JOURNAL OF INFORMATION SCIENCE AND ENGINEERING 19, 159-166 (2003)

Short Paper

An Efficient Algorithm for the Reliability of Consecutive-*k*-*n* Networks

JEN-CHUN CHANG, RONG-JAYE CHEN* AND FRANK K. HWANG+
Department of Information Management
Ming Hsin Institute of Technology
Hsinchu, 304 Taiwan
E-mail: jimmy@mis.mhit.edu.tw
**Department of Computer Science and Information Engineering*
+Department of Applied Mathematics
National Chiao Tung University
Hsinchu, 300 Taiwan
E-mail: rjchen@csie.nctu.edu.tw
E-mail: fhwang@math.nctu.edu.tw

A consecutive-*k*-*n* network is a generalization of the well-known consecutive-*k*-out-of-*n* system, and has many practical applications. This network consists of n + 2 nodes (node 0, the source, nodes 1, 2, …, n, and node $n + 1$, the target) and directed links from node i to node j ($0 \le i < j \le n + 1$, $j - i \le k$). Because all nodes except the source and target, and all links are fallible, the network works if and only if there exists a working path from the source to the target. For the $k = 2$ case, based on identical node reliabilities and some assumptions on link reliabilities, Chen, Hwang and Li (1993) gave a recursive algorithm for the reliability of the consecutive-2-*n* network. In this paper we give a closed form equation for the reliability of the general consecutive-*k*-*n* network by means of a novel Markov chain method. Based on the equation, we propose an algorithm which is more efficient than other published ones for the reliability of the consecutive-*k*-*n* network.

Keywords: consecutive-*k*-*n* network, consecutive-*k*-out-of-*n* system, algorithm, reliability, complexity

1. INTRODUCTION

The consecutive-*k*-*n* network is a generalization of the popular consecutive-*k*-out-of-*n* system [1-3]. A consecutive-*k*-out-of-*n* system consists of n nodes in a line that fails if and only if some k consecutive nodes all fail. In many practical applications, such a system is generalized to a network (see Fig. 1) with n+2 nodes (nodes 0, 1, 2, …, $n + 1$) and directed links from node i to node j ($0 \le i < j \le n + 1$, $j - i \le k$). In this

Received February 15, 2000; revised July 15, 2000; accepted July 31, 2000.
Communicated by Gen-Huey Chen.

network the source (node 0) and the target (node $n + 1$) are infallible, but all other nodes and links may fail. We call this network the consecutive-k-n network. The network works if and only if there exists a working path from the source going through working nodes and links to the target.

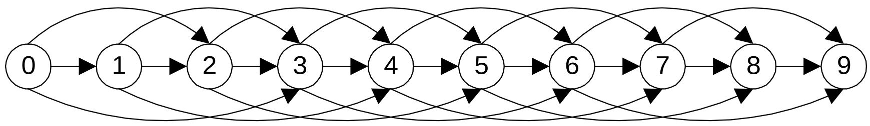

Fig. 1. A consecutive-3-8 network.

Previous research [2-5] claims that the reliability of the consecutive-k-n network is more difficult to compute than that of the consecutive-k-out-of-n system. For the k=2 case, Chen, Hwang and Li [5] studied the consecutive-2-n network with the following classical assumptions:

1. The reliabilities of nodes 1 to n are identical.
2. The reliability of a link is a function only of the distance between the two nodes linked.

In their research, a system of recursive equations for the reliability are derived and solved messily. They also claimed that, for a general consecutive-k-n network, 2^{k-1} recursive equations are needed and these equations are more difficult to derive and solve as k becomes larger.

The Markov chain method was first employed by Fu [6], Fu and Hu [7], Chao and Fu [8, 9], and subsequently by Fu and Koutras [10] in the study of system reliability. Hwang and Wright [11] proposed a Markov chain method to efficiently compute the reliability of the consecutive-k-out-of-n system. They claimed that their method can be extended to compute the reliability of the consecutive-k-n network, but the order of the state transition matrix will be $2^{k(k+1)}$.

