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GLOBAL OPTIMIZATION FOR SIGNOMIAL DISCRETE PROGRAMMING PROBLEMS IN ENGINEERING DESIGN

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This paper proposes a novel method to solve signomial discrete programming (SDP) problems frequently occurring in engineering design. Various signomial terms are first convexified following different strategies. The original SDP program is then converted into a convex integer program solvable by commercialized packages to obtain globally optimal solutions. Compared with current SDP methods, the proposed method is guaranteed to converge to a global optimum, is computationally more efficient, and is capable of treating zero boundary problems. Numerical examples are presented to demonstrate the usefulness of the proposed method in engineering design.

Keywords: Signomial discrete programming problem; Global optimization; Convexification

1 INTRODUCTION

Signomial discrete programming (SDP) problems occur quite frequently in various fields such as civil and material engineering design, chemical engineering, location-allocation, inventory control, production planning, and scheduling etc. These applications are extensively reviewed in Floudas and Pardalos [9] and Floudas [6]. The developed methods for SDP can be divided into three approaches. The first SDP approach includes various heuristic techniques. For instance, Salcedo *et al.* [18] propose an improved random search algorithm for solving nonlinear optimization problems. Cardoso *et al.* [2] solve nonconvex nonlinear integer programming problems with simulated annealing. Wang and Liao [21] develop methods for solving polynomial integer programs by the genetic algorithm. Their methods, however, can only guarantee to find local optima. Moreover, the probability of finding the global solution decreases when the problem size increases.

The second SDP approach for global optimization is the use of stochastic methods such as the Multi-Level Single Linkage method proposed by Rinnooy and Timmer [17] and the Multistart method proposed by Li and Chou [12]. These techniques have a high probability of finding a global optimum for a SDP problem. However, since this approach requires the evaluation of a large number of starting points, it can only be applied to solve small size problems.

The third approach is the deterministic method. Duran and Grossmann [4] treat a class of SDP problems by outer approximation techniques. Michelon and Maculan [15] solve SDP problems by

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Lagrangean decomposition techniques. Li and Chang [11] solve SDP problems, where all signomial terms have integer power values, by piecewise linearization techniques. Pörn *et al.* [16] introduce different convexification strategies for SDP problems with both posynomial and negative binomial terms in the constraints. The above methods, however, can only handle some specially-structured SDP problems. Recently, Floudas and Pardalos [9], Maranas and Floudas [14], and Floudas [7, 8] have proposed more general methods to treat SDP problems. Their methods have been applied widely to solve engineering design problems. The core concept of Floudas's approach is to convert a SDP problem into a new problem in which both the constraints and the objective are decomposed into the difference of two convex functions. By utilizing exponential variable transformation, Floudas's method transform each signomial term $z = x_1^{\alpha}x_2^{\beta}$, where x_1 and x_2 are positive integers, into an exponential term $z' = e^{\alpha \ln x_1 + \beta \ln x_2}$. Since (i) the exponentiation of a linear expression is convex, and (ii) $\ln x_1$ and $\ln x_2$ can be conveniently expressed using 0–1 variables, the signomial term can then be fully expressed as the combination of convex integer terms. Floudas's method therefore can find the global optimum of a SDP problem successfully. However, since Floudas's method performs exponential transformation for all product terms, it requires the use of a large number of 0–1 variables to piecewisely linearize the logarithmic terms. In addition, the exponential transformation technique can only be applied to positive variables and is unable to treat zero boundary problems where variables might have zero value.

This paper proposes another method to treat SDP problems and develops several strategies for convexifying a signomial term. The advantages of the proposed methods over the current SDP methods mentioned above are given below:

(i) Compared with the heuristic approaches and the stochastic methods of Duran and Grossmann [4], and Michelon and Maculan [15], the proposed method is guaranteed to find a global optimum of a SDP problem.
(ii) Compared with Floudas's method, for many cases, the proposed method uses fewer extra 0–1 variables to linearize a signomial term. In addition, the proposed method can treat non-negative integer variables while Floudas's method can only treat positive integer variables.

