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Generalization of Yang et al.'s method for fuzzy programming with piecewise linear membership functions

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Abstract

Li and Yu (Fuzzy Sets and Systems 101 (1999) 109) argued that the model of Yang et al. for fuzzy programming (Fuzzy Sets and Systems 41 (1991) 39) is correct only for a specific type of piecewise linear membership function and proposed their model for other type of membership functions. This study generalizes the model of Yang et al. A numerical example indicates that the model of Li and Yu is inapplicable to a fuzzy programming problem that involves more than one membership function, whereas the proposed model gives the optimal solution. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Fuzzy programming; Piecewise linear membership functions; Quasi-concave membership functions

1. Introduction

A fuzzy programming problem with n objectives is usually represented as follows.

$$\begin{aligned} &\text{Maximize } \lambda \\ &\text{subject to } \lambda \leqslant \mu_i((\mathbf{ax})_i) \quad \text{for } i = 1, 2, \ldots, n, \\ &\qquad \mathbf{x} \in F \text{ (a feasible set)}, \end{aligned} \tag{1}$$

where $\mu_i((\mathbf{ax})_i)$ is the membership function of the ith objective. The most widespread fuzzy sets, the ramp-type and the triangular fuzzy sets, are easily handled in fuzzy programming because of their simplicity. However, the decision-maker might feel that ramp-type and triangular fuzzy sets are too simple to represent fully his/her attitudes. Thus, Hannan [1] introduced interpolated piecewise linear fuzzy sets into fuzzy programming. The most important advantage of the interpolated membership function is that it produces a computationally tractable membership function that closely describes the real structure of a decision-maker's subjective concept of goals or constraints. Before Nakamura [5], however, interpolated membership functions were restricted to be only concave. Later, Nakamura [5], Inuiguchi et al. [3] and Yang et al. [6] proposed their methods for solving fuzzy programming problems with quasi-concave membership functions. Yang et al. used 0–1 variables instead of complicated transforming process to solve fuzzy programs with S-shaped piecewise linear membership functions, as depicted in Fig. 1. Yang et al. proposed the following model.

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Model 1 (Yang et al.).

$$\begin{aligned}&\text{Maximize } \lambda\\&\text{subject to } \left.\begin{aligned}&\lambda \leqslant \mu_{s1}((\mathbf{ax})_s)+M(1-\delta_s)\\&\lambda \leqslant \mu_{s2}((\mathbf{ax})_s)+M\delta_s\\&\lambda \leqslant \mu_{s3}((\mathbf{ax})_s)+M\delta_s\\&\delta_s\in\{0,1\}\end{aligned}\right\} \text{ for } s\in S\subset N,\\&\lambda \leqslant \mu_t((\mathrm{ax})_t) \quad \text{for } t\in N-S,\\&\mathbf{x}\in F \text{ (a feasible set)},\end{aligned}$$

where $N=\{1,2,\dots,n\}$, and M is a large positive number. The S-shaped membership function μ_s is approximated by the intersection and union of three ramp-type functions, μ_{s1}, μ_{s2} and μ_{s3}. μ_s can be represented as

$$\mu_s = \mu_{s1} \cup (\mu_{s2} \cap \mu_{s3}). \tag{2}$$

The ramp-type membership functions, μ_{s1}, μ_{s2} and μ_{s3} can be expressed by

$$\mu_{sj}((\mathbf{ax})_s) = \begin{cases} 0 & \text{if } (\mathbf{ax})_s < l_{sj},\\ \mu_s(p_{s,j-1})+m_{sj}((\mathbf{ax})_s - p_{s,j-1}) & \text{if } l_{sj} \leqslant (\mathbf{ax})_s < r_{sj},\\ 1 & \text{if } r_{sj} \leqslant (\mathbf{ax})_s, \end{cases} \tag{3}$$

where the slope,

$$m_{sj} = \frac{\mu_s(p_{sj}) - \mu_s(p_{s,j-1})}{p_{sj} - p_{s,j-1}}, \tag{4}$$

$$l_{sj} = p_{s,j-1} - \frac{\mu_s(p_{s,j-1})}{m_{sj}} \tag{5}$$

and

$$r_{sj} = p_{s,j-1} + \frac{1-\mu_s(p_{s,j-1})}{m_{sj}}. \tag{6}$$

However, Li and Yu [4] argued that the method of Yang et al. is valid only for a specific type of S-shaped membership function and proposed another model to solve fuzzy programs that cannot be solved by Yang et al.'s model. Li and Yu distinguish S-shaped membership functions into Types 1 and 2. A Type 1 membership function is a union of two concave sub-functions, as depicted in Fig. 2, and a Type 2 membership function is a union of a convex sub-function and a concave sub-function, as depicted in Fig. 3.