In this paper we propose a new Markov chain method and give a closed form equation for the reliability of the general consecutive-k-n network where each node or link has its own reliability. The order of the state transition matrix in our approach is only 2^k. Based on the closed form equation, we also propose an efficient algorithm for the reliability of the consecutive-k-n network.

2. MARKOV CHAIN FOR THE CONSECUTIVE-*k*-*n* NETWORK

Let the reliability (working probability) of node i in the consecutive-k-n network be p_i, and the reliability of the link from node i to node j be $q_{i,j}$. Thus we have $p_0 = p_{n+1} = 1$, $0 \le p_i \le 1$ ($1 \le i \le n$), and $0 \le q_{i,j} \le 1$ ($0 \le i < j \le n + 1$, $j - i \le k$). First, we try to embed the consecutive-k-n network in to a Markov chain $\{Y(t)\}$ defined on the state space $S = \{0, 1, 2, \ldots, n + 1\}$ and the discrete index space $T = \{k, k + 1, \ldots, n + 1\}$. Then, we will show that the reliability of the consecutive-k-n network is equal to the probability that $Y(n+1)$ is odd.

Definition 2.1 A working path is a path through only working nodes and links. ❑

Definition 2.2 For $1 \le i \le n+1$, the binary digit d_i is a random variable which is 1 if and only if node i works and there is a working path from node 0 to node i. ❑

In the following definition, bracketed binary strings followed by the subscript 2 are interpreted as binary numbers. For example, $(1101)_2 = 13$.

Definition 2.3 $\{Y(t)\}$ is a Markov chain defined on the state space S and the discrete index space T, where

$$S = \{0, 1, 2, \ldots, 2^k - 1\},$$
$$T = \{k, k+1, \ldots, n+1\}.$$

Initially,

$$\Pr\{Y(k) = 0\} = \Pr\{(d_1 d_2 \ldots d_k)_2 = 0\},$$
$$\Pr\{Y(k) = 1\} = \Pr\{(d_1 d_2 \ldots d_k)_2 = 1\},$$
$$\Pr\{Y(k) = 2\} = \Pr\{(d_1 d_2 \ldots d_k)_2 = 2\},$$
$$\ldots$$
$$\Pr\{Y(k) = 2^k - 1\} = \Pr\{(d_1 d_2 \ldots d_k)_2 = 2^k - 1\}.$$

The state transition matrix of the Markov chain $\{Y(t)\}$ is

$$\mathbf{M}_t = \begin{bmatrix} m_{0,0,t} & m_{0,1,t} & \cdots & m_{0,2^k-1,t} \\ m_{1,0,t} & m_{1,1,t} & \cdots & m_{1,2^k-1,t} \\ \vdots & & & \vdots \\ m_{2^k-1,0,t} & m_{2^k-1,1,t} & \cdots & m_{2^k-1,2^k-1,t} \end{bmatrix},$$

where for $0 \le i \le 2^k - 1$, $0 \le j \le 2^k - 1$, $k < t \le n+1$,

$$\begin{aligned} & m_{i,j,t} \\ & = \Pr\{Y(t) = j \mid Y(t-1) = i\} \\ & = \begin{cases} 1 & \text{if } i = j = 0 \\ (1 - \prod\limits_{\substack{1 \le x \le k \\ (i \bmod 2^x) \ge 2^{x-1}}} (1 - q_{t-x,t}))\, p_t & \text{if } j = 2(i \bmod 2^{k-1}) + 1 \\ 1 - (1 - \prod\limits_{\substack{1 \le x \le k \\ (i \bmod 2^x) \ge 2^{x-1}}} (1 - q_{t-x,t}))\, p_t & \text{if } j = 2(i \bmod 2^{k-1}) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

❑

Next, we derive the following lemma.

Lemma 2.1 For t ∈ T, $0 \le j \le 2^k - 1$, $\Pr\{Y(t) = j\} = \Pr\{(d_{t-k+1}d_{t-k+2} \ldots d_t)_2 = j\}$.

