DEVIL'S STAIRCASE OF GAP MAPS

JUNG-CHAO BAN
Department of Applied Mathematics,
National Chiao Tung University, Hsinchu 300, Taiwan

CHENG-HSIUNG HSU*
Department of Mathematics,
National Central University, Chung-Li 320, Taiwan

SONG-SUN LIN†
Department of Applied Mathematics,
National Chiao Tung University, Hsinchu 300, Taiwan

This study demonstrates the devil's staircase structure of topological entropy functions for one-dimensional symmetric unimodal maps with a gap inside. The results are obtained by using kneading theory and are helpful in investigating the communication of chaos.

Keywords: Devil's staircase; gap map; kneading theory.

1. Introduction

Mappings of an interval to itself provide complex behavior, which is important in many applications, such as physics, engineering and so on. One method of quantitatively characterizing the complex behavior of a map is topological entropy. The topological entropy can be considered to be the growth rate of distinct states of a map. If the entropy of a map is positive, then the map behaves chaotically.

This study investigates the chaotic behavior of an one-dimensional symmetric unimodal map which contains one symmetric gap inside the domain. Such maps are termed gap maps herein, and are defined in Sec. 2. This work is motivated by studying the models of cellular neural networks (CNN) [Chua & Yang, 1988a, 1988b] and communication with chaos [Bollt & Lai, 1998]. CNN have

been extensively studied and applied, mainly in image processing and pattern recognition, for details see [Chua, 1998] and [Chua & Yang, 1988a, 1988b]. Previous studies [Ban *et al.*, 2001, 2002; Hsu, 2000, etc.] considered the complexity of stable stationary solutions of CNN with an unbounded piecewise-linear output function. With the iteration method, all stable stationary solutions are equivalent to orbits of certain gap maps. The structure of Smale's horseshoe and the devil's staircase of entropy functions for solutions like those above are obtained in [Hsu, 2000; Hsu & Lin, 1999, etc.] Meanwhile, the gaps in a map can be considered a means of providing noise immunity in schemes for communication with chaos. Any possible motion can be achieved by manipulating the system using slight perturbation. Therefore, the relative likelihood of this motion in the uncontrolled system is unimportant and

the topological entropy is the most relevant measure of complexity in this case.

The dependence of the topological entropy of gap maps on gap size is demonstrated herein by applying kneading theory, and exhibits the devil's staircase structure. Kneading theory was developed by Minlor and Thurston [1988], and reveals the importance of the itineraries of turning points and so-called kneading sequences, which are defined in Sec. 2. Two fundamentals in applying kneading theory to a map are kneading determinant and kneading polynomial. Investigating the relationship between the kneading determinant and periodic points of gap maps, obtains the following main results:

Main Theorem. *Assume $f(x)$ is a strongly transitive one-dimensional symmetric unimodal map (see Definition 2.1) from a compact interval I to I with positive entropy. Let f_μ denote the gap map of f as defined in Sec. 2, then there exists $\bar{\mu} > 0$ such that the entropy function $h(\mu)$ of f_μ possesses the following properties*:

(i) *$h(\mu)$ is a continuous, monotonic, and non-increasing function of μ in $[0, \bar{\mu}]$.*
(ii) *$h(\mu) > 0$ for $\mu \in [0, \bar{\mu})$ and $h(\mu) = 0$ for $\mu \geq \bar{\mu}$.*
(iii) *Furthermore, the subset Ω of $[0, \bar{\mu}]$ consists of μ such that $h(\mu)$ is locally constant, is open and dense in $[0, \bar{\mu}]$.*

Therefore, the above results prove the devil's staircase structure of $h(\mu)$.

