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Ring embedding in faulty honeycomb rectangular torus ☆

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Abstract

Assume that m and n are positive even integers with $n \geqslant 4$. The honeycomb rectangular torus $\mathrm{HReT}(m,n)$ is recognized as another attractive alternative to existing torus interconnection networks in parallel and distributed applications. It is known that any $\mathrm{HReT}(m,n)$ is a 3-regular bipartite graph. We prove that any $\mathrm{HReT}(m,n) - e$ is hamiltonian for any edge $e \in E(\mathrm{HReT}(m,n))$. Moreover, any $\mathrm{HReT}(m,n) - F$ is hamiltonian for any $F = \{a,b\}$ with $a \in A$ and $b \in B$ where A and B are the bipartition of $\mathrm{HReT}(m,n)$, if $n \geqslant 6$ or $m = 2$.

Keywords: Interconnection networks; Honeycomb torus; Hamiltonian cycle; Ring embedding

1. Introduction

Network topology is a crucial factor for interconnection networks since it determines the performance of the network. Many interconnection network topologies have been proposed in the literature for the purpose of connecting a large number of processing elements [6]. Network topology is always represented by a graph where nodes represent processors and edges represent links between processors. One of the most popular architectures is the mesh connected computers [6]. Each processor is placed in a square or rectangular grid and is connected by a communication link to its neighbors up to four directions.

It is well known that there are three possible tessellations of a plane with regular polygons of the same kind: square, triangular, and hexagonal, corresponding to the division of a plane into regular squares, triangles, and hexagons, respectively. Based on this observation, some computer and communication networks have been built. The square tessellation is the basis for mesh-connected computers. The triangle tessellation is the basis for defining hexagonal mesh multiprocessors [3,11]. The hexagonal tessellation is the basis for defining the honeycomb meshes [2,10].

Tori are meshes with wraparound connections to achieve vertex and edge symmetry. Meshes and tori are among the most frequent multiprocessor networks available on the market. Stojmenovic [10] also introduced honeycomb

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tori by adding wraparound edges on honeycomb meshes. Recently, the honeycomb torus has been recognized as an attractive alternative to existing torus interconnection networks in parallel and distributed applications. Thus, there are a lot of studies on topological properties of honeycomb torus [7,8,10]. The hamiltonian properties constitute one of the major requirements in designing the topology of networks. For example, the "Token Passing" approach is used in some distributed systems. Interconnection network requires the presence of hamiltonian cycle in the structure to meet the "token ring" requirement. Fault tolerance is also desired in massive parallel systems that have a relative high probability of failure. The hamiltonian properties of the honeycomb tori were studied in [7,8]. It is proved that all the honeycomb tori are hamiltonian [7]. Moreover, there exists a hamiltonian cycle in any honeycomb torus with two adjacent faulty nodes [8]. Recently, Cho and Hsu [5] have generalized the honeycomb torus into generalized honeycomb torus, and some generalized honeycomb tori are proved to be hamiltonian.

Throughout this paper, we assume that m and n are positive even integers with $n \geqslant 4$. The honeycomb rectangular torus $\mathrm{HReT}(m,n)$ is another attractive alternative to existing torus, which is also introduced by Stojmenovic [10]. Its topological properties and its generalization are studied by Parhami and Kwai [9]. Some applications of the honeycomb rectangular torus are studied in [4]. In this paper, we study the hamiltonian properties of honeycomb rectangular torus. We will prove that the honeycomb rectangular torus $\mathrm{HReT}(m,n)$ is hamiltonian. Moreover, any $\mathrm{HReT}(m,n)$ remains hamiltonian when any edge is faulty. The honeycomb rectangular torus we proposed is a bipartite graph with bipartition A and B. Thus, any cycle of it contains the same number of vertices in each part. For this observation, we will prove that any $\mathrm{HReT}(m,n) - F$, with $n > 4$ or $m = 2$, remains hamiltonian for any $F = \{a, b\}$ with $a \in A$ and $b \in B$.