Proof:

1. By definition 2.3, when $t = k$ the lemma holds.
2. Suppose the lemma holds when $k \le t \le t_0$.
3. When $t = t_0 + 1$,

$$\begin{aligned}
&\Pr\{Y(t) = j\} \\
&= \sum_{i=0}^{2^k-1} \Pr\{Y(t) = j \mid Y(t-1) = i\} \Pr\{Y(t-1) = i\} \\
&= \sum_{i=0}^{2^k-1} m_{i,j,t} \Pr\{(d_{t-k} d_{t-k+1} \ldots d_{t-1})_2 = i\} \\
&= \sum_{i=0}^{2^k-1} \Pr\{(d_{t-k+1} d_{t-k+2} \ldots d_t)_2 = j \mid (d_{t-k} d_{t-k+1} \ldots d_{t-1})_2 = i\} \Pr\{(d_{t-k} d_{t-k+1} \ldots d_{t-1})_2 = i\} \\
&= \Pr\{(d_{t-k+1} d_{t-k+2} \ldots d_t)_2 = j\}.
\end{aligned}$$

❑

The next lemma shows that the reliability, $R_N(k, n)$, of the consecutive-k-n network is equal to the probability that $Y(n + 1)$ is odd.

Lemma 2.2

$R_N(k, n) = \Pr\{Y(n + 1) \text{ is odd.}\}$

Proof:

$R_N(k, n)$
$= \Pr\{d_{n+1} = 1\}$
$= \Pr\{(d_{n-k+2}d_{n-k+3} \ldots d_{n+1})_2 \text{ is odd.}\}$
$= \Pr\{Y(n + 1) \text{ is odd.}\}$ ❑

If we define π_k as the initial probability vector consisting of the initial probabilities,

$$\pi_k = [\Pr\{Y(k) = 0\}, \Pr\{Y(k) = 1\}, \ldots, \Pr\{Y(k) = 2^k - 1\}],$$

and define U as a column vector of 2^k elements,

$$U = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

then the following theorem provides a closed form equation for the reliability of the consecutive-k-n network.

Theorem 2.3

$$R_N(k,n) = \pi_k \Big(\prod_{t=k+1}^{n+1} \mathbf{M}_t\Big)U \; .$$

Proof:

$$\begin{aligned}
&R_N(k,n) \\
&= \Pr\{Y(n+1) \text{ is odd.}\} \\
&= \Pr\{Y(n+1)=1\} + \Pr\{Y(n+1)=3\} + \Pr\{Y(n+1)=5\} + \cdots + \Pr\{Y(n+1)=2^k-1\} \\
&= [\Pr\{Y(n+1)=0\}, \Pr\{Y(n+1)=1\}, \Pr\{Y(n+1)=2\}, \cdots, \Pr\{Y(n+1)=2^k-1\}]U \\
&= \pi_k \Big(\prod_{t=k+1}^{n+1} \mathbf{M}_t\Big)U
\end{aligned}$$

❑

3. EFFICIENT COMPUTATION FOR INITIAL PROBABILITIES

Based on theorem 2.3, once the initial probability vector π_k is obtained, we can easily get $R_N(k, n)$. But, if we employ a brute-force algorithm to compute the initial probabilities, the computational complexity is at least $\Omega(2^{k(k+3)/2})$. In order to simplify the computation, we transform the consecutive-*k*-*n* network into an extended consecutive-*k*-*n* network (see Fig. 2) by adding $k-1$ dummy nodes with related links to the left of node 0. The reliability of the extended consecutive-*k*-*n* network is defined as the probability that there is a working path from "node 0" to node $n + 1$. Therefore, dummy nodes and their related links cannot help make working path from node 0 to node $n + 1$. The reliability of the extended network is identical to that of the original network.