The theorem is proved by using kneading theory to characterize the different periodic orbits which never fall into the gap of the map under iteration. Kneading theory reveals that the maximal value of the roots of the kneading polynomial decreases with gap size μ increased, while the corresponding kneading sequence decreases in lexicographic ordering. The devil's structure of the entropy function for gap-tent maps (see Sec. 2) was similarly obtained in [Życzkowski & Bollt, 1999] via kneading theory. Moreover, Collet–Eckman [1980] uses similar ideas to prove the sensitivity of certain unimodal maps to initial conditions. Craczyk and Światek [1997] considered the logistic map $f_\mu(x) = \mu x(1-x)$ using the box mapping method and proved that the mapping f_μ has an hyperbolic attracting cycle for an open and dense set of parameters μ in $(0, 4]$.

This paper is organized as follows. Section 2 introduces some preliminary definitions, notations and results for the dynamical system. Next, Sec. 3 generalizes the results of Sec. 2 to gap maps. Finally, kneading theory is applied to prove the main theorem in Sec. 4.

2. Preliminary

This section introduces some preliminary notations and results from the dynamical system considered herein, and which are used in proving the main theorem. Hereafter, we assume that $I = [0, 1]$.

Definition 2.1. Let $f : I \to I$ be a continuous map. Additionally, we say that

(i) point $c \in \operatorname{int} I$ where f has a local extremum is called a turning point.
(ii) f is unimodal if f is piecewise monotone, $f(\partial I) \subset \partial I$ and has precisely one turning point c in I. Moreover, f is said to be symmetric if it is symmetric with respect to c.
(iii) f is said to be strongly transitive if for any subinterval $J \subset I \setminus \{c\}$, there exists $k \geq 0$ such that $\operatorname{int} I \subseteq f^k(J)$.
(iv) Assume f denotes an unimodal map with turning point c. Let $\mu > 0$ and subinterval J of I be defined by $J = (c - \mu, c + \mu) \subset I$, then the map $f_\mu \equiv f|_{I \setminus J}$ is termed a gap map of f with size μ.

Strong transitivity is well known to be important for a map with chaotic behavior. Birkhoff's theorem (see [Ott, 1994]) is as follows.

Proposition 2.1. *Let f be a strongly transitive map, then the periodic points of f are dense in I and unstable.*

The dense and instability results in Proposition 2.1 play the key role in investigating the behavior of periodic orbits related to kneading theory. Some useful gap maps which are strongly transitive are illustrated below.

Example 2.1. Lorentz-type gap map. Let $f : I \to I$ represent a strongly transitive Lorentz-type map, that is, f satisfies

(1) $c \in \operatorname{int} I$ exists such that f is monotonic on $[0, c)$ and $(c, 1]$.

(2) $\lim_{x \to c^-} f(x) = 1$ and $\lim_{x \to c^+} f(x) = 0$.
(3) The set $D = \bigcup_{n \geq 0} f^{-n}(c)$ is dense in I.

The gap map f_μ of f is termed the Lorentz-type gap map.

Example 2.2. Unimodal-type gap map. Let f_μ denote a gap map of f with turning point c. If

(1) f_μ is monotonic increasing for $[0, c - \mu]$ and decreasing for $[c + \mu, 1]$.
(2) $\lim_{x \to c - \mu} f_\mu(x) = \lim_{x \to c + \mu} f_\mu(x)$.

The gap map f_μ of f is called the unimodal-type gap map.

One type of unimodal-type gap map is the gap-tent map defined by

$$T_\mu(x; \lambda) = \begin{cases} \lambda x & \text{if } x \in \left[0, \frac{1}{2} - \mu\right], \\ \lambda(1 - x) & \text{if } x \in \left[\frac{1}{2} + \mu, 1\right], \end{cases}$$

where $\lambda > 0$ and $\mu < 1/2$. The following results display the strong transitivity of certain gap maps.

