SIAM J. DISCRETE MATH.
Vol. 16, No. 2, pp. 173–178




# THE EXISTENCE OF $2 \times 4$ GRID-BLOCK DESIGNS AND THEIR APPLICATIONS*

YUKIYASU MUTOH†, TOSHIO MORIHARA†, MASAKAZU JIMBO†, AND HUNG-LIN FU‡

**Abstract.** Fu, Hwang, Jimbo, Mutoh, and Shiue [*J. Statist. Plann. Inference*, to appear] introduced the concept of a grid-block design, which is defined as follows: For a $v$-set $V$, let $\mathcal{A}$ be a collection of $r \times c$ arrays with elements in $V$. A pair $(V, \mathcal{A})$ is called an $r \times c$ *grid-block design* if every two distinct points $i$ and $j$ in $V$ occur exactly once in the same row or in the same column. This design has originated from the use of DNA library screening. They gave some general constructions and proved the existence of $3 \times 3$ grid-block designs. Meanwhile, the existence of $2 \times 3$ grid-block designs was shown by Carter [*Designs on Cubic Multigraphs*, Ph.D. thesis, McMaster University, Hamilton, ON, Canada, 1989] by decomposing $K_v$ into cubic graphs. In this paper, we show the existence of $2 \times 4$ grid-block designs.

**Key words.** graph decomposition, graph design, grid-block

**AMS subject classifications.** 05B05, 05C70

**PII.** S0895480101387364

**1. Introduction.** A *graph* $G$ is a pair of sets $(V, E)$, where $V$ is a finite set, and $E$ is a set of unordered pairs of elements of $V$. The elements of $V$ are called *vertices* of $G$ and the elements of $E$ are called *edges* of $G$. If $x$ and $y$ are vertices of a graph $G$, we say that $x$ is *adjacent* to $y$ if there is an edge between $x$ and $y$. $K_v$ is the graph with $v$ vertices such that every vertex is adjacent to every other vertex. For a $v$-set $V$, let $\mathcal{A}$ be a collection of $r \times c$ arrays with elements in $V$. Each array in $\mathcal{A}$ is called a *grid-block*. For a graph $G = (V, E)$, a pair $(V, \mathcal{A})$ is called an $r \times c$ *grid-block design with respect to* $G$ denoted by $D_{r \times c}(G)$ if every two distinct points $i$ and $j$ in $V$ such that $\{i, j\} \in E$ occur exactly once in the same row or in the same column. We used the terminology "grid-block design" to avoid the confusion with the "grid design" defined by Lamken and Wilson [9]. Here we show an example of a $D_{3\times 3}(K_9)$.

EXAMPLE 1. *The following two grid-blocks form a* $D_{3\times 3}(K_9)$.

$$\boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}} \qquad \boxed{\begin{array}{ccc} 1 & 6 & 8 \\ 9 & 2 & 4 \\ 5 & 7 & 3 \end{array}}$$

A grid-block design was introduced by Fu et al. [7]. It is easy to show the following necessary conditions for the existence of a $D_{r\times c}(K_v)$.

LEMMA 1.1. *Necessary conditions for the existence of a* $D_{r\times c}(K_v)$ *are*
(i) $(r + c - 2)|(v - 1)$ *and*
(ii) $rc(r + c - 2)|v(v - 1)$.

Combinatorial designs were used as an efficient way of group testing in fields such as medical science and pharmaceutical science (see Du and Hwang [6]). Recently, a

*Received by the editors April 25, 2001; accepted for publication (in revised form) July 24, 2002; published electronically February 20, 2003. This work was supported in part by the Institute of Statistical Mathematics in Japan.
http://www.siam.org/journals/sidma/16-2/38736.html
†Department of Mathematics, Keio University, Yokohama, Kanagawa, Japan 223-8522 (yukiyasu@jim.math.keio.ac.jp, toshio@jim.math.keio.ac.jp, jimbo@math.keio.ac.jp).
‡Department of Applied Mathematics, National Chiao Tung University, Hsin Chu, Taiwan, R.O.C. (hlfu@math.nctu.edu.tw).

combinatorial design has come to be applied to DNA library screening to discover the required DNA sequences by testing every row and every column in a microtiter plate at the same time.