This study first discusses some theoretical propositions about SDP programs. The rules of convexification are then proposed. Following that, some numerical examples of engineering design are solved to demonstrate the usefulness of the proposed method.

2 THEORETICAL DEVELOPMENT

A Signomial Discrete Programming (SDP) problem discussed here is formulated below:

P1

$$\begin{aligned}
\text{Minimize}\quad & Z(X) = \sum_P c_p z_p \\
\text{Subject to}\quad & \sum_q h_{kq} z_{kq} \le l_k, \quad k = 1, 2, \ldots, K \\
& \prod_{j=1}^{m} f_j(x) = 0, \quad j = 1, 2, \ldots, m \\
& z_p = x_{p_1}^{\alpha_{p1}} x_{p_2}^{\alpha_{p2}} \cdots x_{p_{m(p)}}^{\alpha_{pm(p)}} \\
& z_{kq} = x_{kq1}^{\beta_{kq1}} x_{kq2}^{\beta_{kq2}} \cdots x_{kqm(kq)}^{\beta_{kqm(kq)}} \\
& X = (x_1, x_2, \ldots, x_n), \quad 0 \le \underline{x}_i \le x_i \le \bar{x}_i, x_i \in X
\end{aligned}$$

are non-negative discrete variables.

In problem P1, c_p, α_{pi}, β_{kqi}, h_{kq}, l_k are constants and unrestricted in sign, $\underline{x}_i$ and $\bar{x}_i$ are respectively the lower and upper bounds of discrete variables x_i.

P1 is a nonconvex integer problem which can only be solved to obtain the local optimum. In order to obtain its global optimum, P1 must be converted into a convex integer problem. The conventional convex integer techniques proposed by Borchers and Mitchell [1] and Floudas [5] have traditionally been used to solve convex integer programs to obtain global optima. This paper proposes various techniques for convexifying signomial terms z_p, $-z_p$, z_{kq}, and $-z_{kq}$. The convexified SDP program can be expressed as a linear integer programming problem solvable by many commercialized optimization packages to obtain a globally optimal solution. Some propositions related to convexification techniques are described as follows.

PROPOSITION 1 *For positive discrete variables* $x_i \in \{d_{i1}, d_{i2}, \ldots, d_{in_i}\}$ *where* $d_{i,j+1} > d_{ij} > 0$ *for* $j = 1, 2, \ldots, n_i - 1$, *a product term* $x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$ *with* $r_1, r_2, \ldots, r_n$ *real constants can be transformed to a function* $e^{r_1 y_1 + \cdots + r_n y_n}$ *where* $y_i = \ln d_{i1} + \sum_{j=1}^{n_i-1} u_{ij}(\ln d_{i,j+1} - \ln d_{i1})$, $\sum_{j=1}^{n_i-1} u_{ij} \leq 1$ *for* $u_{ij} \in \{0, 1\}$.

Proof Let $x_i = e^{y_i}$ and $y_i = \ln x_i$, expressing x_i as

$$x_i = d_{i1} + \sum_{j=1}^{n_i-1} u_{ij}(d_{i,j+1} - d_{i1}), \quad \sum_{j=1}^{n_i-1} u_{ij} \leq 1, \quad \text{where } u_{ij} \in \{0, 1\}.$$

We then have $x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} = e^{r_1 y_1 + \cdots + r_n y_n}$ and $y_i = \ln d_{i1} + \sum_{j=1}^{n_i-1} u_{ij}(\ln d_{i,j+1} - \ln d_{i1})$, $\sum_{j=1}^{n_i-1} u_{ij} \leq 1$, for $u_{ij} \in \{0, 1\}$.

