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# LOW-TEMPERATURE SERIES EXPANSIONS FOR SQUARE-LATTICE ISING MODEL WITH FIRST AND SECOND NEIGHBOUR INTERACTIONS

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We have calculated the low-temperature series expansions of the spontaneous magnetization and the zero-field susceptibility of the square-lattice ferromagnetic Ising model with first-neighbour interaction $J_1$ and second-neighbour interaction $J_2$ to the 30th and 26th order respectively by computer. Our results extend the previous calculations by Lee and Lin to six more orders. We use the Padé approximants to estimate the critical exponents and the critical temperature for different ratios of $R = J_2/J_1$. The estimated critical temperature as a function of $R$ agrees with the estimation by Oitmaa from high-temperature series expansions.

*Keywords*: Ising model; low temperature series expansions.



The partition function of the square-lattice Ising model with nearest neighbour interactions in the absence of magnetic field was derived by Onsager.[1] The spontaneous magnetization of this model was also obtained by Onsager but he never published his derivation. Yang[2] was the first to publish a detailed derivation of the spontaneous magnetization and the critical exponent $\beta$ is 0.125. The zero-field susceptibility of the square-lattice Ising model is still unsolved, but the critical exponents $\gamma(T > T_c)$ and $\gamma'(T < T_c)$, where $T_c$ is the critical temperature of the Ising model, are known[3] to be 1.75. The square-lattice Ising models with interactions beyond first neighbours are unsolvable by existing methods. According to the universality hypothesis, the critical exponents are expected to be the same.

The square-lattice Ising model with first and second neighbour interactions was first studied by Dalton and Wood.[4] They derived the low-temperature series expansions of the spontaneous magnetization and the zero-field susceptibility up to the 17th and 13th orders respectively. They used the Padé approximants to estimate the critical exponents and found that $0.122 < \beta < 0.134$. However their series for the susceptibility is too short and no conclusion was obtained for $\gamma'$. Lee and Lin[5] calculated the low-temperature series expansions of the spontaneous magnetization and the zero-field susceptibility up to the 24th and 20th order respectively and found that $0.122 < \beta < 0.128$ and $1.75 < \gamma' < 1.85$. In this paper, we further extend the result of Lee and Lin to six more orders and use the Padé approximants to estimate the critical temperature and the critical exponents.

The Hamiltonian of the square-lattice Ising model with first and second neighbour interactions is

$$\mathbf{H} = -J_1 \sum_{nn} \sigma_i \sigma_j - J_2 \sum_{nnn} \sigma_i \sigma_j \tag{1}$$

where the first and second summations are over nearest neighbour ($nn$) pairs and next nearest neighbour ($nnn$) pairs respectively.

The method of low-temperature series expansions is well known.[6] The spontaneous magnetization $M(x, y)$ and the zero-field susceptibility $\chi(x, y)$ are expanded in power series of $x$ and $y$ where

$$x = \exp\left(\frac{-4J_1}{kT}\right), \qquad y = \exp\left(\frac{-4J_2}{kT}\right). \tag{2}$$

We have calculated the low-temperature series expansions of the spontaneous magnetization $M(x, y)$ and the susceptibility $\chi(x, y)$ for the square-lattice Ising model with first and second neighbour interactions up to the 30th order and 26th order respectively. The results are

$$M(x, y) = 1 - 2x^2y^2 + \sum_{m,n} a(m, n)x^m y^n \tag{3}$$

$$\chi(x, y) = 1 + \sum_{m,n} b(m, n)x^m y^n \tag{4}$$

where the coefficients $a(m, n)$ and $b(m, n)$ are given in Tables 1 and 2 respectively.

In the special case of $x = y$, we have

$$\begin{aligned} M(x, x) = 1 &- 2x^4 - 16x^7 + 18x^8 - 24x^9 - 104x^{10} + 248x^{11} - 516x^{12} - 328x^{13} \\ &+ 2292x^{14} - 7200x^{15} + 4676x^{16} + 13720x^{17} - 81148x^{18} + 128960x^{19} \\ &- 32368x^{20} - 713056x^{21} + 1945752x^{22} - 2600056x^{23} - 3692616x^{24} \\ &+ 21775624x^{25} - 49678052x^{26} + 21759936x^{27} + 173941332x^{28} \\ &- 668655828x^{29} + 971318640x^{30} + \cdots \end{aligned} \tag{5}$$