Proposition 2.2. *Assume f is a Lorentz-type map of I. Let f_μ be the related Lorentz-type gap map of f. If there exists $q > 1$, $n_1 > 0$ and $n_2 > 0$ such that*

(1) $f'(x) \geq q$ for all $x \in I \setminus \{c\}$,
(2) $f_\mu^i(0) \in [0, c - \mu)$, for $i = 1, \cdots n_1$ and $f_\mu^{n_1+1}(0) \notin [0, c - \mu)$,
(3) $f_\mu^j(1) \in (c + \mu, 1]$, for $i = 1, \ldots, n_2$ and $f_\mu^{n_2+1}(1) \notin (c + \mu, 1]$,
(4) $\min\{q^{n_1+1}, q^{n_2+1}\} > 2$,

then f_μ is strongly transitive.

Proof. See [Afraimovich & Hsu, 1998]. ∎

The approach used to prove Proposition 2.2 can also be used to prove the strong transitivity of the gap-tent map in the following.

Corollary 2.1. *If $\sqrt{2} \leq \lambda \leq 2$, then the gap-tent map T_λ is strongly transitive.*

Using Proposition 2.1, the property of strong transitivity for a map describes the complex behavior of a dynamic system. However, complex behavior must be described more precisely. One

important quantity characterizing the complex behavior of a map is entropy. The devil's staircase structure of entropy function for certain gap-tent maps were studied in [Życzkowski & Bollt, 1999]. Since the gap-tent maps are symmetric, it gives us the motivation to generalize the similar results to symmetric unimodal-gap maps. The entropy describes the growth rate of different states of a map and is defined as follows.

Definition 2.2

(i) Let $f \colon X \to f$ denote a continuous map on space X with metric d. A set $S \subset X$ is termed (n, ϵ)-separated for f for n a positive integer and $\epsilon > 0$, provided that for every pair of distinct points $x, y \in S$, there is at least one k with $0 \leq k < n$ such that $d(f^k(x), f^k(y)) > \epsilon$.

(ii) The number of different orbits of length n (as measured by ϵ) is defined by

$$\gamma(n, \epsilon, f) = \max\{\sharp(S)|S \subset X \text{ is } (n, \epsilon) \\ - \text{separated set for } f\} \quad (1)$$

where $\sharp(S)$ represents the number of elements in S.

(iii) The topological entropy of f is defined as

$$h(f) = \lim_{\epsilon \to 0, \, \epsilon > 0} \overline{\lim_{n \to \infty}} \frac{\ln \gamma(n, \epsilon, f)}{n}. \quad (2)$$

Several methods exist for computing the entropy of a dynamical system. This study applies the kneading theory developed by Minlor and Thurston [1988] to calculate the entropies of gap maps. First, the symbolic dynamics associated with a unimodal map are introduced. Let $\Sigma = \{0, c, 1\}^N$, that is, an element of Σ is a sequence $\underline{x} = (x_0, x_1, \ldots, x_n, \ldots)$ where each $x_i \in \{0, c, 1\}$. Meanwhile, let $\sigma : \Sigma \to \Sigma$ represent the shift map $\sigma \underline{x} = \underline{y}$ where $y_i = x_{i+1}$.

Definition 2.3. Let f denote an unimodal map with turning point c.

(i) The itinerary of x with respect to f, denoted by $I(x)$ is the sequence $\underline{i} = (i_0(x), i_1(x), \ldots, i_n(x), \ldots)$ where

$$i_j(x) = \begin{cases} 0, & \text{if } f^j(x) < c, \\ 1, & \text{if } f^j(x) > c, \\ c, & \text{if } f^j(x) = c. \end{cases}$$

(ii) A signed lexicographic ordering $\prec$ on Σ is defined as follows. Let $s_i = t_i$ for $i = 1$ to $n-1$, then $\underline{s} \prec \underline{t}$ if either

   (a) $\tau_{n-1}(s)$ is even and $s_n < t_n$

   or

   (b) $\tau_{n-1}(s)$ is odd and $s_n > t_n$,

   where $\tau_k = \sum_{i=0}^{k} s_i$. We also write $\underline{s} \preceq \underline{t}$ if $\underline{s} \prec \underline{t}$ or $\underline{s} = \underline{t}$.