Suppose a variable x_i in Proposition 1 may have zero value, then Proposition 1 needs to be modified as in the following proposition:

PROPOSITION 2 *For non-negative discrete variables* $x_i \in \{0, d_{i1}, d_{i2}, \ldots, d_{in_i}\}$ *where* $d_{i,j+1} > d_{ij} > 0$ *for* $j = 1, 2, \ldots, n_i - 1$, *then a product term* $z = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$ *can be expressed as*

(i) $0 \leq z \leq \bar{z}(\sum_{j=1}^{n_i} u_{ij})$,
(ii) $\bar{z}(\sum_{i=1}^{n} \sum_{j=1}^{n_i} u_{ij} - n) + e^{r_1 y_1 + \cdots + r_n y_n} \leq z \leq \bar{z}(n - \sum_{i=1}^{n} \sum_{j=1}^{n_i} u_{ij}) + L(e^{r_1 y_1 + \cdots + r_n y_n})$,
where $x_i = \sum_{j=1}^{n_i} u_{ij} d_{ij}$, $y_i = \sum_{j=1}^{n_i} u_{ij}(\ln d_{ij})$, $\sum_{j=1}^{n_i} u_{ij} \leq 1$, $u_{ij} \in \{0, 1\}$, $L(e^{r_1 y_1 + \cdots + r_n y_n})$ *is a piecewisely linearized expression of* $e^{r_1 y_1 + \cdots + r_n y_n}$, *and* $\bar{z}$ *is the upper bound of* z.

Proof If there is $x_i = 0$ for some i, then $\sum_{j=1}^{n_i} u_{ij} = 0$ and $z = 0$. If $x_i > 0$ for all $i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} \sum_{j=1}^{n_i} u_{ij} = n$ and $e^{r_1 y_1 + \cdots + r_n y_n} \leq z \leq L(e^{r_1 y_1 + \cdots + r_n y_n})$. Therefore, the proposition is then proven.

Remark 1 For a discrete variable x, $x \in \{d_1, d_2, \ldots, d_n\}$, $d_1, d_2, \ldots, d_n$ are positive values, the exponential term x^α where α is a real constant can be represented as

$$x^\alpha = d_1^\alpha + \sum_{j=1}^{n-1} u_j(d_{j+1}^\alpha - d_1^\alpha) \text{ where } \sum_{j=1}^{n-1} u_j \leq 1, \quad u_j \in \{0, 1\}.$$

PROPOSITION 3 *A product term $z = uf(x)$ is equivalent to the following linear inequalities*

(i) $M(u-1) + f(x) \leq z \leq M(1-u) + f(x)$,
(ii) $-Mu \leq z \leq Mu$,

$u \in \{0, 1\}$, *z is an unrestricted in sign variable, and $M = \max f(x)$ is a large constant.*

Proof If $u = 1$ then $z = f(x)$, and if $u = 0$ then $z = 0$.

Remark 2 The product term $u_1 u_2 \cdots u_m$ where $u_i \in \{0, 1\}$ for $i = 1, 2, \ldots, m$ can be replaced by a variable u expressed as

(i) $0 \leq u \leq u_i$, for $i = 1, 2, \ldots, m$,
(ii) $u \geq \sum_{i=1}^{m} u_i - m + 1$.

Proof If $u_i = 0$ for any i, then $u = 0$. If $u_i = 1$ for all i, then $u = 1$.

PROPOSITION 4 *A twice-differentiable function $f(x_1, x_2, x_3) = -x_1^{\alpha} x_2^{\beta} x_3^{\gamma}$ is convex for $\alpha + \beta + \gamma \leq 1$ where $x_1, x_2, x_3, \alpha, \beta, \gamma \geq 0$.*

Proof Denote $H(x_1, x_2, x_3)$ as the Hessian matrix of $f(x_1, x_2, x_3)$.