In DNA library screening, a popular group testing method is a two-stage test. In this method, every row and every column in a microtiter plate is tested at the same time in the first stage, and each individual segment with positive response is tested in the second stage. See Figure 1.1 for demonstration. To reduce the number of tests and to improve the efficiency of experiments, several methods of screening have been studied by many authors.

Berger, Mandell, and Subrahmanya [1] evaluated the efficiency for the two-stage test from the point of view of information theory, while Fu et al. [7] introduced a combinatorial method based on a grid-block design.

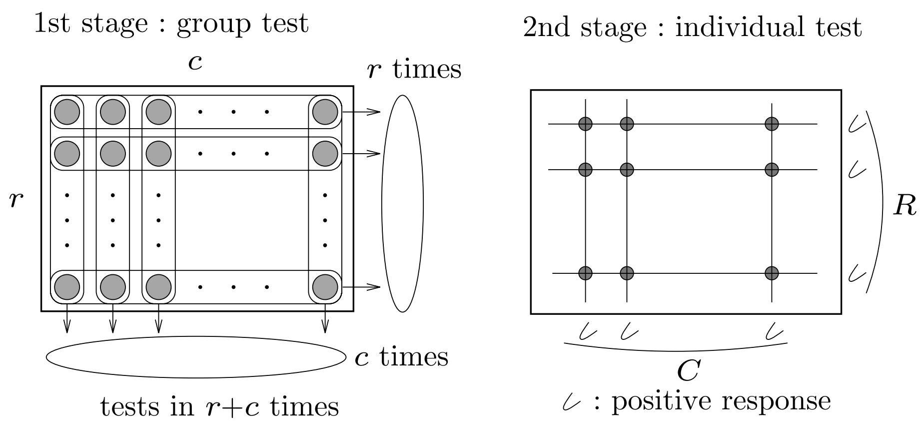


Fig. 1.1. *The demonstration of DNA library screening.*

In this paper, we start with a recursive construction for a grid-block design. Then, by utilizing this recursive construction together with those given by Fu et al. [7], we will prove the existence of $2\times 4$ grid-block designs which satisfy the necessary condition $v \equiv 1 \pmod{32}$.

**2. General constructions.** In this section, we prepare a proposition and lemmas to use in the next section. First, we define a block design. For sets of positive integers $K$ and $M$, let $V$ be a set of $v$ *points*, let $\mathcal{G}$ be a partition of $V$ such that each $G$ has $m$ points for $m \in M$, and let $\mathcal{B}$ be a collection of $k$-subsets (*blocks*) of $V$ for $k \in K$. A triple $(V, \mathcal{G}, \mathcal{B})$ is called a *group divisible design*, denoted by $GD[K, \lambda, M; v]$, if every two distinct points contained in different groups occur in exactly $\lambda$ blocks and if every two distinct points contained in the same group do not occur together in any blocks. Especially, a $GD[\{k\}, \lambda, \{m\}; v]$ is written by $GD[k, \lambda, m; v]$ for simplicity of notation.

Suppose that the set of $st$ vertices are partitioned into $s$ subsets of size $t$ each. Let $K_s(t)$ be the complete multipartite graph such that $(i, j)$ is an edge if $i$ and $j$ are not in the same subset. A grid-block design $D_{r\times c}(K_s(t))$ is called a *group divisible grid-block design*. It is easy to see that the following lemma holds.

Lemma 2.1. *Necessary conditions for a* $D_{r\times c}(K_s(t))$ *to exist are*

(i) $(r + c - 2)|(s - 1)t$ *and*

(ii) $rc(r + c - 2)|(s - 1)st^2$.

Fu et al. [7] proved the following construction.