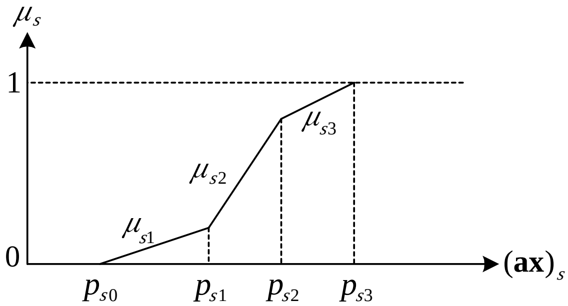

Fig. 1. An S-shaped membership function.

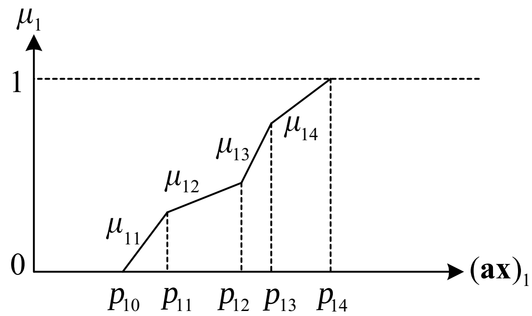

Fig. 2. A Type 1 S-shaped membership function.

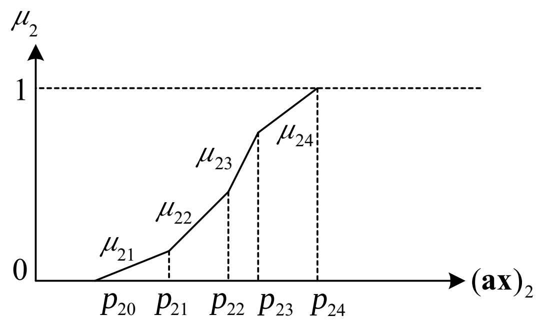

Fig. 3. A Type 2 S-shaped membership function.

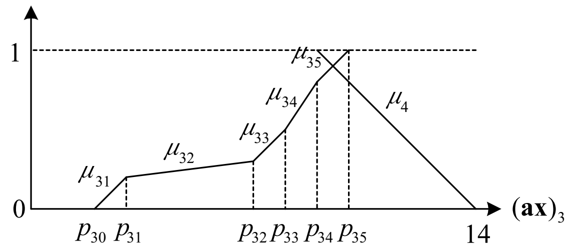

Fig. 4. Membership function $\mu_3((\mathbf{ax})_3)$ and $\mu_4((\mathbf{ax})_3)$.

The Type 1 membership function, μ_1, can be expressed as

$$\mu_1 = (\mu_{11} \cap \mu_{12}) \cup (\mu_{13} \cap \mu_{14}) \quad (7)$$

and the Type 2 membership function, μ_2, can be expressed as

$$\mu_2 = (\mu_{21} \cup \mu_{22}) \cup (\mu_{23} \cap \mu_{24}). \quad (8)$$

Li and Yu indicate that Yang et al.'s model is valid only for Type 1 membership functions and that Type 2 membership functions must involve two 0–1 variables, if represented by Yang et al.'s model. Li and Yu proposed the following model for Type 2 membership functions.

Model 2 (Li and Yu).

Minimize $z - \lambda_2$

subject to $z \geqslant \lambda_1 + \lambda_2 + M(\delta - 1)$,

$z \geqslant 0$,

$\lambda_1 \geqslant \mu_{21}((\mathbf{ax})_2) + M(\delta - 1)$,

$\lambda_1 \geqslant \mu_{22}((\mathbf{ax})_2) + M(\delta - 1)$,

$\lambda_2 \leqslant \mu_{23}((\mathbf{ax})_2) + M\delta$,

$\lambda_2 \leqslant \mu_{24}((\mathbf{ax})_2) + M\delta$,

$x \in F, \quad \delta \in \{0, 1\}$.

They also state that any S-shaped membership function, for example the one shown in Fig. 4, can be regarded as a combination of Types 1 and 2 functions. The membership function, μ_3, in Fig. 4 can be expressed in either of the following ways.

$$\mu_3 = \mu_{31} \cap (\mu_{32} \cup \mu_{33}) \cup (\mu_{34} \cap \mu_{35}) \quad (9)$$

or

$$\mu_3 = (\mu_{31} \cap \mu_{32}) \cup \mu_{33} \cup (\mu_{34} \cap \mu_{35}). \quad (10)$$

The associated fuzzy program with a membership function as expressed in (9), can be formulated as follows.