In the following section, we give some graph terms that are used in this paper and a formal definition of honeycomb rectangular torus. In Section 3, we present a recursive property of the ring embeddings in $\mathrm{HReT}(m,n)$. In Section 4, we discuss the ring embedding properties of $\mathrm{HReT}(2,n)$. With the recursive property presented in Section 3, we can prove that any $\mathrm{HReT}(m,n)$ remains hamiltonian when any edge is faulty. In Section 5, we discuss the ring embedding property of $\mathrm{HReT}(4,n) - F$ for any $F = \{a, b\}$ with $a \in A$ and $b \in B$. In the final section, we discuss the ring embedding properties of any $\mathrm{HReT}(m,n) - F$ where $F = \{a, b\}$ with $a \in A$ and $b \in B$.

2. Honeycomb rectangular torus

Usually, computer and communication networks are represented by graphs where nodes represent processors and edges represent links between processors. In this paper, a network is represented as an undirected graph. For the graph definition and notation we follow [1]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(a, b) \mid (a, b) \text{ is an unordered pair of } V\}$. We say that V is the *node set* and E is the *edge set* of G. Two nodes a and b are *adjacent* if $(a, b) \in E$. A *path* is a sequence of nodes such that two consecutive nodes are adjacent. A path is delimited by $\langle x_0, x_1, x_2, \ldots, x_{n-1}\rangle$. We use P^{-1} to denote the path $\langle x_{n-1}, \ldots, x_2, x_1, x_0\rangle$ if P is the path $\langle x_0, x_1, x_2, \ldots, x_{n-1}\rangle$. A path is called a *hamiltonian path* if its nodes are distinct and span V. A *cycle* is a path of at least three nodes such that the first node is the same as the last node. A cycle is called a *hamiltonian cycle* if its nodes are distinct except for the first node and the last node and if they span V. A graph is called *hamiltonian* if it has a hamiltonian cycle. A graph $G = (V, E)$ is 1-*edge hamiltonian* if $G - e$ is hamiltonian for any $e \in E$. A hamiltonian bipartite graph G is 1_p-*hamiltonian* if $G - F$ remains hamiltonian for any $F = \{a, b\}$ with $a \in A$ and $b \in B$ where A and B are the bipartition of G.

For any two positive integers r and s, we use $[r]_s$ to denote $r \pmod s$. We use the brick drawing, proposed in [10], to define the honeycomb rectangular torus. The honeycomb rectangular torus $\mathrm{HReT}(m,n)$ is the graph with the vertex set $\{(i, j) \mid 0 \leqslant i < m,\ 0 \leqslant j < n\}$ such that (i, j) and (k, l) are adjacent if they satisfy one of the following conditions:

(1) $i = k$ and $j = [l \pm 1]_n$;
(2) $j = l$ and $k = [i - 1]_m$ if $i + j$ is even; and
(3) $j = l$ and $k = [i + 1]_m$ if $i + j$ is odd.

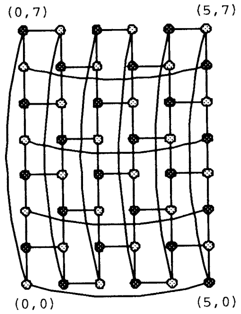

Fig. 1. The graph HReT(6, 8).

For example, the graph HReT(6, 8) is shown in Fig. 1. From the illustration, it is easy to see that HReT(m,n) is a subgraph of the torus $T(m,n)$ [6]. Obviously, any honeycomb rectangular torus is a 3-regular bipartite graph. We set A as $\{(i,j) \mid (i,j) \in V(\mathrm{HReT}(m,n))$, and $i+j$ is even$\}$ and set B as $\{(i,j) \mid (i,j) \in V(\mathrm{HReT}(m,n))$, and $i+j$ is odd$\}$. Moreover, any honeycomb rectangular torus is vertex transitive. The recursive structure of HReT(m,n) can easily be observed by inserting a pair of rows and/or a pair of columns.

Since the honeycomb rectangular torus is a bipartite graph, any spanning cycle of it contains the same number of vertices in each part. We will prove that any HReT(m,n) is 1-edge hamiltonian. Moreover, HReT(m,n) is 1_p-hamiltonian if and only if $n > 4$ or $m = 2$.