TABLE 3.1
*Table of the existence of group divisible designs.*

| $v$ | $K$ | Group type | $u$ | Exceptions | Ref. |
|---|---|---|---|---|---|
| $0, 1 \pmod 4$ | $\{4, 5\}$ | $1^u$ | $0, 1 \pmod 4$ | 12 | [2] |
| 12 | 4 | $3^4$ | – | – | [4] |
| $12 \pmod{12}$ | 4 | $2^u$ | $1 \pmod 3$ | – | [4] |
| $3 \pmod{12}$ | 4 | $3^u$ | $1 \pmod 4$ | – | [4] |
| $6 \pmod{12}$ | 4 | $6^u$ | Anything | 18 | [4] |
| $7 \pmod{12}$ | 4 | $7^1 1^u$ | $0 \pmod{12}$ | 19 | [3] |
| $10 \pmod{12}$ | 4 | $7^1 1^u$ | $3 \pmod{12}$ | – | [3] |
| $11 \pmod{12}$ | 4 | $5^1 2^u$ | $0 \pmod 3$ | – | [3] |

PROPOSITION 2.2 (Fu et al. [7]). *A $D_{r\times c}(K_{st+1})$ exists if a $D_{r\times c}(K_{t+1})$ and a $D_{r\times c}(K_s(t))$ exist.*

We give a recursive construction by utilizing a group divisible design, group divisible grid-block designs, and grid-block designs.

LEMMA 2.3. *A $D_{r\times c}(K_{vt+1})$ exists if a $GD[K, 1, M; v]$ exists and if a $D_{r\times c}(K_k(t))$ and a $D_{r\times c}(K_{mt+1})$ exist for any $k \in K$ and for any $m \in M$.*

*Proof.* For a $v$-set $V$, let a triple $(V, \mathcal{G}, \mathcal{B})$ be a $GD[K, 1, M; v]$, where $\mathcal{B} = \{B_1, B_2, \dots, B_b\}$ is a collection of blocks and $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$ is a family of group sets. Let $T = \{0, 1, \dots, t-1\}$ and $V^* = (V \times T) \cup \{\infty\}$. For each block $B_i$ of size $k \in K$, let $(B_i \times T, \mathcal{H}_i, \mathcal{E}_i)$ be the ingredient design $D_{r\times c}(K_k(t))$, where $\mathcal{E}_i$ is a collection of grid-blocks and $\mathcal{H}_i$ is a family of group sets $\{\{b_{i1}\} \times T, \{b_{i2}\} \times T, \dots, \{b_{ik}\} \times T\}$ for $b_{ij} \in B_i$. We define a collection of grid-blocks $\mathcal{A}_1 = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \cdots \cup \mathcal{E}_b$. Also, for each group $G_i$ of size $m \in M$, let $((G_i \times T) \cup \{\infty\}, \mathcal{F}_i)$ be the ingredient design $D_{r\times c}(K_{mt+1})$, where $\mathcal{F}_i$ is a collection of grid-blocks. We define another collection of grid-blocks $\mathcal{A}_2 = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_n$ and let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. Then a pair $(V^*, \mathcal{A})$ is the desired $D_{r\times c}(K_{vt+1})$.

In fact, if two distinct elements $x$ and $y$ in $V$ are not contained in the same group set $G_j$, then $x$ and $y$ occur together exactly once in a $B_i$. Hence $(x, \alpha_1)$ and $(y, \alpha_2)$ occur exactly once in the same row or in the same column of a grid-block in $\mathcal{A}_1$ and do not occur in $\mathcal{A}_2$ for any $\alpha_1, \alpha_2 \in T$. Otherwise, two elements $x$ and $y$ in $V$ are contained in the same group set $G_j$ including the case of $x = y$. In this case, $(x, \alpha_1)$ and $(y, \alpha_2)$ occur exactly once in the same row or in the same column of a grid-block in $\mathcal{A}_2$ and do not occur in $\mathcal{A}_1$. Finally, $\infty$ and $(x, \alpha)$ for any $x \in V$ and $\alpha \in T$ occur exactly once in the same row and in the same column of a grid-block in $\mathcal{A}_2$. ∎

## 3. The existence of a $2 \times 4$ grid-block design.

In this section we apply the results obtained in the previous section to prove the following theorem.

THEOREM 3.1. *The necessary condition $v \equiv 1 \pmod{32}$ for the existence of a $D_{2\times 4}(K_v)$ is also sufficient.*

This existence theorem is shown by utilizing a recursive construction. First, we give an existence of a group divisible design.