Table 1. Coefficients of the series expansions for spontaneous magnetization.

| $m, n$ | $a(m, n)$ | $m, n$ | $a(m, n)$ | $m, n$ | $a(m, n)$ | $m, n$ | $a(m, n)$ |
|---|---|---|---|---|---|---|---|
| 3,4 | −8 | 4,3 | −8 | 4,4 | 18 | 4,5 | −24 |
| 4,6 | −20 | 5,5 | −48 | 6,4 | −36 | 5,6 | 80 |
| 6,5 | 168 | 5,7 | −144 | 6,6 | −364 | 8,4 | −8 |
| 5,8 | −40 | 6,7 | 144 | 7,6 | −288 | 8,5 | −144 |
| 6,8 | −52 | 7,7 | 1184 | 8,6 | 1160 | 6,9 | −504 |
| 7,8 | −2704 | 8,7 | −3872 | 9,6 | −40 | 10,5 | −80 |
| 6,10 | −70 | 7,9 | 1440 | 8,8 | 5358 | 9,7 | −1712 |
| 10,6 | −340 | 7,10 | −1648 | 8,9 | −2704 | 9,8 | 11464 |
| 10,7 | 6632 | 12,5 | −24 | 7,11 | −1344 | 8,10 | −8064 |
| 9,9 | −37328 | 10,8 | −33356 | 11,7 | −576 | 12,6 | −480 |
| 7,12 | −112 | 8,11 | 2672 | 9,10 | 56368 | 10,9 | 76880 |
| 11,8 | −7664 | 12,7 | 816 | 8,12 | −8524 | 9,11 | −57136 |
| 10,10 | −84864 | 11,9 | 88608 | 12,8 | 29968 | 13,7 | −112 |
| 14,6 | −308 | 8,13 | −3024 | 9,12 | −10696 | 10,11 | −45808 |
| 11,10 | −407200 | 12,9 | −239176 | 13,8 | −5584 | 14,7 | −1568 |
| 8,14 | −168 | 9,13 | −13168 | 10,12 | 167404 | 11,11 | 963392 |
| 12,10 | 828616 | 13,9 | −16384 | 14,8 | 16156 | 16,6 | −96 |
| 9,14 | −29512 | 10,13 | −311072 | 11,12 | −1445272 | 12,11 | −1448928 |
| 13,10 | 551920 | 14,9 | 87168 | 15,8 | −2216 | 16,7 | −2144 |
| 9,15 | −6048 | 10,14 | −23220 | 11,13 | 956384 | 12,12 | 496260 |
| 13,11 | −3643744 | 14,10 | −1439614 | 15,9 | −34752 | 16,8 | 2136 |
| 18,6 | −18 | 9,16 | −240 | 10,15 | −118792 | 11,14 | −298432 |
| 12,13 | 2618528 | 13,12 | 12257544 | 14,11 | 7089248 | 15,10 | 118184 |
| 16,9 | 111384 | 17,8 | −504 | 18,7 | −1296 | 9,17 | −1497984 |
| 10,16 | −7901600 | 11,15 | −10183888 | 12,14 | −3557904 | 13,13 | −15541072 |
| 14,12 | −14323744 | 15,11 | 3088944 | 16,10 | 272480 | 17,9 | −25024 |
| 18,8 | −8260 | 10,17 | −11088 | 11,16 | −286264 | 12,15 | 5900896 |
| 13,14 | 30843176 | 14,13 | 19729328 | 15,12 | −27347504 | 16,11 | −7023008 |
| 17,10 | −117680 | 18,9 | 72760 | 19,8 | −80 | 20,7 | −600 |
| 10,18 | −330 | 11,17 | −533632 | 12,16 | −5363752 | 13,15 | −20101024 |
| 14,14 | 19749266 | 15,13 | 125751456 | 16,12 | 52267388 | 17,11 | 1747920 |
| 18,10 | 443756 | 19,9 | −10816 | 20,8 | −8900 | 11,18 | −197888 |
| 12,17 | −3854936 | 13,16 | −11201256 | 14,15 | −119303842 | 15,14 | −351894966 |
| 16,13 | −185818284 | 17,12 | 6993368 | 18,11 | −3204304 | 19,10 | −174528 |
| 20,9 | 984 | 22,7 | −176 | 11,19 | −19008 | 12,18 | −1804944 |
| 13,17 | 17185120 | 14,16 | 201958068 | 15,15 | 618347936 | 16,14 | 329430948 |
| 17,13 | −170524608 | 18,12 | −24134784 | 19,11 | 303408 | 20,10 | 587212 |
| 21,9 | −3888 | 22,8 | −6820 | | | | |