To discuss the 1_p-hamiltonian property of HReT(m,n), let $F = \{a,b\}$ with $a \in A$ and $b \in B$. We may assume that $(0,0) \in F$ because HReT(m,n) is vertex transitive. For this reason, we use $\mathcal{F}(m,n)$ to denote $\{F \mid F = \{(0,0),(x,y)\} \mid (x,y) \in B\}$. We use (x,y) to denote the unique element in $F - \{(0,0)\}$. By the assumption, $x+y$ is odd. We use $P(i,j,k)$ to denote the path $\langle (i,j), (i,[j+1]_n)), (i,[j+2]_n)), \ldots, (i,k) \rangle$ and use $Q(i,k,j)$ to denote the path $P^{-1}(i,j,k)$.

3. A recursive property

In this section, we use F' to denote a subset of $V(\mathrm{HReT}(m,n)) \cup E(\mathrm{HReT}(m,n))$. We will present a recursive algorithm to obtain hamiltonian cycle of HReT$(m,n) - F'$.

Assume that $0 \leqslant i < m$. We define a function from the vertex set of HReT(m,n) into the vertex set of HReT$(m+2,n)$ by assigning $f_i((k,l)) = (k,l)$ if $k \leqslant i$ and $f_i((k,l)) = (k+2,l)$ otherwise. We define $f_i(F')$ to be the set

$$\begin{aligned}
&\bigl\{ f_i(k,l) \mid (k,l) \in V\bigl(\mathrm{HReT}(m,n)\bigr) \cap F' \bigr\} \\
&\quad \cup \bigl\{ \bigl(f_i(k,l), f_i(k',l')\bigr) \mid \bigl((k,l),(k',l')\bigr) \in E\bigl(\mathrm{HReT}(m,n)\bigr) \cap F' \text{ with } \{k,k'\} \neq \bigl\{i,[i+1]_m\bigr\}\bigr\} \\
&\quad \cup \bigl\{ \bigl((i,l),(i+1,l)\bigr) \mid \bigl((i,l),([i+1]_m,l)\bigr) \in E\bigl(\mathrm{HReT}(m,n)\bigr) \cap F' \bigr\}.
\end{aligned}$$

Let H be a hamiltonian cycle of HReT$(m,n) - F'$ such that there are some edges of H joining vertices of column i to vertices of column $[i+1]_m$); i.e., $((i,j),([i+1]_m,j) \in E(H)$ for some j. Now, we construct a hamiltonian cycle $f_i(H)$ of HReT$(m+2,n) - f_i(F')$ as follows:

Let $0 \leqslant k_0 < k_1 < \cdots < k_{t-1} \leqslant n-1$ be the indices such that $((i,k_j),([i+1]_m,k_j))$ is an edge of H. Let $\overline{H_i}$ be the image of $H - \{(i,j),([i+1]_m,j)) \mid 0 \leqslant j < n\}$ under f_i. For $0 \leqslant j < t$, we set Q_j as the path

$$\begin{aligned}
&\bigl\langle (i,k_j), \bigl([i+1]_{m+2},k_j\bigr) \xrightarrow{P([i+1]_{m+2},k_j,[k_{[j+1]_t}-1]_n)} \bigl([i+1]_{m+2},[k_{[j+1]_t}-1]_n\bigr), \\
&\quad \bigl([i+2]_{m+2},[k_{[j+1]_t}-1]_n\bigr) \xrightarrow{Q([i+2]_{m+2},[k_{[j+1]_t}-1]_n,k_j)} \bigl([i+2]_{m+2},k_j\bigr), \bigl([i+3]_{m+2},k_j\bigr) \bigr\rangle.
\end{aligned}$$