$$H(x_1,x_2,x_3) = \begin{bmatrix} \dfrac{\partial^2 f(x_1,x_2,x_3)}{\partial x_1 \partial x_1} & \dfrac{\partial^2 f(x_1,x_2,x_3)}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f(x_1,x_2,x_3)}{\partial x_1 \partial x_3} \\ \dfrac{\partial^2 f(x_1,x_2,x_3)}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f(x_1,x_2,x_3)}{\partial x_2 \partial x_2} & \dfrac{\partial^2 f(x_1,x_2,x_3)}{\partial x_2 \partial x_3} \\ \dfrac{\partial^2 f(x_1,x_2,x_3)}{\partial x_3 \partial x_1} & \dfrac{\partial^2 f(x_1,x_2,x_3)}{\partial x_3 \partial x_2} & \dfrac{\partial^2 f(x_1,x_2,x_3)}{\partial x_3 \partial x_3} \end{bmatrix}$$
$$= \begin{bmatrix} -\alpha(\alpha-1)x_1^{\alpha-2}x_2^{\beta}x_3^{\gamma} & -\alpha\beta x_1^{\alpha-1}x_2^{\beta-1}x_3^{\gamma} & -\alpha\gamma x_1^{\alpha-1}x_2^{\beta}x_3^{\gamma-1} \\ -\alpha\beta x_1^{\alpha-1}x_2^{\beta-1}x_3^{\gamma} & -\beta(\beta-1)x_1^{\alpha}x_2^{\beta-2}x_3^{\gamma} & -\beta\gamma x_1^{\alpha}x_2^{\beta-1}x_3^{\gamma-1} \\ -\alpha\gamma x_1^{\alpha-1}x_2^{\beta}x_3^{\gamma-1} & -\gamma\beta x_1^{\alpha}x_2^{\beta-1}x_3^{\gamma-1} & -\gamma(\gamma-1)x_1^{\alpha}x_2^{\beta}x_3^{\gamma-2} \end{bmatrix}$$

The ith principal minor, denoted by H_i, of a $n \times n$ matrix is the $i \times i$ matrix obtained by deleting the last $n - i$ rows and columns of the matrix. It is clear that if $\det H_1 \geq 0$, $\det H_2 \geq 0$, and $\det H_3 \geq 0$, then $f(x_1, x_2, x_3)$ is convex.

Check:

(i) $\det H_1 \geq 0$ ($\because x_1, x_2, x_3, \alpha, \beta, \gamma \geq 0$ and $-\alpha(\alpha-1)x_1^{\alpha-2}x_2^{\beta}x_3^{\gamma} \geq 0$).
(ii) $\det H_2 \geq 0$ ($\because \det H_2 = \alpha\beta x_1^{2\alpha-2}x_2^{2\beta-2}x_3^{2\gamma}(-\alpha-\beta+1) \geq 0$).
(iii) $\det H_3 \geq 0$ ($\because \det H_3 = \alpha\beta\gamma x_1^{3\alpha-2}x_2^{3\beta-2}x_3^{3\gamma-2}(-\alpha-\beta-\gamma+1) \geq 0$).

Following (i), (ii), and (iii), the proposition is proven.

PROPOSITION 5 *An equality constraint $\prod_{j=1}^{m} f_j(x) = 0$ can be replaced by following expressions.*

(i) $-M(1-u_j) < f_j(x) < M(1-u_j)$,
(ii) $\sum_{j=1}^{m} u_j \geq 1$,

where M is a large constant, $M = \max\{0, f_j(x)\}$, and $u_j \in \{0, 1\}$ for $j \in \{1, 2, \ldots, m\}$.

Proof Expression (i) means if and only if $u_j = 1$ then $f_j(x) = 0$. Expression (ii) means there is at least one $j \in \{1, 2, \ldots, m\}$ such that $u_j = 1$. Both expressions ensure $\prod_{j=1}^{m} f_j(x) = 0$.

3 CONVEXIFICATION STRATEGIES

Following the above discussion, a signomial term with three variables is here used as an example to describe the strategy of convexification. The strategy can be extended to convexify a signomial term containing n variables.