LEMMA 3.2. *For any integer $v \geq 12$, there exists a $GD[K, 1, M; v]$, where $K = \{4, 5\}$ and $M = \{1, 2, \dots, 7\}$.*

*Proof.* According to Brouwer [3], Brouwer, Schrijver, and Hanani [4], and Beth, Jungnickel, and Lenz [2], we know the existence of a $GD[K, 1; M; v]$ for any $v \geq 12$ except for $v = 18$ and 19 as is listed in Table 3.1 (see also Kreher and Stinson [8] and Mullin and Gronau [10]). In Table 3.1, the notation $t_1^{u_1} t_2^{u_2}$ of a group type implies that $V$ is divided into $u_1$ groups with group size $t_1$ and $u_2$ groups with group size $t_2$.

TABLE 3.2
*Table of the base grid-blocks of group divisible grid-block designs.*

| | Base grid-blocks | | |
|---|---|---|---|
| $D_{2\times4}(K_4(32))$ | 0 1 6 15<br>13 30 3 48 | 0 21 58 47<br>22 63 20 97 | 0 25 74 55<br>63 56 17 122 |
| $D_{2\times4}(K_5(32))$ | 0 1 7 3<br>11 27 48 39 | 0 31 17 63<br>22 73 129 30 | 0 66 47 133<br>13 149 105 51 |
| | 0 111 52 23<br>84 15 141 102 | | |

TABLE 3.3
*Table of the base grid-blocks of grid-block designs.*

| | Base grid-blocks | | |
|---|---|---|---|
| $D_{2\times4}(K_{33})$ | 0 1 3 9<br>12 5 23 28 | | |
| $D_{2\times4}(K_{65})$ | 0 1 3 7<br>5 13 22 38 | 0 10 21 45<br>47 32 60 9 | |
| $D_{2\times4}(K_{97})$ | 0 1 3 7<br>5 13 22 33 | 0 10 23 41<br>33 65 86 3 | 0 15 37 61<br>39 55 84 12 |
| $D_{2\times4}(K_{193})$ | 0 36 65 60<br>89 155 152 153 | 0 46 180 153<br>186 23 71 169 | 0 55 108 73<br>114 77 133 81 |
| | 0 14 97 165<br>102 52 40 134 | 0 105 54 44<br>75 34 178 55 | 0 76 67 148<br>39 189 73 174 |
| $D_{2\times4}(K_{225})$ | 0 104 76 167<br>67 137 121 209 | 0 189 223 92<br>156 74 167 199 | 0 221 77 194<br>41 94 42 16 |
| | 0 122 177 140<br>212 190 106 67 | 0 15 220 111<br>95 76 55 46 | 0 7 10 24<br>38 82 206 32 |
| | 0 87 161 99<br>79 192 102 13 | | |

Moreover, it is known that $GD[5,1,4;20]$ exists, which is obtained by deleting one parallel class of lines and five points on a line in the parallel class from $AG(2,5)$. By deleting a single point of a $GD[5,1,4;20]$, we can show the existence of a $GD[\{4,5\},1,\{3,4\};19]$. Similarly, by deleting two points from the same group of a $GD[5,1,4;20]$, we obtain a $GD[\{4,5\},1,\{2,4\};18]$, which proves the case of $v=18$ and 19. Thus, the lemma is proved. $\square$

Second, we give two group divisible grid-block designs which are obtained by computer.

LEMMA 3.3. *There exists a* $D_{2\times4}(K_k(32))$ *for* $k=4$ *and* 5.