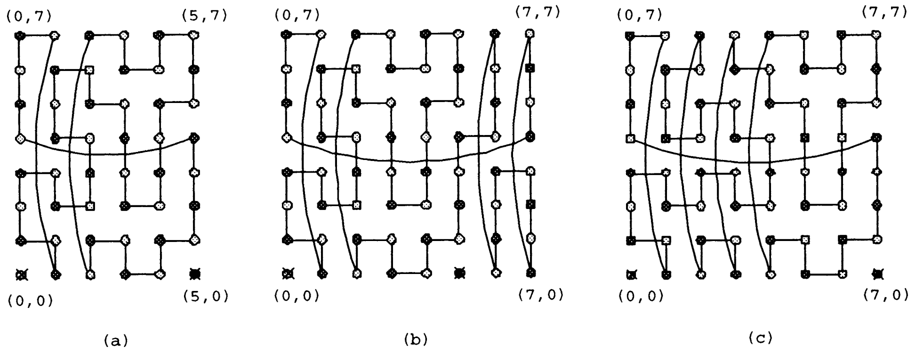

Fig. 2. (a) A hamiltonian cycle H in $\mathrm{HReT}(6,8) - \{(0,0),(5,0)\}$, (b) $f_5(H)$, and (c) $f_1(H)$.

Obviously, Q_j is a path joining (i,k_j) and $([i+3]_{m+2},k_j)$ for $0 \leqslant j < t$. It is easy to see that edges of $\overline{H_i}$ together with edges of Q_j, with $0 \leqslant j < t$, form a hamiltonian cycle of $\mathrm{HReT}(m+2,n) - f_i(F')$. We denote this cycle as $f_i(H)$. For example, a hamiltonian cycle H of $\mathrm{HReT}(6,8) - \{(0,0),(5,0)\}$ is shown in Fig. 2(a). The corresponding $f_5(H)$ and $f_1(H)$ are shown in Figs. 2(b) and 2(c). We have following lemmas.

Lemma 1. *Assume that $0 \leqslant i < m$. Let H be a hamiltonian cycle of* $\mathrm{HReT}(m,n) - F'$ *such that there are some edges of H joining vertices of column i to vertices of column $[i+1]_m$. Then, $f_i(H)$ is a hamiltonian cycle of* $\mathrm{HReT}(m+2,n) - f_i(F')$. *Moreover, $f_i(H)$ contains some edges joining column t to column $[t+1]_m + 2$ for any t in* $\{i,[i+1]_m,[i+2]_m\}$.

Lemma 2.

(1) *Suppose that H is a hamiltonian cycle of* $\mathrm{HReT}(2,n) - F'$ *such that H contains some edges in* $\{((0,j),(1,j)) \mid j \text{ is odd}\}$. *Then $f_0(H)$ is a hamiltonian cycle of* $\mathrm{HReT}(4,n) - f_0(F')$. *Moreover, $f_0(H)$ contains some edges joining column t to column $t+1$ for any t in* $\{0,1,2\}$.
(2) *Suppose that H is a hamiltonian cycle of* $\mathrm{HReT}(2,n) - F'$ *such that H contains some edges in* $\{((0,j),(1,j)) \mid j \text{ is even}\}$. *Then $f_1(H)$ is a hamiltonian cycle of* $\mathrm{HReT}(4,n) - f_1(F')$. *Moreover, $f_1(H)$ contains some edges joining column t to column $t+1$ for any t in* $\{1,2,3\}$.

We say a hamiltonian cycle of $\mathrm{HReT}(2,n) - F'$ is *regular* if H contains some edges in $\{((0,j),(1,j)) \mid j \text{ is odd}\}$ and some edges in $\{((0,j),(1,j)) \mid j \text{ is even}\}$. Assume that $m \geqslant 4$. A hamiltonian cycle H of $\mathrm{HReT}(m,n) - F'$ is *regular* if H contains some edges joining column i to column $[i+1]_m$ for $0 \leqslant i < m$. The following lemma is derived from the above two lemmas.

Lemma 3. *Suppose that H is a regular hamiltonian cycle for* $\mathrm{HReT}(m,n) - F'$. *Then $f_i(H)$ is a regular hamiltonian cycle of* $\mathrm{HReT}(m+2,n) - f_i(F')$ *for every* $0 \leqslant i < m$.