Consider a signomial term $cx_1^{\alpha}x_2^{\beta}x_3^{\gamma}$ composed of three positive discrete variables x_1, x_2, x_3, where $x_i = d_{i1} + \sum_{j=1}^{n_i-1} u_{ij}(d_{i,j+1} - d_{i1})$, $\sum_{j=1}^{n_i-1} u_{ij} \leq 1$. This term can be convexified by following rules:

Rule 1 If $c > 0$, then let $cx_1^{\alpha}x_2^{\beta}x_3^{\gamma} = ce^{\alpha \ln x_1 + \beta \ln x_2 + \gamma \ln x_3}$ where $\ln x_i = \ln d_{i1} + \sum_{j=1}^{n_i-1} u_{ij}(\ln d_{i,j+1} - \ln d_{i1})$, $\sum_{j=1}^{n_i-1} u_{ij} \leq 1$, for $u_{ij} \in \{0, 1\}$.

Rule 2 If $c < 0$, $\alpha, \beta, \gamma \geq 0$, and $\alpha + \beta + \gamma \leq 1$, then $cx_1^{\alpha}x_2^{\beta}x_3^{\gamma}$ is already a convex term following Proposition 4. No convexification is required.

Rule 3 If $c < 0$, $0 \leq \alpha, \beta < 1$, $\gamma \geq 0$, $\alpha + \beta < 1$, and $\alpha + \beta + \gamma > 1$, then let $cx_1^{\alpha}x_2^{\beta}x_3^{\gamma} = cx_1^{\alpha}x_2^{\beta}y_3^{1-\alpha-\beta}$ and $y_3 = x_3^{\gamma/(1-\alpha-\beta)}$ where $cx_1^{\alpha}x_2^{\beta}y_3^{1-\alpha-\beta}$ is regarded as a convex term, and y_3 is a discrete variable, $y_3 = h_{31} + \sum_{j=1}^{n_3-1} u_{3j}(h_{3,j+1} - h_{31})$, $h_{3j} = (d_{3j})^{\gamma/(\alpha+\beta+\gamma)}$ for $j \in \{1, 2, \ldots, n_i - 1\}$.

Rule 4 If $c < 0$, $\alpha, \beta, \gamma > 0$, and $\alpha + \beta + \gamma > 1$, then let $cx_1^{\alpha}x_2^{\beta}x_3^{\gamma} = cy_1^{\alpha/(\alpha+\beta+\gamma)}y_2^{\beta/(\alpha+\beta+\gamma)} \times y_3^{\gamma/(\alpha+\beta+\gamma)}$ where $y_1 = x_1^{\alpha+\beta+\gamma}$, $y_2 = x_2^{\alpha+\beta+\gamma}$, $y_3 = x_3^{\alpha+\beta+\gamma}$. $cy_1^{\alpha/(\alpha+\beta+\gamma)}y_2^{\beta/(\alpha+\beta+\gamma)}y_3^{\gamma/(\alpha+\beta+\gamma)}$ is a convex term, and $y_i = h_{i1} + \sum_{j=1}^{n_i-1} u_{ij}(h_{i,j+1} - h_{i1})$, $h_{ij} = (d_{ij})^{\gamma/(\alpha+\beta+\gamma)}$ for $i = 1, 2, 3$, and $j \in \{1, 2, \ldots, n_i - 1\}$.

Rule 5 If $\alpha, \beta > 0$, $x_3 = 1$, and $\alpha + \beta > 1$, then let $cx_1^{\alpha}x_2^{\beta} = c[d_{11}^{\alpha} + \sum_{j=1}^{n_1-1} u_{1j}(d_{1,j+1}^{\alpha} - d_{11}^{\alpha})]x_2^{\beta}$ for $j \in \{1, 2, \ldots, n_1 - 1\}$. By Proposition 3, the product term $u_{1j}x_2^{\beta}$ can be transformed into linear inequalities.

4 NUMERICAL EXAMPLES

According to the convexification strategies described above, several examples are presented in the following to demonstrate its usefulness in engineering design.