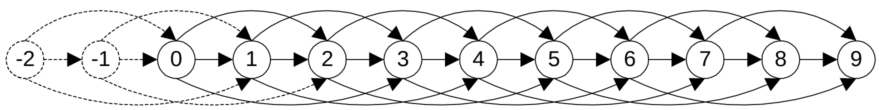

Fig. 2. The extended consecutive-3-8 network transformed from the consecutive-3-8 network in Fig. 1.

For computing the reliability of the extended network, the previous definitions can be extended for the extended consecutive-*k*-*n* network. In definition 2.2, d_i can be defined in a similar way for $-k + 1 \le i \le n + 1$; in definition 2.3, $\{Y'(t)\}$ can be defined in a similar way except that T is replaced by $T' = \{0, 1, 2, \ldots, n + 1\}$; $\mathbf{M}_t$, is defined for $0 < t \le n + 1$, and initially

$\Pr\{Y'(0) = 0\} = \Pr\{(d_{-k+1}d_{-k+2} \ldots d_0)_2 = 0\}$,
$\Pr\{Y'(0) = 1\} = \Pr\{(d_{-k+1}d_{-k+2} \ldots d_0)_2 = 1\}$,
$\Pr\{Y'(0) = 2\} = \Pr\{(d_{-k+1}d_{-k+2} \ldots d_0)_2 = 2\}$,
…
$\Pr\{Y'(0) = 2^k - 1\} = \Pr\{(d_{-k+1}d_{-k+2} \ldots d_0)_2 = 2^k - 1\}$.

Furthermore, in the extended consecutive-*k*-*n* network, we can set

$p_i = 0, -(k-1) \le i < 0$
$q_{i,j} = 0, -(k-1) \le i < 0, 0 < j - i \le k.$

Again, we embed the extended consecutive-*k*-*n* network into the Markov chain $\{Y'(t)\}$. Considering nodes $-k+1, -k+2, \dots, -1$, we find that they always fail and there is no working path from node 0 to them. Thus the binary number $(d_{-k+1}d_{-k+2} \dots d_0)_2$ is always equal to 1.

If we define π_0 as the initial probability vector for the extended consecutive-*k*-*n* network, then

$$\begin{aligned}\pi_0 &= [\Pr\{Y'(0) = 0\}, \Pr\{Y'(0) = 1\}, \dots, \Pr\{Y'(0) = 2^k - 1\}] \\ &= [0, 1, 0, 0, \dots, 0].\end{aligned}$$

The following theorem gives a simplified equation for the reliability of the consecutive-*k*-*n* network.

Thorem 3.1

$$R_N(k,n) = \pi_0 (\prod_{t=1}^{n+1} \mathbf{M}_t) U . \tag{1}$$

Proof:

$$R_N(k,n) = \pi_k (\prod_{t=k+1}^{n+1} \mathbf{M}_t) U ,$$

$$\pi_k = \pi_0 (\prod_{t=1}^{k} \mathbf{M}_t) \cdot$$ ❑

4. THE ALGORITHM AND COMPUTATIONAL COMPLEXITY

For a consecutive-*k*-*n* network with independent node and link reliabilities, the following algorithm uses equation (1) in theorem 3.1 to efficiently compute the network reliability $R_N(k, n)$.

Algorithm A

Step 1: $\pi \leftarrow \pi_0$, $t \leftarrow 1$.
Step 2: Construct $\mathbf{M}_t$ in a data structure for sparse matrices.
Step 3: $\pi \leftarrow \pi\mathbf{M}_t$.
Step 4: If $t < n + 1$ then
 $t \leftarrow t + 1$,
 go to step 2.
Step 5: Return πU. ❑

Based on the sparseness of $\mathbf{M}_t$, we can employ efficient data structures for sparse matrices [12] in steps 2 and 3. Thus, for each t, step 2 costs only $O(2^k)$ operations, and step 3 costs only $2(2^k - 1)$ multiplications and $2^k - 1$ additions. Thus the total computational complexity is $O(2^k n)$. This algorithm is very efficient and practical because, when n is large and k is a small constant, it is an $O(n)$ complexity algorithm.