4. Hamiltonian properties of HReT$(2,n)$

Obviously,

$$\langle (0,0),(1,0),(1,1),(0,1),(0,2),\ldots,(0,n-2),(1,n-2),(1,n-1),(0,n-1),(0,0)\rangle$$

and

$$\langle (0,0) \xrightarrow{P(0,0,n-1)} (0,n-1),(1,n-1) \xrightarrow{Q(1,n-1,0)} (1,0),(0,0)\rangle$$

are regular hamiltonian cycles of $\mathrm{HReT}(2,n)$. With these two hamiltonian cycles and the symmetric property of $\mathrm{HReT}(2,n)$, $\mathrm{HReT}(2,n)$ is 1-edge hamiltonian.

Now, we discuss the 1_p-hamiltonian property of $\mathrm{HReT}(2,n)$. Assume that $F \in \mathcal{F}(2,n)$ and that (x,y) is the unique element in $F - \{(0,0)\}$.

Suppose that $x = 0$. Then

$$\big\langle (0,1), (0,2), (1,2), (1,3), \ldots, (0,y-1), (1,y-1), (1,y), (1,y+1), (0,y+1), \ldots,$$
$$(0,n-3), (1,n-3), (1,n-2), (0,n-2), (0,n-1), (1,n-1), (1,0), (1,1), (0,1) \big\rangle$$

forms a hamiltonian cycle of $\mathrm{HReT}(2,n) - F$.

Suppose that $x = 1$. Then

$$\big\langle (0,1), (0,2), (1,2), (1,3), \ldots, (1,y-1), (0,y-1), (0,y), (0,y+1), (1,y+1), \ldots,$$
$$(0,n-3), (1,n-3), (1,n-2), (0,n-2), (0,n-1), (1,n-1), (1,0), (1,1), (0,1) \big\rangle$$

forms a hamiltonian cycle of $\mathrm{HReT}(2,n) - F$.

Lemma 4. $\mathrm{HReT}(2,n)$ *is* 1*-edge hamiltonian and* 1_p*-hamiltonian. Moreover, there exists a regular hamiltonian cycle in* $\mathrm{HReT}(2,n) - e$ *for any* $e \in E(\mathrm{HReT}(2,n))$*. Furthermore, there exists a regular hamiltonian cycle in* $\mathrm{HReT}(2,n) - F$ *for any* $F \in \mathcal{F}(2,n)$ *with* $F \neq \{(0,0),(1,0)\}$.

5. 1_p-hamiltonian property of $\mathrm{HReT}(4,n)$

We first consider the case $\mathrm{HReT}(4,4)$. Suppose that $F = \{(0,0),(1,0)\}$. Obviously, $\deg_{G-F}(v) = 2$ if $v \in \{(0,1),(0,3),(1,1),(1,3)\}$. For this reason, any hamiltonian cycle of $\mathrm{HReT}(4,4) - F$ must include the following edge set:

$$\big\{\big((0,1),(0,2)\big), \big((0,2),(0,3)\big), \big((0,3),(1,3)\big), \big((1,3),(1,2)\big), \big((1,2),(1,1)\big), \big((1,1),(0,1)\big)\big\}.$$

However, this edge set induces a cycle of length 6. Thus, $\mathrm{HReT}(4,4) - F$ is not hamiltonian.

In the following, we will prove that every $\mathrm{HReT}(4,n)$ with $n \geqslant 6$ is 1_p-hamiltonian. Assume t is an integer with $0 \leqslant t < (\frac{1}{4}n - 1)$. For $0 \leqslant i \leqslant 2t$, let D_i denote the path $\langle (3,2i), (3,2i+1), (2,2i+1), (2,2i+2), (1,2i+2), (1,2i+3), (0,2i+3), (0,2i+4) \rangle$.

We set R_t as the path

$$\big\langle (3,0) \xrightarrow{D_0} (0,4), (3,4) \xrightarrow{D_2} (0,8), (3,8) \cdots \xrightarrow{D_{2t-2}} (0,4t) \big\rangle$$

and set S_t as the path

$$\big\langle (3,2) \xrightarrow{D_1} (0,6), (3,6) \xrightarrow{D_3} (0,10), (3,10) \cdots \xrightarrow{D_{2t-1}} (0,4t+2) \big\rangle.$$

Let $F \in \mathcal{F}(4,n)$ and let (x,y) be the unique element in $F - \{(0,0)\}$.