Example 1 Consider the following nonconvex minimization problem containing three integer variables.

$$\begin{aligned} \text{Minimize} \quad & x_1^2 x_2^{3.5} x_3 - x_2 x_3^{2.6} - x_1^3 \\ \text{Subject to} \quad & x_1 + x_2 + x_3 \leq 10 \\ & 1 \leq x_1 \leq 5, 1 \leq x_2 \leq 5, 1 \leq x_3 \leq 5, x_1, x_2, x_3 \text{ are inter variables.} \end{aligned}$$

This program is a nonconvex integer program. Solving it by LINGO 7.0 [13], the obtained solution is $(x_1, x_2, x_3) = (1, 2, 5)$ and the objective value is -75.7579. This is, however, a local optimum. In order to obtain a global optimum, all signomial terms are transformed into convex terms as follows:

(i) $x_1^2 x_2^{3.5} x_3$ is convexified as $e^{2y_1+3.5y_2+y_3}$ by Rule 1.
(ii) $-x_2 x_3^{2.6}$ is convexified as

$$\begin{aligned}-x_2 x_3^{2.6} &= -(1+u_{21}+2u_{22}+3u_{23}+4u_{24})x_3^{2.6}\\ &= -h_1 - z_1 - 2z_2 - 3z_3 - 4z_4 \text{ by Rule 5.}\end{aligned}$$

(iii) $-x_1^3$ is treated directly as

$$\begin{aligned}-x_1^3 &= -1-(2^3-1)u_{11}-(3^3-1)u_{12}-(4^3-1)u_{13}-(5^3-1)u_{14}\\ &= -1-7u_{11}-26u_{12}-63u_{13}-124u_{14}.\end{aligned}$$

The transformed program is then presented as a convex integer program below:

$$\begin{aligned}
\text{Minimize}\quad & e^{2y_1+3.5y_2+y_3} - h_1 - z_1 - 2z_2 - 3z_3 - 4z_4 - h_2\\
\text{Subject to}\quad & x_1+x_2+x_3 \le 10\\
& x_1 = 1+u_{11}+2u_{12}+3u_{13}+4u_{14}\\
& y_1 = u_{11}\cdot\ln 2+u_{12}\cdot\ln 3+u_{13}\cdot\ln 4+u_{14}\cdot\ln 5\\
& u_{11}+u_{12}+u_{13}+u_{14}\le 1\\
& x_2 = 1+u_{21}+2u_{22}+3u_{23}+4u_{24}\\
& y_2 = u_{21}\cdot\ln 2+u_{22}\cdot\ln 3+u_{23}\cdot\ln 4+u_{24}\cdot\ln 5\\
& u_{21}+u_{22}+u_{23}+u_{24}\le 1\\
& x_3 = 1+u_{31}+2u_{32}+3u_{33}+4u_{34}\\
& y_3 = u_{31}\cdot\ln 2+u_{32}\cdot\ln 3+u_{33}\cdot\ln 4+u_{34}\cdot\ln 5\\
& u_{31}+u_{32}+u_{33}+u_{34}\le 1\\
& h_1 = 1+(2^{2.6}-1)u_{31}+(3^{2.6}-1)u_{32}+(4^{2.6}-1)u_{33}+(5^{2.6}-1)u_{34}\\
& h_2 = 1+(2^3-1)u_{11}+(3^3-1)u_{12}+(4^3-1)u_{13}+(5^3-1)u_{14}\\
& M(u_{21}-1)+h_1\le z_1\le M(1-u_{21})+h_1 \qquad 0\le z_1\le Mu_{21}\\
& M(u_{22}-1)+h_1\le z_2\le M(1-u_{22})+h_1 \qquad 0\le z_2\le Mu_{22}\\
& M(u_{23}-1)+h_1\le z_3\le M(1-u_{23})+h_1 \qquad 0\le z_3\le Mu_{23}\\
& M(u_{24}-1)+h_1\le z_4\le M(1-u_{24})+h_1 \qquad 0\le z_4\le Mu_{24}\\
& (1,1,1,0,0,0)\le(x_1,x_2,x_3,y_1,y_2,y_3)\le(5,5,5,\ln 5,\ln 5,\ln 5)\\
& \text{where } u_{ij}\in\{0,1\}, M \text{ is a large constant.}
\end{aligned}$$

Solving the above convex integer program by LINGO 7.0 [13], the obtained global optimal solution is $(x_1, x_2, x_3) = (5, 1, 1)$ and the objective value is -101.

