_f)] = 0. \quad \text{(A.8)}$$

This implies that some columns of $H(\sigma_{f+1})$ cannot be expressed as a linear combination of columns of matrix (A.5), so that it is true that

$$\begin{aligned} &\operatorname{rank}[H(\sigma_1) \colon H(\sigma_2) \colon \cdots \colon H(\sigma_f) \colon H(\sigma_{f+1})] \\ &\qquad \geq \operatorname{rank}[H(\sigma_1) \colon H(\sigma_2) \colon \cdots \colon H(\sigma_f)] + 1. \quad \text{(A.9)} \end{aligned}$$

By giving the extending procedure at most n_1 times, one can finally find $\{\sigma_1, \sigma_2, \cdots, \sigma_f, \sigma_{f+1}, \cdots, \sigma_{f+q}\} \subset (0, T]$, such that

$$\operatorname{rank}[H(\sigma_1) \colon H(\sigma_2) \colon \cdots \colon H(\sigma_f) \colon H(\sigma_{f+1}) \colon \cdots \colon H(\sigma_{f+q})] = n_1.$$

□

References

[1] H. M. Al-Rahmani and G. F. Franklin, "A new optimal multirate control of linear periodic and time-invariant systems," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 406–415, Apr. 1990.

[2] S. Bittanti, P. Colaneri, and G. Guardabassi, "Periodic solutions of periodic Riccati equations," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 665–667, July 1984.

[3] S. Bittanti, P. Bolzern and P. Colaneri, "The extended periodic Lyapunov lemma," *Automatica*, vol. 21, pp. 603–605, 1985.

[4] S. Bittanti, P. Colaneri, and G. De Nicolao, "Periodic regulators with incomplete and noise measurements," in *Proc. 29th Conf. Decisions Contr.*, Hawaii, 1990, pp. 3648–3649.

[5] H. Kano and T. Nishimura, "Controllability, stabilizability, and matrix Riccati equation for Periodic system," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 1129–1131, Nov. 1985.

[6] P. T. Kabamba, "Control of linear systems using generalized sampled-data hold functions," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 772–782, Sept. 1987.

[7] D. H. Chyung, "State variable reconstruction," *Int. J. Contr.*, vol. 39, pp. 955–963, 1984.

[8] N. Z. Yen and Y. C. Wu, "On a general optimal algorithm for multirate output feedback controllers of linear stochastic periodic systems," *IEEE Trans. Automat. Contr.*, 1993, to be published.

On Discrete Spectral Factorizations—A Unify Approach

M. C. Tsai

Abstract—**This note summarizes state-space formulae for all key spectral factorizations appearing in the discrete-time H^2/H^∞ optimization. The factorization problems are categorized into three groups. The construction of solutions is formulated into finding special coprime factors of a given transfer matrix by the associated discrete algebraic Riccati equation. Solution procedures for the three groups are in general the same, and under that we may lead to yield a unify approach.**

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