(iii) The kneading sequence of f, denoted by Kf, is defined as the itinerary of point $f(c)$, i.e.

$$Kf = I(f(c)).$$

(iv) Sequence $\underline{s}$ is n-periodic if $s_i = s_{i+n}$ for $i \geq 0$, and denoted by $\underline{s} = (s_0, s_1, \ldots, s_{n-2}, s_{n-1})^\infty$.

Since $f(c)$ is the extremun of f over the interval I, it is known that for a given unimodal map, the most important one, among all its itineraries, is the itinerary of point $f(c)$, i.e. the kneading sequence of f. The sufficient and necessary conditions for a sequence in Σ to be an itinerary of some point in I is recalled as follows.

Proposition 2.3 (see [Xie, 1996]). *Let* $Kf = e_1 \cdots e_n \cdots$ *be the kneading sequence of a unimodal map* f, *then*

(i) *If* $s = I(x)$ *is an itinerary of a point* $x \in I$, *then*

$$\sigma^i(s) \preceq Kf \quad \text{for all} \quad i \geq 1.$$

(ii) *If* $s \in \Sigma$ *and* Kf *does not contain the symbol* c, *and* $s = s_1, \ldots, s_n \ldots$, *satisfies inequalities*

$$\sigma^i(s) \prec Kf \quad \text{for all} \quad i \geq 1,$$

then there exists a point $x \in I$ *such that* $t = I(x)$.

The main theorem is established by examining the relationship between entropy and kneading sequence. The results of [Molnor & Thurston, 1988], can be stated as follows:

Proposition 2.4. *Let* f *be an unimodal map with the kneading sequence* $Kf = (e_1, e_2, \ldots, e_{n-1}, e_n)^\infty$, *then the topological entropy of* f *is the logarithm of the largest positive root of the following equation*

$$P(\lambda) = \lambda^{n-1} + \varepsilon_1' \lambda^{n-2} + \cdots + \varepsilon_{n-1}' = 0. \quad (3)$$

Here $e_i \in \{0, 1\}$, $\varepsilon(0) = 1$, $\varepsilon(1) = -1$, $\varepsilon_i = \varepsilon(e_i)$ *and* $\varepsilon_i' = \prod_{k=1}^{i} \varepsilon_k$.

From [Molnor & Thurston, 1988], the polynomial $P(\lambda)$ is closely related to the kneading determinant defined by Thurston and Milnor. In fact, $P(\lambda)$ is the characteristic polynomial of the Stefan transition matrix. The coefficients of $P(\lambda)$ are uniquely determined by the periodic kneading sequence which represents the itinerary of a periodic turning point. Kneading determinants are considered an important tool for computing the entropy of one-dimension maps. However, Proposition 2.4 is enough for us to consider the polynomial $P(\lambda)$.

3. Properties of Gap Maps

The definitions and results from Sec. 2 relating to gap maps are generated in this section.

Definition 3.1. Let f be an unimodal map with turning point c and f_μ is the gap map of f.

(i) Define set $G_\infty \subset I$ by $G_\infty(\mu) = \bigcup_{i=1}^{\infty} f_\mu^{-i}(c - \mu, c + \mu)$.

(ii) The kneading sequence of f_μ, written as Kf_μ, is defined by either

   (a) if $f^i(c + \mu) \notin (c - \mu, c + \mu)$ for all $i \geq 1$, then $Kf_\mu = I(f(c + \mu))$

   or

   (b) if $f^i(c + \mu) \in (c - \mu, c + \mu)$ for some $i \geq 1$, let $\hat{\mu}$ be the μ-nearest number such that $f^k(c + \hat{\mu})$ is equal to $c - \hat{\mu}$ or $c + \hat{\mu}$ for some $k \geq 1$, then

$$Kf_\mu = Kf_{\hat{\mu}} = I(f(c + \hat{\mu})).$$

(iii) A sequence $\underline{s} \in \{0, c, 1\}^N$ is admissible for f_μ if $x \in I \setminus (c - \mu, c + \mu)$ exists such that $I(x) = \underline{s}$.