If we let $0 \le x_1 \le 5$, $0 \le x_2 \le 5$, $0 \le x_3 \le 5$, Example 1 becomes a nonconvex integer problem with non-negative variables. Floudas's method, however, cannot be used to solve this kind of problem. By Proposition 2, we can treat zero boundary problems effectively. Solving a modified Example 1 with non-negative variables by LINGO 7.0 [13] yields the global solution $(x_1, x_2, x_3) = (5, 4, 0)$ and the objective value is -125.

Example 2 Consider the optimal design problem of a pressure vessel given in Sandgren [19] depicted in Figure 1 where x_1 (the spherical head thickness) and x_2 (the shell thickness) are discrete variables and x_3 (the radius of the shell) and x_4 (the length of the shell) are continuous variables. This problem was solved by Sandgren [19] and Fu *et al.* [10] to obtain a locally optimal solution. Li and Chou [12] and Li and Chang [11] solved this problem to obtain an approximate solution. In order to illustrate the applicability of the present method in solving signomial discrete programs, all variables x_1, x_2, x_3, and x_4 are treated as discrete variables. The problem is formulated below:

$$\begin{array}{ll} \text{Minimize} & 0.6224x_1x_3x_4 + 1.7781x_2x_3^2 + 3.1661x_1^2x_4 + 19.84x_1^2x_3 \\ \text{Subject to} & -x_1 + 0.0193x_3 \le 0 \\ & -x_2 + 0.00954x_3 \le 0 \\ & -\pi x_3^2x_4 - \dfrac{4}{3}\pi x_3^3 + 750 * 1728 \le 0 \\ & -240 + x_4 \le 0 \\ & 1 \le x_1 \le 1.375 \\ & 0.625 \le x_2 \le 1 \\ & 48 \le x_3 \le 52 \\ & 90 \le x_4 \le 112 \end{array}$$

where x_1 and x_2 are discrete variables with discreteness 0.0625, and x_3 and x_4 are integer variables.

x_1 is the spherical head thickness, x_2 is the shell thickness, x_3 is the radius and x_4 is the length of the shell. The product term $x_1x_3x_4$ can be treated by Rule 1; product terms $x_2x_3^2$, $x_1^2x_4$, and $x_1^2x_3$ can be treated by Rule 5. x_1, x_2, x_3 and x_4 can be completely expressed by binary variables as follows:

$$\begin{aligned} x_1 &= 1 + 0.0625u_{11} + 0.125u_{12} + 0.25u_{13} \\ x_2 &= 0.625 + 0.0625u_{21} + 0.125u_{22} + 0.25u_{23} \\ x_3 &= 48 + u_{31} + 2u_{32} + 4u_{33} \\ x_4 &= 90 + u_{41} + 2u_{42} + 4u_{43} + 8u_{44} + 16u_{45}, \quad u_{ij} \in \{0, 1\} \end{aligned}$$

This program can then be completely transformed to a convex 0–1 program solvable to obtain a globally optimal solution. A detail description about how to solve a convex 0–1

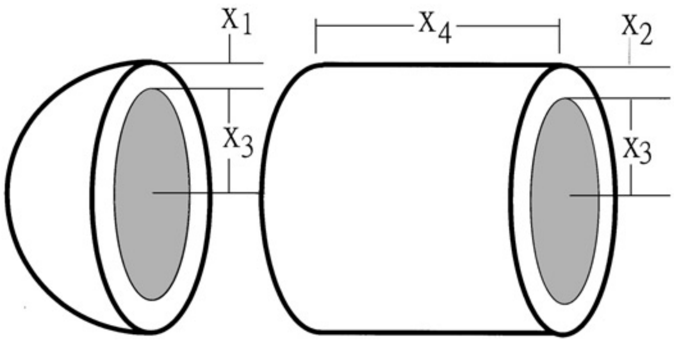

FIGURE 1 Tube and pressure vessel (Sandgren [19]).

program can be found in Borchers and Mitchell [1] and Floudas [5]. By utilizing a branch-bound algorithm or an outer approximation algorithm, a convex 0–1 program can be solved conveniently to reach a global optimum. A commercialized optimization package (*i.e.* LINGO [13]) is also available for solving convex integer programs. Solving this program by LINGO 7.0 [13], the obtained global solution is $(x_1, x_2, x_3, x_4) = (1, 0.625, 51, 91)$ and the objective is 7079.037. The comparison of the solutions for this example is given in Table I. Table I illustrates that even with the extra restriction of the discreteness requirements on the variables x_3 and x_4, the present method obtains a better solution than other methods do.