Model 3 (Li and Yu).

Minimize $z - \lambda_0 + \lambda_2$

subject to $z \geqslant \lambda_1 + \lambda_2 + M(\delta_2 - 1)$,

$z \geqslant 0$,

$\lambda_0 \leqslant \mu_{31}((\mathbf{ax})_3) + M(1 - \delta_1)$,

$\lambda_1 \geqslant \mu_{32}((\mathbf{ax})_3) + M(\delta_2 - 1)$,

$\lambda_1 \geqslant \mu_{33}((\mathbf{ax})_3) + M(\delta_2 - 1)$,

$\lambda_2 \leqslant \mu_{34}((\mathbf{ax})_3) + M\delta_2$,

$\lambda_2 \leqslant \mu_{35}((\mathbf{ax})_3) + M\delta_2$,

$\delta_1 + \delta_2 \leqslant 1$,

$x \in F$.

Several questions remain concerning the model of Li and Yu. First, no generalized procedure or algorithm is proposed for constructing an auxiliary model for S-shaped membership functions, other than the two membership functions elucidated by Li and Yu, as shown in Figs. 3 and 4. Thus, *Model* 3 is difficult to infer from *Model* 2 and applying approach to other S-shaped membership functions is also tricky. Secondly, the advantage of Li and Yu's model, that it requires only one 0–1 variable seems to apply only for S-shaped membership functions that can be expressed by Eq. (8), as can be verified by considering the membership function, μ_3, that needs two 0–1 variables in *Model* 3. Thirdly, the actual aspiration level cannot be directly obtained by solving *Model* 2 and *Model* 3. Only the optimal solution can be obtained, and this, in turn, is used to find the actual aspiration level. Furthermore, when different types of membership functions, say μ_2 and μ_3, are to be considered in a single fuzzy program can *Model* 2 and *Model* 3 be combined to establish a model that solves the fuzzy program?

In fact, all these problems can be resolved using Yang et al.'s approach, the primary advantage of which is the use of only one aspiration level, λ. Therefore, the actual aspiration level achieved can be obtained directly, and different types of membership functions are easily considered simultaneously in a single fuzzy program. The only omission is a generalized procedure that deals with very complicated membership functions.

2. Generalization of Yang et al.'s method

Considering the membership functions in the previous section, any S-shaped piecewise linear membership function can be expressed as a union of several concave sub-functions. A concave sub-function is that part of an S-shaped membership function which is the intersection of perhaps several linear functions. A single linear function can also be considered to be concave. Type 1 membership functions are already unions of concave sub-functions. A convex sub-function can be considered as a union of concave functions because it is a union of at least two linear functions. Type 2 membership functions, which are unions of a convex sub-function and a concave sub-function, can therefore be expressed as unions of concave sub-functions:

$$\mu_2 = \mu_{21} \cup \mu_{22} \cup (\mu_{23} \cap \mu_{24}). \tag{11}$$

Furthermore, as pointed out by Li and Yu, any S-shaped piecewise linear membership function can be considered as a combination of concave sub-functions and convex sub-functions. Therefore, an S-shaped piecewise linear membership function can be expressed as a union only of concave sub-functions. The expression of a piecewise linear membership function as a union of concave sub-functions is unique.

A generalized procedure for transforming an S-shaped piecewise linear membership function into several simultaneous linear constraints consists of the following two procedures.

Procedure 1: If μ_s is the intersection of two sub-functions μ_a and μ_b, that is,

$$\mu_s = \mu_a \cap \mu_b, \tag{12}$$

then solving a fuzzy program that includes μ_s is equivalent to solving the following problem.

$$\begin{aligned} &\text{Maximize } \lambda \\ &\text{subject to } \lambda \leqslant \mu_a((\mathbf{ax})_s), \\ &\qquad \lambda \leqslant \mu_b((\mathbf{ax})_s), \\ &\qquad \mathbf{x} \in F. \end{aligned} \tag{13}$$

Procedure 2: If μ_s is the union of two sub-functions, μ_a and μ_b, that is,

$$\mu_s = \mu_a \cup \mu_b, \tag{14}$$

then solving a fuzzy program that includes μ_s is equivalent to solving the following problem.