Case 1: $x = 0$. By Lemma 4, there exists a hamiltonian cycle H of $\mathrm{HReT}(2,n) - F$. By Lemma 2, $f_0(H)$ is a regular hamiltonian cycle of $\mathrm{HReT}(4,n) - F$.

Case 2: $x = 1$. Assume that $(x,y) \neq (1,0)$. By Lemma 4, there exists a regular hamiltonian cycle H of $\mathrm{HReT}(2,n) - F$. By Lemma 2, $f_1(H)$ is a regular hamiltonian cycle of $\mathrm{HReT}(4,n) - F$. Suppose that $(x,y) = (1,0)$. It can be checked that

$$\big\langle (0,1), (0,2), (0,3), (1,3) \xrightarrow{P(1,3,n-1)} (1,n-1), (0,n-1) \xrightarrow{Q(0,n-1,4)} (0,4),$$
$$(3,4) \xrightarrow{P(3,4,3)} (3,3), (2,3) \xrightarrow{P(2,3,2)} (2,2), (1,2), (1,1), (0,1) \big\rangle$$

forms a regular hamiltonian cycle of $\mathrm{HReT}(4,n) - F$. See Fig. 3(a) for illustration.

Case 3: $x = 2$. By the symmetric property of HReT$(4, n)$, we may assume that $1 \leqslant y \leqslant \frac{1}{2}n$. Since $x + y$ is odd, y is odd.

Subcase 3.1: $y = 1$. It can be checked that

$$\langle (0,1), (0,2), (3,2) \xrightarrow{Q(3,2,3)} (3,3), (2,3), (2,2), (1,2), (1,3), (0,3) \xrightarrow{P(0,3,n-1)} (0, n-1),$$
$$(1, n-1) \xrightarrow{Q(1,n-1,4)} (1,4), (2,4) \xrightarrow{P(2,4,0)} (2,0), (1,0), (1,1), (0,1) \rangle$$

forms a regular hamiltonian cycle of HReT$(4, n) - F$. See Fig. 3(b) for illustration.

Subcase 3.2: $y = 4t + 1$ for some positive integer t. Then the path

$$\langle (3,0) \xrightarrow{R_t} (0,4t), (3,4t)(3,4t+1), (3,4t+2), (0,4t+2) \xrightarrow{S_t^{-1}} (3,2), (0,2),$$
$$(0,1), (1,1), (1,0), (2,0) \xrightarrow{Q(2,0,4t+4)} (2,4t+4), (1,4t+4) \xrightarrow{P(1,4t+4,n-1)} (1, n-1),$$
$$(0, n-1) \xrightarrow{Q(0,n-1,4t+3)} (0,4t+3), (1,4t+3), (1,4t+2), (2,4t+2),$$
$$(2,4t+3), (3,4t+3) \xrightarrow{P(3,4t+3,0)} (3,0) \rangle$$

forms a hamiltonian cycle of HReT$(4, n) - F$. See Fig. 3(c) for illustration.

Subcase 3.3: $y = 4t + 3$ for some nonnegative integer t. Then the path

$$\langle (3,0) \xrightarrow{R_t} (0,4t) \xrightarrow{D_{2t}} (0,4t+4) \xrightarrow{P(0,4t+4,n-1)} (0, n-1),$$
$$(1, n-1) \xrightarrow{Q(1,n-1,4t+4)} (1,4t+4), (2,4t+4) \xrightarrow{P(2,4t+4,0)} (2,0), (1,0), (1,1),$$
$$(0,1), (0,2), (3,2) \xrightarrow{S_t} (0,4t+2), (3,4t+2) \xrightarrow{P(3,4t+2,0)} (3,0) \rangle$$

forms a hamiltonian cycle of HReT$(4, n) - F$. See Fig. 3(d) for illustration.

