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Numerical range of Aluthge transform of operator ☆

Pei Yuan Wu

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan, ROC

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Dedicated to Professor Tsuyoshi Ando on his 71st birthday

Abstract

For any operator A on a Hilbert space, let $\widetilde{A}$ denote its Aluthge transform. In this paper, we prove that the closure of the numerical range of $\widetilde{A}$ is always contained in that of A. This supplements the recently proved case for $\dim\ker A \leqslant \dim\ker A^*$ by Yamazaki, and partially confirms a conjecture of Jung, Ko and Pearcy.

Keywords: Numerical range; Aluthge transform; Polar decomposition

Let A be a bounded linear operator on a complex Hilbert space H. If $A = V|A|$ is any polar decomposition of A with V a partial isometry and $|A| = (A^*A)^{1/2}$, then the Aluthge transform $\widetilde{A}$ of A is the operator $|A|^{1/2}V|A|^{1/2}$. Note that $\widetilde{A}$ is independent of the choice of the partial isometry V in the polar decomposition of A. This is first defined by Aluthge [1] in his study of p-hyponormal operators. In recent years, properties of the transform have been investigated by several authors [6–8]. Some of them are concerned with the relation between the numerical ranges of A and $\widetilde{A}$. Recall that the *numerical range* $W(A)$ of A is the subset $\{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$ of the plane, where $\langle\cdot,\cdot\rangle$ is the inner product in H. It is known that $W(A)$ is always convex and $\overline{W(A)}$ contains $\sigma(A)$, the spectrum of A. (For properties of numerical ranges, see [3, Chapter 22] or [2].) In [6, Proposition 1.8], it was proved that $W(\widetilde{A}) \subseteq W(A)$ for

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E-mail address: pywu@cc.nctu.edu.tw (P.Y. Wu).

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any A on a two-dimensional space, and was conjectured that this should be the case for any operator A. Recently, Yamazaki [8, Theorem 5] showed that $\overline{W(\widetilde{A})} \subseteq \overline{W(A)}$ for operator A with $\dim\ker A \leqslant \dim\ker A^*$. The purpose of this paper is to prove that the containment $\overline{W(\widetilde{A})} \subseteq \overline{W(A)}$ holds for any A.

Theorem 1. $\overline{W(\widetilde{A})} \subseteq \overline{W(A)}$ *for any operator* A.

For its proof, we need another dual notion of the Aluthge transform defined by Yamazaki [7, Definition 2]. Let $A = V|A|$ be any polar decomposition of A. The $*$-*Aluthge transform* $\widetilde{A}^{(*)}$ of A is the operator $|A^*|^{1/2}V|A^*|^{1/2}$. It is easily seen that $\widetilde{A}^{(*)}$ is again independent of the choice of V in A. Yamazaki showed in [7, Theorem 1 (ii)] that the numerical radii of $\widetilde{A}$ and $\widetilde{A}^{(*)}$ are equal. Recall that the *numerical radius* $w(A)$ of operator A is the quantity $\sup\{|z| : z \in W(A)\}$. The next theorem says that more is true.

Theorem 2. $\overline{W(\widetilde{A})} = \overline{W(\widetilde{A}^{(*)})}$ *for any operator* A.

To prove this, we need the following two lemmas.

Lemma 3. *Let* $A = V|A|$ *be any polar decomposition of* A. *Then*
(a) $A^* = V^*|A^*|$ *is a polar decomposition of* A^*,
(b) $(\widetilde{A^*})^* = \widetilde{A}^{(*)}$, *and*
(c) $\widetilde{A}^{(*)} = V\widetilde{A}V^*$.

The assertions can be proved by delving into the construction of the polar decomposition and using the properties of V, $|A|$ and $|A^*|$, which we leave to the reader. In fact, (b) and (c) here have already been used in [7, Definition 2] and in the proof of [7, Theorem 1 (ii)], respectively. For any subset $\triangle$ of the plane, let $\widehat{\triangle}$ denote its convex hull.

Lemma 4. *If* A *and* B *are operators such that* $A = X^*BX$ *for some contraction* X, *then* $W(A) \subseteq (W(B) \cup \{0\})^\wedge$. *If, in addition,* X *is a coisometry* $(XX^* = 1)$, *then we also have* $W(B) \subseteq W(A)$.

Proof. If x is a unit vector with $Xx = 0$, then $\langle Ax, x\rangle = \langle X^*BXx, x\rangle = 0$, which is in $(W(B) \cup \{0\})^\wedge$. On the other hand, if $Xx \neq 0$, then

$$\begin{aligned}\langle Ax, x\rangle &= \langle BXx, Xx\rangle \\ &= \|Xx\|^2 \cdot \left\langle B\left(\frac{Xx}{\|Xx\|}\right), \frac{Xx}{\|Xx\|}\right\rangle + (1 - \|Xx\|^2)\cdot 0,\end{aligned}$$

which shows that $\langle Ax, x\rangle$ is again in $(W(B) \cup \{0\})^\wedge$. Hence $W(A) \subseteq (W(B) \cup \{0\})^\wedge$.