$$\begin{aligned} &\text{Maximize } \lambda \\ &\text{subject to } \lambda \leqslant \mu_a((\mathbf{ax})_s) + M(1-\delta), \\ &\qquad \lambda \leqslant \mu_b((\mathbf{ax})_s) + M\delta, \\ &\qquad \delta \in \{0,1\}, \quad \mathbf{x} \in F \end{aligned} \tag{15}$$

or equivalently,

$$\begin{aligned} &\text{Maximize } \lambda \\ &\text{subject to } \lambda \leqslant \mu_a((\mathbf{ax})_s) + M\delta, \\ &\qquad \lambda \leqslant \mu_b((\mathbf{ax})_s) + M(1-\delta), \\ &\qquad \delta \in \{0,1\}, \quad \mathbf{x} \in F. \end{aligned} \tag{16}$$

The two procedures can be alternately used to transform an S-shaped piecewise linear membership function into several simultaneous linear constraints by a "divide and conquer" method. Consider μ_3, for example. First, randomly choose the first $\cup$ operator to be divided and represent μ_3 as

$$\mu_3 = (\mu_{31} \cap \mu_{32}) \cup (\mu_{33} \cup (\mu_{34} \cap \mu_{35})). \tag{17}$$

Procedure 1 then yields the following program.

Maximize λ

$$\text{subject to} \quad \lambda \leqslant (\mu_{31} \cap \mu_{32}) + M(1-\delta_1),$$
$$\lambda \leqslant (\mu_{33} \cup (\mu_{34} \cap \mu_{35})) + M\delta_1,$$
$$\delta \in \{0,1\}, \quad \mathbf{x} \in F. \tag{18}$$

Secondly, divide (17) at the second $\cup$ operator and obtain the following program using *Procedure* 1.

Maximize λ

$$\text{subject to} \quad \lambda \leqslant (\mu_{31} \cap \mu_{32}) + M(1-\delta_1),$$
$$\lambda \leqslant \mu_{33} + M(1-\delta_2) + M\delta_1,$$
$$\lambda \leqslant (\mu_{34} \cap \mu_{35}) + M\delta_2 + M\delta_1,$$
$$\delta_1, \delta_2 \in \{0,1\}, \quad x \in F. \tag{19}$$

Finally, use *Procedure* 2 to deal with the $\cap$ operators and obtain the final program.

Maximize λ

$$\text{subject to} \quad \lambda \leqslant \mu_{31} + M(1-\delta_1),$$
$$\lambda \leqslant \mu_{32} + M(1-\delta_1),$$
$$\lambda \leqslant \mu_{33} + M(1-\delta_2) + M\delta_1,$$
$$\lambda \leqslant \mu_{34} + M\delta_2 + M\delta_1,$$
$$\lambda \leqslant \mu_{35} + M\delta_2 + M\delta_1,$$
$$\delta_1, \delta_2 \in \{0,1\}, \quad \mathbf{x} \in F. \tag{20}$$

Any S-shaped membership function, expressed with a union of concave sub-functions, can be represented using a binary tree with non-terminal nodes that are $\cup$ operators and terminal nodes that are concave sub-functions. For example, μ_3 can be represented by the binary tree depicted in Fig. 5. This tree representation is not unique, and thus the number of 0–1 variables varies. Nevertheless, representing a membership function as an AVL-tree is particularly useful. An AVL-tree is a height-balanced tree in which the maximal height difference is unity for each sub-tree [2]. Representing an S-shaped membership function by an AVL-tree yields the advantage of minimizing the number of required 0–1 variables. The number of required 0–1 variables equals the height of the AVL-tree subtract one. For an S-shaped membership function that consists of n concave sub-functions, the number of 0–1 variables is $\lceil \log_2 n \rceil$, which is the smallest integer above $\log_2 n$.

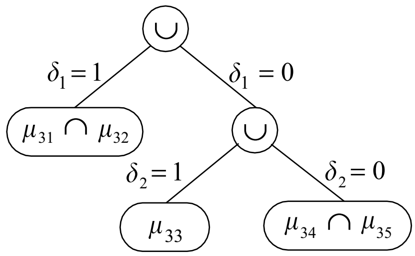

Fig. 5. A binary tree representation for membership function μ_3.

3. Numerical example

As stated at the end of Section 1, it is unclear whether Li and Yu's model is applicable to fuzzy programming problems with more than one membership function. The following example proves this suspicion.

Consider the following fuzzy program with two membership functions.