$$\begin{aligned}\chi(x,x) = {} & 1 + 16x^3 - 18x^4 + 36x^5 + 160x^6 - 340x^7 + 980x^8 + 808x^9 - 3802x^{10} \\ & + 16464x^{11} - 7175x^{12} - 24432x^{13} + 221346x^{14} - 300192x^{15} \\ & + 180712x^{16} + 2314428x^{17} - 5521804x^{18} + 9422824x^{19} + 14985996x^{20} \\ & - 71270104x^{21} + 193417806x^{22} - 54203752x^{23} - 629440122x^{24} \\ & + 2864059974x^{25} - 3814925920x^{26} + \cdots . \end{aligned} \tag{6}$$

The lattice constant $c(n, r)$ is the number of connected graphs with $r$ vertices and $b = 4r - n$ bonds. The lattice constants for $n \leq 30$ are given in Tables 3 and 4.

Table 2. Coefficients $b(m,n)$ of the series expansions for susceptibility.

| $m,n$ | $b(m,n)$ | $m,n$ | $b(m,n)$ | $m,n$ | $b(m,n)$ | $m,n$ | $b(m,n)$ |
|---|---:|---|---:|---|---:|---|---:|
| 1,2 | 8 | 2,1 | 8 | 2,2 | −18 | 2,3 | 36 |
| 2,4 | 34 | 3,3 | 72 | 4,2 | 54 | 3,4 | −88 |
| 4,3 | −252 | 3,5 | 328 | 4,4 | 636 | 6,2 | 16 |
| 3,6 | 104 | 4,5 | −160 | 5,4 | 576 | 6,3 | 288 |
| 4,6 | 626 | 5,5 | −2128 | 6,4 | −2300 | 4,7 | 1628 |
| 5,6 | 6256 | 6,5 | 8280 | 7,4 | 100 | 8,3 | 200 |
| 4,8 | 259 | 5,7 | −1360 | 6,6 | −11300 | 7,5 | 4376 |
| 8,4 | 850 | 4,9 | 8536 | 6,7 | 10416 | 7,6 | −27148 |
| 8,5 | −16308 | 10,3 | 72 | 5,9 | 5856 | 6,8 | 28452 |
| 7,7 | 97464 | 8,6 | 86406 | 9,5 | 1728 | 10,4 | 1440 |
| 5,10 | 560 | 6,9 | 15392 | 7,8 | −137160 | 8,7 | −200648 |
| 9,6 | 24028 | 10,5 | −2364 | 6,10 | 50568 | 7,9 | 213704 |
| 8,8 | 259222 | 9,7 | −256656 | 10,6 | −87596 | 11,5 | 392 |
| 12,4 | 1078 | 6,11 | 17112 | 7,10 | 128932 | 8,9 | 172792 |
| 9,8 | 1240288 | 10,7 | 729888 | 11,6 | 19928 | 12,5 | 5488 |
| 6,12 | 1092 | 7,11 | 216152 | 8,10 | −282466 | 9,9 | −2903456 |
| 10,8 | −2564024 | 11,7 | 65680 | 12,6 | −55166 | 14,4 | 384 |
| 7,12 | 210468 | 8,11 | 1555840 | 9,10 | 5026084 | 10,9 | 4785740 |
| 11,8 | −1882952 | 12,7 | −289832 | 13,6 | 8900 | 14,5 | 8576 |
| 7,13 | 43152 | 8,12 | 981880 | 9,11 | −2601680 | 10,10 | −1431332 |
| 11,9 | 12792400 | 12,8 | 5065695 | 13,7 | 143904 | 14,6 | −8104 |
| 16,4 | 81 | 7,14 | 1968 | 8,13 | 1404308 | 9,12 | 3905488 |
| 10,11 | −7335936 | 11,10 | −43165120 | 12,9 | −25242080 | 13,8 | −414684 |
| 14,7 | −432148 | 15,6 | 2268 | 16,5 | 5832 | 7,15 | 7092736 |
| 8,14 | 35470520 | 9,13 | 46834296 | 10,12 | 12209208 | 11,11 | 53215208 |
| 12,10 | 51602088 | 13,9 | −12136984 | 14,8 | −1021124 | 15,7 | 114608 |
| 16,6 | 37250 | 8,15 | 97416 | 9,14 | 7376136 | 10,13 | −12267944 |
| 11,12 | −114028180 | 12,11 | −72783160 | 13,10 | 109160104 | 14,9 | 27983744 |
| 15,8 | 574544 | 16,7 | −319812 | 17,6 | 400 | 18,5 | 3000 |
| 8,16 | 3333 | 9,15 | 6466912 | 10,14 | 48215780 | 11,13 | 106268896 |
| 12,12 | −65961853 | 13,11 | −504320608 | 14,10 | −210794544 | 15,9 | −7512968 |
| 16,8 | −1904178 | 17,7 | 54608 | 18,6 | 44500 | 9,16 | 2017056 |
| 10,15 | 52559808 | 11,14 | 86278308 | 12,13 | 506072141 | 13,12 | 1463559527 |
| 14,11 | 768993890 | 15,10 | −30879880 | 16,9 | 14560120 | 17,8 | 899488 |
| 18,7 | −1452 | 20,5 | 968 | 9,17 | 201696 | 10,16 | 40996948 |
| 11,15 | 39152672 | 12,14 | −776591430 | 13,13 | −2610404376 | 14,12 | −1374685066 |
| 15,11 | 763348848 | 16,10 | 107041820 | 17,9 | −1184072 | 18,8 | −2861950 |
| 19,7 | 21480 | 20,6 | 37510 | | | | |