From Proposition 2.4, it is natural to define the language $L_\mu(f)$ of f_μ by

$$L_\mu(f) = \{x \in I \setminus G_\infty | \sigma^i(x) \prec Kf_\mu, \quad \text{for} \quad i \geq 0\}. \quad (4)$$

Then, the entropy function of f_μ can be defined by the language as follows:

Definition 3.2. Let f_μ be an unimodal-type gap

map with

$$S_n(\mu) \equiv \text{card}\{x \in L_\mu(f) | |x| = n\} \quad \text{and}$$

$$S(\mu) \equiv \lim_{n \to \infty} \{S_n(\mu)\}^{\frac{1}{n}} .$$

The topological entropy $h(\mu)$ of f_μ is defined by $h(\mu) = ln S(\mu)$. Here, $|x|$ denotes the length of string x.

From [Bowen, 1973] and [Robinson, 1995], the above definition of topological entropy is consistent with that in Definition 2.2. Additionally, $S_n(\mu)$ can be considered the growth number of f_μ as length n increases. If we replace $S_n(\mu)$ and $S(\mu)$ with $\tilde{S}_n(\mu)$ and $\tilde{S}(\mu)$, respectively, where

$$\tilde{S}_n(\mu) = \{x \in S_n(\mu) | I(x) = 1\} \quad \text{and}$$
$$\tilde{S}(\mu) \equiv \lim_{n \to \infty} \{\tilde{S}_n(\mu)\}^{\frac{1}{n}} , \quad (5)$$

then it is easy to check that $h(\mu) = \ln \tilde{S}(\mu)$ owing to $S_n(\mu) = \sum_{k=1}^n \tilde{S}_k(\mu) + 1$. Therefore, we assume $I(x) = 1$ for all $x \in S_n(\mu)$ hereafter and recall the results of [Xie, 1996] as follows.

Proposition 3.1

(i) *Let f_μ denote an unimodal type gap map with kneading sequence $Kf_\mu = \{e_1, \ldots, e_n, \ldots\}$ and $e_i \in \{0, 1\}$, then*

$$S_n(\mu) + \varepsilon_1' S_{n-1}(\mu) + \varepsilon_2' S_{n-2}(\mu)$$
$$+ \cdots + \varepsilon_{n-1}' S_1(\mu) + \frac{1}{2}\varepsilon_n' = \frac{1}{2}. \quad (6)$$

(ii) *If $n > 1$ and $Kf_\mu = \{e_1, \ldots, e_n\}^\infty$ with $n > 1$, then $h(\mu)$ is equal to the logarithm of the largest positive root of $P(\lambda)$.*

(iii) *If Kf_{μ_1} and Kf_{μ_2} are two kneading sequences with $Kf_{\mu_1} \prec Kf_{\mu_2}$, then $h(\mu_1) < h(\mu_2)$.*

Proof. The proof of the results resembles [Xie, 1996], the only difference being the assumption $\sum_{i=1}^n e_i$ is an even number, an assumption needed for (ii) in [Xie, 1996]. In fact, this assumption is not necessary. Consider the power series $\Im(t)$ by

$$\Im(t) = \frac{1}{2} + \sum_{i=1}^\infty S_i(\mu) t^i . \quad (7)$$

The radius of convergence of $\Im(t)$ is $1/S(\mu)$, and has a pole at $t = 1/S(\mu)$. Multiply (7) by 1 and