Maximize λ

$$\text{subject to} \quad \lambda \leqslant \mu_3((\mathbf{ax})_3),$$
$$\lambda \leqslant \mu_4((\mathbf{ax})_3),$$
$$\mathbf{x} \in F. \tag{21}$$

Fig. 4 shows the S-shaped membership function, μ_3, and the ramp-type membership function, μ_4, with,

$$(p_{30}, p_{31}, p_{32}, p_{33}, p_{34}, p_{35}) = (2, 3, 7, 8, 9, 10) \tag{22}$$

and

$$(\mu_3(p_{30}), \mu_3(p_{31}), \mu_3(p_{32}), \mu_3(p_{33}), \mu_3(p_{34}), \mu_3(p_{35})) = (0, 0.2, 0.3, 0.5, 0.8, 1). \tag{23}$$

Li and Yu's, *Model* 3 formulates the corresponding optimization program as follows.

Li and Yu's Model.

$$\text{Minimize } z - \lambda_0 + \lambda_2$$

$$\begin{aligned}
\text{subject to } & z \geqslant \lambda_1 + \lambda_2 + M(\delta_2 - 1),\\
& z \geqslant 0,\\
& \lambda_0 \leqslant 0.2((\mathbf{ax})_3 - 2) + M(1-\delta_1),\\
& \lambda_1 \geqslant 0.2 + 0.025((\mathbf{ax})_3 - 3) + M(\delta_2 - 1),\\
& \lambda_1 \geqslant 0.3 + 0.2((\mathbf{ax})_3 - 7) + M(\delta_2 - 1),\\
& \lambda_2 \leqslant 0.5 + 0.3((\mathbf{ax})_3 - 8) + M\delta_2,\\
& \lambda_2 \leqslant 0.8 + 0.2((\mathbf{ax})_3 - 9) + M\delta_2,\\
& \lambda_2 \leqslant 1 - 0.2((\mathbf{ax})_3 - 9) + M\delta_2,\\
& \delta_1 + \delta_2 \leqslant 1.
\end{aligned} \tag{24}$$

As verified by Li and Yu in [2], if $\delta_1 = 1$ and $\delta_2 = 0$, then $\lambda = \mu_{31}$; if $\delta_1 = 0$ and $\delta_2 = 1$, then $\lambda = \max\{\mu_{32}, \mu_{33}\}$, and if $\delta_1 = \delta_2 = 0$, then $\lambda = \min\{\mu_{34}, \mu_{35}, \mu_4\}$. However, after the program is solved, $(\mathbf{ax})_3 = 14$. Substituting $(\mathbf{ax})_3 = 14$ into μ_4 yields the aspiration level, zero, which is apparently not the optimal solution ($(\mathbf{ax})_3 = 9.5$), as can be easily observed from Fig. 4.

The model of Eq. (20) yields the following optimization program.

The proposed Model.

$$\text{Maximize } \lambda$$

$$\begin{aligned}
\text{subject to } & \lambda \leqslant 0.2((\mathbf{ax})_3 - 2) + M(1 - \delta_1),\\
& \lambda \leqslant 0.2 + 0.025((\mathbf{ax})_3 - 3) + M(1-\delta_1),\\
& \lambda \leqslant 0.3 + 0.2((\mathbf{ax})_3 - 7) + M(1 - \delta_2)\\
& \qquad + M\delta_1,\\
& \lambda \leqslant 0.5 + 0.3((\mathbf{ax})_3 - 8) + M\delta_2 + M\delta_1,\\
& \lambda \leqslant 0.8 + 0.2((\mathbf{ax})_3 - 9) + M\delta_2 + M\delta_1,\\
& \lambda \leqslant 1 - 0.2((\mathbf{ax})_3 - 9).
\end{aligned} \tag{25}$$

The optimal solution is $(\mathbf{ax})_3 = 9.5$ and $\lambda = 0.9$. Thus, the model of Li and Yu failed to find the optimal solution while the proposed model succeeded.

4. Conclusions

This study generalizes the method of Yang et al. for dealing with fuzzy programming problems with piecewise linear membership functions. Although Li and Yu indicated that Yang et al.'s method is valid only for specific membership functions and proposed a model for other types of membership functions, the numerical example presented here shows that Li and Yu's model benefits only when one membership function that consists of a single concave sub-function and a single convex sub-function, is considered. The model of Li and Yu becomes inapplicable if the number of membership functions exceeds one. The proposed model is not only superior to Li and Yu's model in that it requires fewer variables and fewer constraints, but also universally applicable.

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