Oitmaa[7] calculated the high-temperature series expansion of zero-field susceptibility up to 11th order. His result was extended to two more orders by Soehianie and Oitmaa.[8] From analysis of the high-temperature susceptibility series, Oitmma obtained precise estimates of the ferromagnetic critical temperature as a function of $R = J_2/J_1$.

We have applied the method of Oitmaa to analysis the low-temperature series and to estimate the critical temperature and exponents. The special case of $R = 0.5$ has been discussed in detail by Oitmaa and he obtained $K_c = J_1/kT_c = 0.2628$. We shall consider the same case first so that we can compare our results with

Table 3. Lattice constants $c(n,r)$ for $n \leq 26$.

| $n,r$ | $c(n,r)$ | $n,r$ | $c(n,r)$ | $n,r$ | $c(n,r)$ | $n,r$ | $c(n,r)$ |
|---|---|---|---|---|---|---|---|
| 4,1 | 1 | 18,11 | 30 | 22,12 | 3386 | 25,9 | 119908 |
| 7,2 | 4 | 18,12 | 1 | 22,13 | 1900 | 25,10 | 115816 |
| 9,3 | 4 | 19,6 | 1670 | 22,14 | 924 | 25,11 | 95416 |
| 10,3 | 16 | 19,7 | 1884 | 22,15 | 390 | 25,12 | 72010 |
| 10,4 | 1 | 19,8 | 1250 | 22,16 | 129 | 25,13 | 51096 |
| 11,4 | 8 | 19,9 | 752 | 22,17 | 14 | 25,14 | 34550 |
| 12,4 | 29 | 19,10 | 394 | 23,8 | 31306 | 25,15 | 22660 |
| 12,5 | 9 | 19,11 | 152 | 23,9 | 27616 | 25,16 | 13540 |
| 13,4 | 72 | 19,12 | 54 | 23,10 | 21152 | 25,17 | 7604 |
| 13,5 | 16 | 19,13 | 4 | 23,11 | 15424 | 25,18 | 3814 |
| 13,6 | 6 | 20,7 | 4512 | 23,12 | 10452 | 25,19 | 1732 |
| 14,5 | 80 | 20,8 | 2985 | 23,13 | 6568 | 25,20 | 666 |
| 14,6 | 28 | 20,9 | 1820 | 23,14 | 3646 | 25,21 | 152 |
| 14,7 | 6 | 20,10 | 1128 | 23,15 | 1916 | 25,22 | 16 |
| 15,5 | 192 | 20,11 | 634 | 23,16 | 816 | 26,9 | 211576 |
| 15,6 | 80 | 20,12 | 268 | 23,17 | 320 | 26,10 | 241918 |
| 15,7 | 32 | 20,13 | 105 | 23,18 | 66 | 26,11 | 218188 |
| 15,8 | 6 | 20,14 | 14 | 23,19 | 4 | 26,12 | 177438 |
| 16,5 | 341 | 21,7 | 7552 | 24,8 | 45969 | 26,13 | 135664 |
| 16,6 | 204 | 21,8 | 7344 | 24,9 | 61946 | 26,14 | 97854 |
| 16,7 | 116 | 21,9 | 5380 | 24,10 | 53700 | 26,15 | 67012 |
| 16,8 | 52 | 21,10 | 3324 | 24,11 | 39172 | 26,16 | 43912 |
| 16,9 | 9 | 21,11 | 2012 | 24,12 | 27285 | 26,17 | 27142 |
| 17,6 | 624 | 21,12 | 1118 | 24,13 | 18248 | 26,18 | 15670 |
| 17,7 | 288 | 21,13 | 500 | 24,14 | 11874 | 26,19 | 8340 |
| 17,8 | 132 | 21,14 | 210 | 24,15 | 6772 | 26,20 | 3960 |
| 17,9 | 68 | 21,15 | 48 | 24,16 | 3738 | 26,21 | 1772 |
| 17,10 | 16 | 21,16 | 2 | 24,17 | 1734 | 26,22 | 570 |
| 18,6 | 1220 | 22,7 | 8382 | 24,18 | 738 | 26,23 | 116 |
| 18,7 | 820 | 22,8 | 15551 | 24,19 | 222 | 26,24 | 9 |
| 18,8 | 482 | 22,9 | 12538 | 24,20 | 30 | | |
| 18,9 | 224 | 22,10 | 8826 | 24,21 | 1 | | |
| 18,10 | 94 | 22,11 | 5620 | 25,8 | 42864 | | |