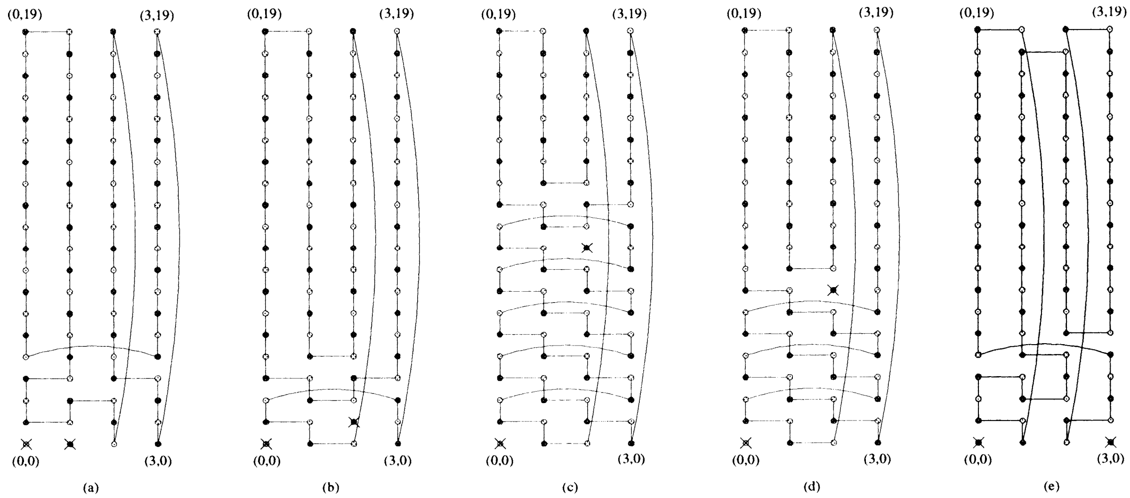

Fig. 3. (a) A hamiltonian cycle H in HReT$(4,20) - \{(0,0), (1,0)\}$, (b) a hamiltonian cycle H in HReT$(4,20) - \{(0,0), (2,1)\}$, (c) a hamiltonian cycle H in HReT$(4,20) - \{(0,0), (2,9)\}$, (d) a hamiltonian cycle H in HReT$(4,20) - \{(0,0), (2,7)\}$, and (e) a hamiltonian cycle H in HReT$(4,20) - \{(0,0), (3,0)\}$.

Case 4: $x = 3$. Assume that $(x, y) \neq (3, 0)$. By Lemma 4, there exists a hamiltonian cycle H of HReT$(2, n) - \{(0, 0), (1, y)\}$. By Lemma 2, $f_0(H)$ is a regular hamiltonian cycle of HReT$(4, n) - F$. Assume that $(x, y) = (3, 0)$. Suppose that $n \geqslant 8$. It can be checked that

$$\begin{aligned}&\big\langle(0,1),(0,2),(0,3),(1,3),(1,2),(2,2),(2,3),(2,4),(1,4)\xrightarrow{P(1,4,n-2)}(1,n-2),\\&(2,n-2)\xrightarrow{Q(2,n-2,5)}(2,5),(3,5)\xrightarrow{P(3,5,n-1)}(3,n-1),(2,n-1),(2,0),(2,1),\\&(3,1)\xrightarrow{P(3,1,4)}(3,4),(0,4)\xrightarrow{P(0,4,n-1)}(0,n-1),(1,n-1),(1,0),(1,1)(0,1)\big\rangle\end{aligned}$$

forms a regular hamiltonian cycle of HReT$(4, n) - F$. See Fig. 3(e) for illustration.

Hence, we have the following lemma.

Lemma 5.

(1) HReT$(4, n)$ *is* 1_p*-hamiltonian if and only if* $n \geqslant 6$.
(2) *Suppose that* $n \geqslant 6$. *There exists a regular hamiltonian cycle in* HReT$(m, n) - F$ *for any* $F \in \mathcal{F}(4, n)$ *except the case that* $F = \{(0, 0), (3, 0)\}$ *and* $n = 6$.

6. Hamiltonian properties of HReT(m, n)

Theorem 1.