$\varepsilon_i' t^i$ for $i = 1$ to $n - 1$, respectively. This operation produces

$$\left(1 + \sum_{i=1}^{n-1} \varepsilon_i' t^i\right) \Im(t) = \frac{1}{2} \sum_{i=0}^{n-1} t^i, \quad \text{for} \quad |t| < \frac{1}{S(\mu)} , \quad (8)$$

and $1/S(\mu)$ is a root of $P(\lambda)$. Let $R(0) = 1$, $R(1) = 0$, $R(e_1, \ldots, e_n) = \prod_{i=1}^n R(e_i)$ and rewrite

$$L_\mu(f) = \{x \in I \setminus G_\infty | \sigma^i(x) \succ R(Kf_\mu), \quad \text{for} \quad i \geq 0\},$$

then $\sum_{i=1}^n R(e_1, \ldots, e_i)$ is an odd number. Hence, using similar arguments, we also obtain

$$\left(1 + \sum_{i=1}^{n-1} \varepsilon_i' t^i\right) \Im(t) = \frac{1}{2} \sum_{i=0}^{n-1} (-1)^i t^i, \quad \text{for} \quad |t| < \frac{1}{S(\mu)}. \quad (9)$$

Therefore, the results of Proposition 3.1 hold even when $\sum_{i=1}^n e_i$ is odd, and the proof is complete. ∎

Example 3.1

(i) Suppose the kneading sequence $Kf_\mu = (1, 0, 1)^\infty$ is as in Fig. 1, then $P(\lambda) = \lambda^2 - \lambda - 1$. Hence, the entropy $h(\mu) = \ln((1 + \sqrt{5})/2)$.

On the other hand, consider the covering relationship between I_1 and I_2 in Fig. 1, the Stefan graph is

$$I_1 \to I_2, \ I_2 \to I_1 \ \text{ and } \ I_2 \to I_2 .$$

Hence, the polynomial $P(\lambda)$ is the characteristic polynomial of transition matrix M by

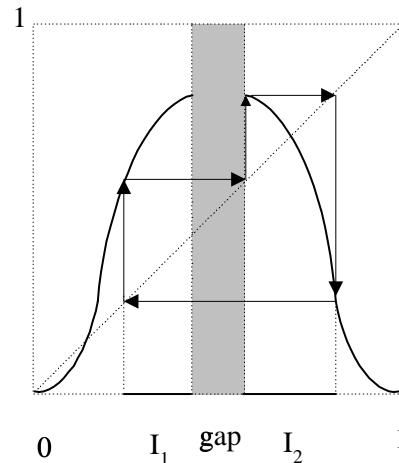$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$
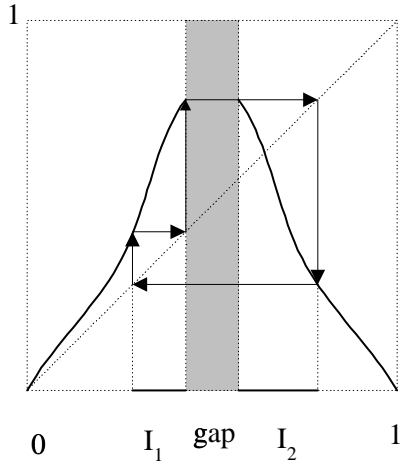

Fig. 1.

Fig. 2.

(ii) Suppose the kneading sequence $Kf_\mu = (0, 1, 0)^\infty$ is as in Fig. 2, then the number $\sum_{i=1}^{n} e_i$ is odd. The entropy is equal to $\ln((1 + \sqrt{5})/2)$ is as in Fig. 1.

4. Proof of the Main Theorem

To prove the continuity of entropy function, we need to consider the following two cases:

case (i): $f_\mu(c + \mu) \in \overline{G_\infty}(\mu) \setminus G_\infty(\mu)$,
case (ii): $f_\mu(c + \mu) \in G_\infty(\mu)$.

For case (i), there are also two subcases to be considered, i.e.

(a) $c + \mu$ is a periodic point

or

(b) $c + \mu$ is an aperiodic point, i.e. neither periodic nor eventually periodic point.