his result. The case $R = 0.5$ means $x = y^2$. The estimated critical temperature from low-temperature series should be the same as (or very close to) the estimated temperature from high-temperature series. We have

$$\begin{aligned} M(y^2, y) = 1 &- 2y^6 - 8y^{10} - 8y^{11} + 18y^{12} - 24y^{13} - 20y^{14} - 48y^{15} + 44y^{16} \\ &+ 24y^{17} - 404y^{18} + 144y^{19} - 348y^{20} + 536y^{21} - 1614y^{22} - 2432y^{23} \\ &+ 3670y^{24} - 5840y^{25} + 2948y^{26} - 28024y^{27} + 14488y^{28} + 16120y^{29} \\ &- 103872y^{30} + 30448y^{31} - 239340y^{32} + 406984y^{33} + 646008y^{34} \\ &- 2127272y^{35} - 7135696y^{36} - 11133024y^{37} + 6971120y^{38} \\ &- 3121456y^{39} + 11078112y^{40} + \cdots . \end{aligned} \tag{7}$$

Table 4. Lattice constants $c(n, r)$ for connected graphs with $26 \leq n \leq 30$.

| $r$ | $c(26, r)$ | $c(27, r)$ | $c(28, r)$ | $c(29, r)$ | $c(30, r)$ |
|---|---|---|---|---|---|
| 9 | 211576 | 276724 | 222421 | | |
| 10 | 241918 | 496890 | 886380 | 1402474 | 1653338 |
| 11 | 218188 | 499500 | 1021922 | 2022792 | 3843400 |
| 12 | 177438 | 421536 | 958600 | 2092768 | 4405280 |
| 13 | 135664 | 332960 | 812639 | 1888458 | 4182618 |
| 14 | 97854 | 252248 | 647780 | 1586980 | 3655386 |
| 15 | 67012 | 185140 | 493114 | 1266500 | 2977250 |
| 16 | 43912 | 129730 | 364828 | 968132 | 2493596 |
| 17 | 27142 | 86640 | 259347 | 719640 | 1935270 |
| 18 | 15670 | 55162 | 176044 | 517250 | 1458870 |
| 19 | 8340 | 32868 | 114942 | 356996 | 1064002 |
| 20 | 3960 | 18596 | 70421 | 237528 | 750562 |
| 21 | 1772 | 9252 | 41676 | 150452 | 507712 |
| 22 | 570 | 4476 | 22098 | 91870 | 330886 |
| 23 | 116 | 1772 | 11250 | 51920 | 207348 |
| 24 | 9 | 530 | 5098 | 27640 | 123432 |
| 25 | | 92 | 1926 | 13784 | 68576 |
| 26 | | 6 | 536 | 6014 | 36560 |
| 27 | | | 84 | 2228 | 17598 |
| 28 | | | 6 | 574 | 7734 |
| 29 | | | | 92 | 2728 |
| 30 | | | | 6 | 718 |
| 31 | | | | | 116 |
| 32 | | | | | 9 |