*Proof.* For $V=\boldsymbol{Z}_{128}$, let

$$A_0=\boxed{\begin{array}{cccc}0&1&6&15\\13&30&3&48\end{array}}\,,\qquad B_0=\boxed{\begin{array}{cccc}0&21&58&47\\22&63&20&97\end{array}}\,,\ \text{and}$$

$$C_0=\boxed{\begin{array}{cccc}0&25&74&55\\63&56&17&122\end{array}}\,,$$

which are listed in Table 3.2. Here $A_0$, $B_0$, and $C_0$ are called a *base grid-block* or a *starting grid-block*. For each base grid-block, let $A_i=A_0+i \pmod{128}$, $B_i=B_0+i \pmod{128}$, and $C_i=C_0+i \pmod{128}$. Now we define

$$\mathcal{A}=\{A_0,A_1,\dots,A_{127},B_0,B_1,\dots,B_{127},C_0,C_1,\dots,C_{127}\};$$

TABLE 3.4
*Table of the base grid-blocks of grid-block designs (continued).*

| | Base grid-blocks | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $D_{2\times4}(K_{257})$ | 0 | 51 | 168 | 216 | 0 | 22 | 230 | 37 | 0 | 58 | 61 | 234 |
| | 148 | 147 | 81 | 37 | 30 | 211 | 187 | 193 | 200 | 118 | 101 | 154 |
| | 0 | 107 | 73 | 14 | 0 | 169 | 42 | 98 | 0 | 132 | 246 | 124 |
| | 50 | 79 | 202 | 176 | 63 | 61 | 96 | 216 | 20 | 41 | 72 | 162 |
| | 0 | 171 | 210 | 65 | 0 | 75 | 178 | 247 | | | | |
| | 202 | 190 | 197 | 206 | 72 | 255 | 210 | 185 | | | | |
| $D_{2\times4}(K_{289})$ | 0 | 217 | 34 | 207 | 0 | 199 | 54 | 19 | 0 | 228 | 8 | 13 |
| | 28 | 188 | 253 | 168 | 105 | 282 | 236 | 183 | 86 | 35 | 165 | 189 |
| | 0 | 179 | 122 | 4 | 0 | 241 | 47 | 244 | 0 | 27 | 256 | 218 |
| | 209 | 37 | 211 | 284 | 124 | 191 | 110 | 98 | 248 | 182 | 225 | 98 |
| | 0 | 185 | 148 | 163 | 0 | 133 | 271 | 227 | 0 | 25 | 32 | 213 |
| | 128 | 186 | 216 | 180 | 166 | 14 | 150 | 206 | 77 | 255 | 266 | 164 |
| $D_{2\times4}(K_{321})$ | 0 | 235 | 247 | 257 | 0 | 3 | 101 | 281 | 0 | 35 | 186 | 37 |
| | 310 | 101 | 228 | 133 | 76 | 105 | 212 | 309 | 244 | 138 | 264 | 16 |
| | 0 | 160 | 1 | 265 | 0 | 26 | 317 | 9 | 0 | 7 | 157 | 25 |
| | 158 | 66 | 291 | 221 | 269 | 178 | 228 | 315 | 23 | 205 | 143 | 74 |
| | 0 | 146 | 61 | 16 | 0 | 315 | 211 | 33 | 0 | 279 | 200 | 255 |
| | 283 | 288 | 174 | 115 | 206 | 78 | 146 | 254 | 34 | 105 | 272 | 308 |
| | 0 | 240 | 165 | 294 | | | | | | | | |
| | 313 | 59 | 255 | 175 | | | | | | | | |
| $D_{2\times4}(K_{353})$ | 0 | 286 | 267 | 129 | 0 | 133 | 95 | 248 | 0 | 81 | 72 | 26 |
| | 198 | 149 | 219 | 118 | 22 | 20 | 275 | 113 | 82 | 257 | 147 | 261 |
| | 0 | 294 | 142 | 15 | 0 | 88 | 76 | 247 | 0 | 337 | 109 | 217 |
| | 34 | 173 | 198 | 1 | 71 | 222 | 144 | 194 | 66 | 150 | 2 | 211 |
| | 0 | 340 | 7 | 343 | 0 | 169 | 254 | 122 | 0 | 193 | 8 | 44 |
| | 195 | 5 | 234 | 264 | 316 | 229 | 17 | 59 | 352 | 103 | 127 | 76 |
| | 0 | 52 | 23 | 154 | 0 | 186 | 40 | 83 | | | | |
| | 45 | 192 | 134 | 4 | 236 | 298 | 201 | 293 | | | | |

then $(\boldsymbol{Z}_{128}, \mathcal{A})$ is the desired $D_{2\times4}(K_4(32))$. In fact, by calculating the differences of two elements in the same row or in the same column of $A_0$, $B_0$, and $C_0$, any difference except for multiples of 4 occurs exactly once.