If, in addition, X is a coisometry, then from $A = X^*BX$ we obtain $B = XAX^*$. For any unit vector x, we have

$$\langle Bx, x\rangle = \langle XAX^*x, x\rangle = \langle AX^*x, X^*x\rangle.$$

Since X^*x is also a unit vector, this shows that $\langle Bx, x\rangle$ is in $W(A)$. Hence $W(B) \subseteq W(A)$ as asserted. □

We are now ready for the proof of Theorem 2.

Proof of Theorem 2. Two cases are considered separately.

(i) $\dim\ker A \leqslant \dim\ker A^*$. In this case, the partial isometry V in the polar decomposition $A = V|A|$ of A can be taken to be an isometry. Since $\widetilde{A}^{(*)} = V\widetilde{A}V^*$ by Lemma 3(c), we may apply Lemma 4 to obtain

$$W(\widetilde{A}) \subseteq W(\widetilde{A}^{(*)}) \subseteq (W(\widetilde{A}) \cup \{0\})^\wedge.$$

It follows that

$$\overline{W(\widetilde{A})} \subseteq \overline{W(\widetilde{A}^{(*)})} \subseteq (\overline{W(\widetilde{A})} \cup \{0\})^\wedge.$$

If 0 is in $\overline{W(\widetilde{A})}$, then these containments imply that $\overline{W(\widetilde{A})} = \overline{W(\widetilde{A}^{(*)})}$. On the other hand, if 0 is not in $\overline{W(\widetilde{A})}$, then 0 cannot be in $\sigma(\widetilde{A})$. Hence $\widetilde{A} = |A|^{1/2}V|A|^{1/2}$ is invertible. This implies the invertibility of $|A|^{1/2}$ and V. Thus V is a unitary operator, and $\widetilde{A}$ and $\widetilde{A}^{(*)}$ are unitarily equivalent. Hence we obviously have $\overline{W(\widetilde{A})} = \overline{W(\widetilde{A}^{(*)})}$.

(ii) $\dim\ker A^* \leqslant \dim\ker A$. For this case, we apply (i) to A^* to obtain $\overline{W(\widetilde{A^*})} = \overline{W(\widetilde{A^*}^{(*)})}$. By Lemma 3(b), we have

$$(\widetilde{A^*})^* = \widetilde{A}^{(*)} \quad \text{and} \quad (\widetilde{A^*}^{(*)})^* = \widetilde{A}.$$

Thus $\overline{W(\widetilde{A}^{(*)})} = \overline{W(\widetilde{A})}$ as required. □

Finally, we come to prove Theorem 1.

Proof of Theorem 1. Again, we consider two cases separately.

(i) $\dim\ker A \leqslant \dim\ker A^*$. This case is already proved in [8, Theorem 5]. Here, for completeness, we give a rather simplified sketch. As before, we can choose the partial isometry V in $A = V|A|$ to be an isometry. Then

$$\begin{aligned} \|\widetilde{A} - zI\| &\leqslant \||A|V - zI\|^{1/2}\|A - zI\|^{1/2} \\ &= \|V^*(A - zI)V\|^{1/2}\|A - zI\|^{1/2} \\ &\leqslant \|A - zI\| \end{aligned}$$

for any z in $\mathbb{C}$, where the first inequality is a consequence of Heinz inequality ($\|A^{1/2}XB^{1/2}\| \leqslant \|AXB\|^{1/2}\|X\|^{1/2}$ for positive operators A and B and an arbitrary operator X; cf. [4]). This implies $\overline{W(\widetilde{A})} \subseteq \overline{W(A)}$ since

$$\overline{W(A)} = \bigcap_{\lambda \in \mathbb{C}} \{z \in \mathbb{C} : |z - \lambda| \leqslant \|A - \lambda I\|\}$$

and similarly for $\overline{W(\widetilde{A})}$ (cf. [5, Satz 5]).

(ii) $\dim\ker A^* \leqslant \dim\ker A$. For this case, we apply (i) to A^* to obtain $\overline{W(\widetilde{A^*})} \subseteq \overline{W(A^*)}$. Therefore,

$$\overline{W(\widetilde{A})} = \overline{W(\widetilde{A}^{(*)})} = \overline{W((\widetilde{A^*})^*)} \subseteq \overline{W(A)}$$

by Theorem 2 and Lemma 3(b) and by taking the adjoints. This completes the proof. □

It remains to be seen whether $W(\widetilde{A}) = W(\widetilde{A}^{(*)})$ and $W(\widetilde{A}) \subseteq W(A)$ (without the closures) hold for an arbitrary operator A. The former is indeed true when $1 \leqslant \dim\ker A \leqslant \dim\ker A^*$ or $1 \leqslant \dim\ker A^* \leqslant \dim\ker A$. This is because, assuming that $1 \leqslant \dim\ker A \leqslant \dim\ker A^*$, we have 0 as an eigenvalue of A and hence of $\widetilde{A}$, which implies that 0 is in $W(\widetilde{A})$ and hence we obtain $W(\widetilde{A}^{(*)}) \subseteq W(\widetilde{A})$ from $W(\widetilde{A}^{(*)}) \subseteq (W(\widetilde{A}) \cup \{0\})^\wedge$ and the equality $W(\widetilde{A}) = W(\widetilde{A}^{(*)})$ as in the proof of case (i) of Theorem 2.

Note added in proof

Using totally different techniques, both T. Ando and T. Yamazaki have obtained independent proofs of our main result, Theorem 1, here.

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