Table 5. Estimates of the critical point $y_c = \exp(-2J_1/kT)$ from poles of the $[n+j, n]$ Padé approximants to the $(d/dy)\log M(y^2, y)$ series. In brackets are shown corresponding estimates of the exponent $\beta$ from Padé approximants to $(y - y_c)(d/dy)\log M(y^2, y)$ evaluating at $y = y_c = 0.59146$.

| $n$ | $j = -1$ | $j = 0$ | $j = 1$ |
|---|---|---|---|
| 15 | 0.59144 (0.127) | 0.59148 (0.127) | 0.59147 (0.123) |
| 16 | 0.59148 (0.127) | 0.59148 (0.126) | 0.59147 (0.123) |
| 17 | 0.59147 (0.127) | 0.59147 (0.126) | 0.59144 (0.125) |
| 18 | 0.59145 (0.126) | 0.59147 (0.126) | 0.59143 (0.126) |

The first step is to obtain an estimate for $K_c$ from the pole of Padé approximants to the series $d(\log M(y^2, y))/dy$ such that $K_c = -0.5\log y_c$. In Table 5, we show estimates of $y_c$ and find $y_c = 0.59146$. Therefore our estimate for $K_c$ is $-0.5\log 0.59146 = 0.2626$ which is very close to 0.2628. The next step is to make biased estimate of the exponent $\beta$ by forming Padé approximants to $(y - y_c)d(\log M(y^2, y))/dy$ and evaluating these at $y = y_c = 0.59146$. The results are shown in Table 5 and the conclusion is $\beta = 0.125 \pm 0.002$.

In Fig. 2 of Ref. 7, quantitative estimates from the high-temperature series of the critical temperature versus the parameter $R$ are shown. The estimated critical

Table 6. The estimated critical temperature $kT_c/J_1$ for various values of $R$. In brackets are shown corresponding estimates from high-temperature series expansion derived by Oitmaa.

| $R$ | $kT_c/J_1$ | $R$ | $kT_c/J_1$ | $R$ | $kT_c/J_1$ |
|---|---|---|---|---|---|
| 0 | 2.269 (exact) | 1/2 | 3.808 (3.81) | −0.1 | 1.944 (1.94) |
| 1 | 5.256 (5.26) | 1/3 | 3.305 (3.31) | −0.2 | 1.608 (1.61) |
| 2 | 8.012 (8.01) | 1/4 | 3.052 (3.05) | −0.3 | 1.251 (1.25) |
| 3 | 10.692 (10.7) | 1/5 | 2.912 (2.91) | | |
| 4 | 13.355 (13.4) | | | | |
| 5 | 15.886 (15.9) | | | | |



temperatures $kT_c/J_1$ from the low-temperature ferromagnetic series for different values of $R$ are given in Table 6, where the critical temperature for $R = 0$ is exactly known.[9] The corresponding estimates from high-temperature series expansion derived by Oitmaa are shown in brackets and the agreement is remarkable. Our estimates from low-temperature magnetization series fit his curve very well.

The critical temperature can be estimated also from the low-temperature susceptibility series. However the results are not satisfactory, because the series is relatively short.

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## References

1. L. Onsager, *Phys. Rev.* **65**, 117 (1944).
2. C. N. Yang, *Phys. Rev.* **85**, 808 (1952).
3. B. M. McCoy and T. T. Wu, *The Two-Dimensional Ising Model* (Boston, Havard University Press, 1973)
4. N. W. Dalton and D. W. Wood, *J. Math. Phys.* **10**, 1271 (1969).
5. S. F. Lee and K. Y. Lin, *Chin. J. Phys.* **34**, 1261 (1996).
6. C. Domb, *Adv. Phys.* **9**, 149 (1960).
7. J. Oitmaa, *J. Phys. A: Math. Gen.* **14**, 1159 (1981).
8. A. Soehianie and J. Oitmaa, *Mod. Phys. Lett.* **B11**, 609 (1997).
9. H. A. Kramers and G. H. Wannier, *Phys. Rev.* **60**, 252 (1941).