(1) *Any rectangular honeycomb torus* HReT(m, n) *is* 1*-edge hamiltonian.*
(2) HReT(m, n) *is* 1_p*-hamiltonian if and only if either* $n \geqslant 6$ *or* $m = 2$.
(3) *Assume that* $m \geqslant 4$, $n \geqslant 6$. *There exists a regular hamiltonian cycle in* HReT$(m, n) - F$ *for any* $F \in \mathcal{F}(m, n)$ *except the case that* $F = \{(0, 0), (m - 1, 0)\}$ *and* $n = 6$.

Proof. With Lemma 4, there exists a regular hamiltonian cycle in HReT$(2, n) - e$ for any $e \in E(\text{HReT}(2, n))$. Recursively applying Lemma 3, any rectangular honeycomb torus HReT(m, n) is 1-edge hamiltonian.

Now, we discuss the 1_p-hamiltonian property of HReT(m, n). Let $F \in \mathcal{F}(m, n)$ and (x, y) be the unique element in $F - \{(0, 0)\}$. By Lemma 4, HReT$(2, n)$ is 1_p-hamiltonian.

Now, we consider the case $n = 4$. Suppose that $F = \{(0, 0), (1, 0)\}$. Obviously, $\deg_{G-F}(v) = 2$ if $v \in \{(0, 1), (0, 3), (1, 1), (1, 3)\}$. Therefore, any hamiltonian cycle of HReT$(m, n) - F$ must include the following edge set:

$$\big\{\big((0,1),(0,2)\big),\big((0,2),(0,3)\big),\big((0,3),(1,3)\big),\big((1,3),(1,2)\big),\big((1,2),(1,1)\big),\big((1,1),(0,1)\big)\big\}.$$

However, this edge set induces a cycle of length 6. Thus, HReT$(m, n) - F$ is not hamiltonian. Hence, HReT(m, n) is not 1_p-hamiltonian if $m \geqslant 4$ and $n = 4$.

Now, we prove that HReT(m, n) is 1_p-hamiltonian if $n \geqslant 6$. We prove the statement by induction on m. With Lemma 5, our theorem holds for $m = 4$. Hence, we assume that the theorem holds for HReT(m', n) when m' is any even integer with $4 \leqslant m' < m$. Now, we consider the case that $m \geqslant 6$.

We first consider the case that $n \geqslant 8$. Suppose that $x < m - 2$. By induction, there exists a regular hamiltonian cycle H of HReT$(m - 2, n) - F$. By Lemma 2, $f_{m-1}(H)$ is a regular hamiltonian cycle of HReT$(m, n) - F$. Suppose that $x \geqslant m - 2$. By induction, there exists a regular hamiltonian cycle H of HReT$(m - 2, n) - \{(0, 0), (x - 2, y)\}$. By Lemma 2, $f_0(H)$ is a regular hamiltonian cycle of HReT$(m, n) - F$. Hence, the theorem holds for $n \geqslant 8$.

Now, we consider the case that $n = 6$. Suppose that (x, y) is neither $(m - 3, 0)$ nor $(m - 1, 0)$. By induction, there exists a regular hamiltonian cycle H of HReT$(m - 2, n) - F$. By Lemma 2, $f_{m-1}(H)$ is a regular hamiltonian cycle of $\mathrm{HReT}(m,n) - F$. Suppose that $(x, y) = (m - 3, 0)$. By induction, there exists a regular hamiltonian cycle H of $\mathrm{HReT}(m - 2, n) - \{(0, 0), (m - 5, 0)\}$. By Lemma 2, $f_0(H)$ is a regular hamiltonian cycle of $\mathrm{HReT}(m,n) - F$. Suppose that $(x, y) = (m - 1, 0)$. By induction, there exists a hamiltonian cycle H of $\mathrm{HReT}(m - 2, n) - \{(0, 0), (m - 3, 0)\}$. The hamiltonian cycle must contain some edges joining column i to column $i + 1$ for some i with $0 \leqslant i \leqslant m - 2$. By Lemma 2, $f_i(H)$ is a hamiltonian cycle of $\mathrm{HReT}(m,n) - F$.

The theorem is proved. □

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