In (a), $c + \mu$ is an unstable periodic point by the strong transitivity of f, i.e. every periodic point is repelling. Suppose $c + \mu$ is a n periodic point, then there exist a neighborhood U of $c + \mu$ such that for every subset $\Omega \subset U$, there exists a positive integer $p(\Omega)$ such that $f^{nj}|_\Omega$ is a homomorphism and

$$f^{nj}(\Omega) \subset U, \text{ for } 1 \le j \le p, \text{ and } f^{np}(\Omega) \cap U = \emptyset.$$

Hence, without lost of generality, we can choose a sequence $\{(\Omega_i, p_i)\}_{i=1}^{\infty}$ such that
(α) $\Omega_i \bigcap \Omega_j = \emptyset$, for $i \ne j$,
(β) $f^{nk}|_{\Omega_j}$ is a homomorphism for $k = 1$ to p_j,
(γ) $f^{np_j}(\Omega_j) \bigcap U = \emptyset$.

Due to the strong transitivity of f, there exists $c + \mu_j \in \Omega_j$ for each j such that $I(f(c + \mu_j))$ has the following properties:

(1) $I(f(c+\mu_j)) = ((e_1, \ldots, e_n)^{p_i} \cdot I(f^{np_j}(c+\mu_j)))^\infty$, where $e_i = I_i(f(c + \mu_j))$.
(2) $\{c + \mu_j\}_{j=1}^{\infty}$ are distinct and monotonic.
(3) $\lim_{j \to \infty} c + \mu_j = c + \mu$.

Therefore, by kneading theorem, we have

$$h(\mu_j) = \max\{\lambda | \lambda > 0 \text{ and } \Gamma_j(\lambda) = 0\},$$

and

$$h(\mu) = \max\{\lambda | \lambda > 0 \text{ and } \Gamma(\lambda) = 0\},$$

where

$$\Gamma_j(\lambda) = \left(1 + \sum_{i=1}^{n-1} \varepsilon_i' \lambda^i\right) \sum_{j=0}^{p_j} \lambda^{jp_j} + \sum_{j=np_j+1}^{m(p_j)-1} \varepsilon_j' \lambda^j, \tag{10}$$

$$\Gamma(\lambda) = \left(1 + \sum_{i=1}^{n-1} \varepsilon_i' \lambda^i\right) \sum_{j=0}^{\infty} \lambda^j, \tag{11}$$

and $m(p_j)$ is the smallest positive integer that satisfies $f^{m(p_j)}(c + \mu_j) = c + \mu_j$. Since Γ_j and Γ have the same coefficients up to the first np_j terms and $p_j \to \infty$ as $j \to \infty$, by the Cauchy integral formula, we obtain that $h(\mu_j)$ tends to $h(\mu)$.

To prove the results of case (b), we need the following lemma.

Lemma 4.1. *Let $c + \mu$ be the point such that*

(1°) $|I(f(c + \mu))| = \infty$, *here $|\cdot|$ is denoted as the length of $I(f(c + \mu))$,*
(2°) $f^i(c + \mu) \notin \text{int } G_\infty(\mu)$, *for all $i \ge 1$,*

then there exists a sequence of periodic points and positive integers $\{(c + \mu_j, p_j)\}_{j=1}^{\infty}$ such that

$$\lim_{j \to \infty} c + \mu_j = c + \mu$$

and

$$I_i(f(c + \mu_j)) = I_i(f(c + \mu)), \text{ for } 1 \le i \le p_j.$$

Proof. By the strong transitivity of f, without lost of generality, we may choose $n >$ such that

$I^n(f(c+\mu)) \equiv (I_1(f(c+\mu)), \ldots, I_n(f(c+\mu)))$ is even

and

$$I_n(f(c+\mu)) = 1\,.$$

Similar results can also be obtained for the other case. By (2°), we have $f^n(c+\mu) > c+\mu$ and let $c+\mu^*$ be the point such that

$$I_i(f(c+\mu^*)) = I_i((f(c+\mu)), \quad \text{for} \quad i = 1,\dots,n$$

and

$$f^{n+1}(c+\mu^*) = c+\mu\,.$$

Moreover, we have $f^{n+1}(c+\mu) > c+\mu$ by assumptions. Therefore, let $U = (c+\mu, c+\mu^*)$ then $f^i|_U$ is a homeomorphism for $i = 1$ to n and $f^{n+1}(U) \supset U$. Hence, by the contraction mapping principle, there exists a periodic point $c+\mu_n \in U$ such that