REFERENCES

1. C. Derman, G. Lieberman, and S. Ross, "On the consecutive-k-out-of-n:F system," *IEEE Transactions on Reliability*, Vol. R-31, 1982, pp. 57-63.
2. M. Lambiris and S. Papastavridis, "Exact probability formulas for linear and circular consecutive-k-out-of-n:F systems," *IEEE Transactions on Reliability*, Vol. R-34, 1985, pp. 124-126.
3. F. K. Hwang, "Simplified reliabilities for consecutive-k-out-of-n:F system," *SIAM Journal on Algebraic and Discrete Methods*, Vol. 7, 1986, pp. 258-264.
4. F. K. Hwang, "Relayed consecutive-k-out-of-n:F lines," *IEEE Transactions on Reliability*, Vol. 37, 1988, pp. 512-514.
5. R. W. Chen, F. K. Hwang, and W. W. Li, "Consecutive-2-out-of-n: F systems with node and link failures," *IEEE Transactions on Reliability*, Vol. 42, 1993, pp. 497-502.
6. J. C. Fu, "Reliability of large consecutive-k-out-of-n:F systems with (k–1)-step Markov dependence," *IEEE Transactions on Reliability*, Vol. R-35, 1986, pp. 602-606.
7. J. C. Fu and B. Hu, "On reliability of large consecutive-k-out-of-n:F systems with (k–1)-step Markov dependence," *IEEE Transactions on Reliability*, Vol. R-36, 1987, pp. 75-77.
8. M. T. Chao and J. C. Fu, "A limit theorem of certain repairable systems," *Annals Institute of Statistical Mathematics*, Vol. 41, 1989, pp. 809-818.
9. M. T. Chao and J. C. Fu, "The reliability of large series system under Markov structure," *Advanced Applied Probability*, Vol. 41, 1991, pp. 894-908.
10. J. C. Fu and M. V. Koutras, "Distribution theory of runs: a Markov chain approach," *Journal of American Statistic Association*, Vol. 89, 1994, pp. 1050-1058.
11. F. K. Hwang and P. E. Wright, "An $O(k^3 \log(n/k))$ algorithm for the consecutive-k-out-of-n:F system," *IEEE Transactions on Reliability*, Vol. 44, 1995, pp. 128-131.
12. E. Horowitz, S. Sahni, and D. Mehta, *Fundamentals of Data Structures in C++*, Computer Science Press, New York, 1995.

Jen-Chun Chang (張仁俊) was born in I-Lan, Taiwan, 1967. He received the B.S. and M.S. degrees in Computer Science and Information Engineering from the National Taiwan University in 1989 and 1991 respectively. He received his PhD degree from the Department of Computer Science and Information Engineering of National Chiao Tung University, Taiwan, in 2000. He was a senior programmer in 1992-1994, a lecturer in

1995-2000, and is an associate professor at the Ming Hsin University of Science and Technology since 2001. His research interests include information security, coding theory, and algorithms.

Rong-Jaye Chen (陳榮傑) received his BS(1977) in mathematics from Taiwan Tsing Hua University and his PhD(1987) in computer science from the University of Wisconsin-Madison. He is currently a professor of Computer Science and Information Engineering Department at National Chiao Tung University, Taiwan. His research interests include cryptography, coding theory, algorithm design, and theory of computation.

Frank K. Hwang (黃光明) received his BA degree from National Taiwan University (1960), MBA from Baruch College (1964) and PhD in statistics from North Carolina State University (1968). He worked at Bell Laboratories from 1967 to 1996 upon his retirement. He has been teaching at the department of Apllied Mathematics, National Chiao Tung University since 1996, and a university chair professor since 1998. He has co-authored four books: The Steiner Tree Problem (1992), Combinatorial Group and Its Applications (1993, 2nd edition 2000), The Mathematical Theory of Nonblocking Switching Networks (1998), Reliabilities of the Consecutive Systems (2000), and publish about 350 papers.