Similarly, for $V = \boldsymbol{Z}_{160}$, by utilizing four base grid-blocks in Table 3.2, we obtain a $D_{2\times4}(K_5(32))$. In fact, by calculating the differences of two elements in the same row or in the same column of $A_0$, $B_0$, $C_0$, and $D_0$ any difference except for multiples of 5 occurs exactly once. ∎

Third, we give some grid-block designs which are obtained by computer.

LEMMA 3.4. *There exists a* $D_{2\times4}(K_{32m+1})$ *for any* $m = 1, 2, \ldots, 11$.

*Proof.* By utilizing the base grid-blocks in Tables 3.3 and 3.4, we obtain the desired $D_{2\times4}(K_{32m+1})$'s for $m = 1, 2, 3, 6, 7, \ldots, 11$. By applying Proposition 2.2 to a $D_{2\times4}(K_4(32))$ and a $D_{2\times4}(K_5(32))$ in Lemma 3.3 and a $D_{2\times4}(K_{33})$, $D_{2\times4}(K_{32m+1})$'s are obtained for $m = 4$ and 5. ∎

Now we will show the main theorem.

*Proof of Theorem* 3.1. By Lemma 1.1, it is easy to show that the necessary condition for the existence of a $D_{2\times4}(K_v)$ is $v \equiv 1 \pmod{32}$. Now we write $v = 32w + 1$; then there exists a $D(K_{32w+1})$ for $w \le 11$ by Lemma 3.4. By Lemma 3.2, a $GD[K, 1, M; w]$ exists for $w \ge 12$, where $K = \{4, 5\}$ and $M = \{1, 2, \ldots, 7\}$. And a $D(K_k(32))$ exists for $k = 4$ and 5 by Lemma 3.3. Thus by Lemma 2.3 a $D(K_{32w+1})$ exists for any $w \ge 12$, which prove the main theorem. ∎

**Acknowledgments.** The authors would like to thank the referees for their careful readings of this article and for their many constructive comments. One of the referees suggested to us that the proof of the previous version can be shortened. The authors would also like to thank Professor R. C. Mullin for his valuable comments.

## REFERENCES

[1] T. Berger, J. W. Mandell, and P. Subrahmanya, *Maximally efficient two-stage screening*, Biometrics, 56 (2000), pp. 833–840.

[2] T. Beth, D. Jungnickel, and H. Lenz, *Design Theory*, Cambridge University Press, Cambridge, UK, 1986.

[3] A. E. Brouwer, *Optimal packings of $K_4$'s into a $K_n$*, J. Combin. Theory Ser. A, 26 (1979), pp. 278–297.

[4] A. E. Brouwer, A. Schrijver, and H. Hanani, *Group divisible designs with block size* 4, Discrete Math., 20 (1977), pp. 1–10.

[5] J. E. Carter, *Designs on Cubic Multigraphs*, Ph.D. thesis, McMaster University, Hamilton, ON, Canada, 1989.

[6] D.-Z. Du and F. K. Hwang, *Combinatorial Group Testing and Its Application*, World Scientific, River Edge, NJ, 2000.

[7] H. L. Fu, F. K. Hwang, M. Jimbo, Y. Mutoh, and C. L Shiue, *Decomposing complete graphs into $K_r \times K_c$'s*, J. Statist. Plann. Inference, to appear.

[8] D. L. Kreher and D. R. Stinson, *Small group-divisible designs with block size four*, J. Statist. Plann. Inference, 58 (1997), pp. 111–118.

[9] E. R. Lamken and R. M. Wilson, *Decompositions of edged-colored complete graphs*, J. Combin. Theory Ser. A, 89 (2000), pp. 149–200.

[10] R. C. Mullin and H.-D.O.F. Gronau, *PBDs and GDDs: The basics*, in The CRC Handbook of Combinatorial Designs, C. J. Colbourn and J. H. Dinitz, eds., CRC Press, Boca Raton, FL, 1996, pp. 185–193.