$$I_i(f(c+\mu_n)) = I_i(f(c+\mu)), \quad \text{for} \quad i = 1,\dots,n\,.$$

Continue the same process and apply the monotonic property of itinerary, we can find p_j and $c+\mu_j$ such that

$$I_i(f(c+\mu_j)) = I_i(f(c+\mu)), \quad \text{for}$$

$$i = 1,\dots,p_j$$

and

$$\lim_{j\to\infty} c+\mu_j = c+\mu\,.$$

The proof is complete. ∎

By Lemma 4.1 and the same arguments of (a), we also obtain that $h(\mu_j)$ tend to $h(\mu)$ as $j \to \infty$ by taking such μ_j as in above lemma.

Next, we consider case (ii), i.e. $f(c+\mu) \in G_\infty(\mu)$. The following lemma is necessary for the proof.

Lemma 4.2. *If $f_\mu(c+\mu) \in \overline{G_\infty}(\mu)\backslash G_\infty(\mu)$, then there exist $\varepsilon_0 > 0$ such that $f_\mu(c+\mu+\varepsilon) \in G_\infty(\mu)$, for all $0 < \varepsilon < \varepsilon_0$.*

Proof. Suppose the result is false, then there exists a neighborhood U of $[0,1]$ such that $f(c+\mu+\xi) \in \overline{G_\infty}(\mu)\backslash G_\infty(\mu)$ for all ξ in U. Therefore, $f(c+\mu+\xi)$ has the same itinerary for all ξ in U. Without loss of generality, we may assume that $c+\mu$ is a n-periodic point and which gives $f^n(U) = U$. However, this result contradicts the strong transitivity of f. The proof is complete. ∎

By Lemma 4.2, case (ii) can be considered as case (i) by a small perturbation of μ. Indeed, if

$f(c+\mu) \in G_\infty(\mu)$, then there exists $0 < \varepsilon < \varepsilon_0$ such that $f(c+\mu+\varepsilon) \in \overline{G_\infty}(\mu+\varepsilon)\backslash G_\infty(\mu+\varepsilon)$. Since $G_\infty(\mu)$ is an open set, there exists a neighborhood U of μ such that $f(c+\mu+\xi) \in G_\infty(\mu+\xi)$ for all $\mu+\xi \in U$. To prove the continuity of entropy function, we may assume that $f(c+\nu+\varepsilon)$ is a n-periodic point. Denote $\Sigma(\mu)$ by

$$\Sigma(\mu) = \{x | x \in [0,1]\backslash G_\infty(\mu)\}\,,$$

then $\Sigma(\mu) = \Sigma(\mu+\varepsilon)$ for $0 < \varepsilon < \varepsilon_0$. Hence $h(\mu) = h(\mu+\varepsilon)$ for all $\xi \in U$, i.e. the entropy function $h(\mu)$ is a continuous function of μ in case (ii).

To prove the continuity result of case (i), we can construct infinite many periodic points which approach to a given periodic point or aperiodic point. For such infinite many periodic points, there exist small perturbations of these points which will fall into the set $G_\infty(\mu)$, i.e. belong to the case (ii). Hence, by the same arguments, there exist intervals of parameters μ such that f_μ has the same entropy and such intervals form an open and dense set, i.e. the entropy function has the nonuniform Cantor set structure or devil's staircase structure. The proof is complete.

Remark 4.1. Let f be a symmetric unimodal map with negative Schwarzian derivative. If f has a periodic point t with period $2^m p$, where $m \geq 0$, p is a prime number and t is a maximal point, i.e. $\sigma^i(I(t)) \prec I(t)$ for $i \geq 1$. Then the entropy function $h(\mu)$ of f_μ also has the same structure as the main theorem. The results will appear in our further study.

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