Example 3 This example shows the detailed process of solving a global nonlinear mixed discrete programming (GDP) problem with Proposition 5. The problem is modified from Cha and Mayne [3].

$$\begin{aligned}
\text{Minimize}\quad & 2x_1^2 + x_2^3 - 16x_1x_2 - 10x_2 \\
\text{Subject to}\quad & (x_1^2 - 6x_1 + 4x_2 - 11)[(3.25x_1 - 3.1x_2)^2 + (x_1 + x_2 - 6.35)^2] \\
& [(3.55x_1 - 3.3x_2)^2 + (x_1 + x_2 - 6.85)^2][(3.6x_1 - 3.5x_2)^2 + \\
& (x_1 + x_2 - 7.1)^2][(3.8x_1 - 4.1x_2)^2 + (x_1 + x_2 - 7.9)^2]^2 = 0 \\
& -x_1x_2 + 3x_2 + e^{x_1 - 3} - 1 \leq 0 \\
& 3 \leq x_1 \leq 6 \\
& 3 \leq x_2 \leq 5
\end{aligned}$$

where x_1 is an integer variable and x_2 is a discrete variable with discreteness 0.2.
x_1 and x_2 are expressed as:

$$x_1 = 3 + u_{11} + 2u_{12}, u_{11}, u_{12} \in \{0, 1\}$$
$$x_2 = 3 + 0.2u_{21} + 0.4u_{22} + 0.8u_{23} + 1.6u_{24}, u_{21}, u_{22}, u_{23}, u_{24} \in \{0, 1\}$$

Here the product term $-x_1x_2$ can be treated by Rule 5, and the first constraint can be treated by Proposition 5. This program can then be converted into a linear integer program. By Proposition 5, the first constraint in the program can be reformulated with following inequality constraints.

$$-M(1 - u_1) \leq x_1^2 - 6x_1 + 4x_2 - 11 \leq M(1 - u_1)$$
$$-M(1 - u_2) \leq (3.25x_1 - 3.1x_2)^2 + (x_1 + x_2 - 6.35)^2 \leq M(1 - u_2)$$
$$-M(1 - u_3) \leq (3.55x_1 - 3.3x_2)^2 + (x_1 + x_2 - 6.85)^2 \leq M(1 - u_3)$$
$$-M(1 - u_4) \leq (3.6x_1 - 3.5x_2)^2 + (x_1 + x_2 - 7.1)^2 \leq M(1 - u_4)$$
$$-M(1 - u_5) \leq [(3.8x_1 - 4.1x_2)^2 + (x_1 + x_2 - 7.9)^2]^2 \leq M(1 - u_5)$$

$u_1 + u_2 + u_3 + u_4 + u_5 \geq 1$, where M is a large constant, $u_j \in \{0, 1\}$, $j = 1, 2, \ldots, 5$.

TABLE I A Comparison of Optimum Solutions for Example 2.

Items	*Sandgren*	*Fu et al.*	*Li and Chou*	*Li and Chang*	*The proposed method*
x_1	1.125	1.125	1	1	1
x_2	0.625	0.625	0.625	0.625	0.625
x_3	48.95	48.38	51.25	51.25	51
x_4	106.72	111.745	90.991	90.991	91
Objective	7982.5	8048.6	7127.3	7127.3	7079.037

Solving the transformed program with LINGO 7.0 [13], the global optimal solution is found as $(x_1, x_2) = (5, 4)$ and the objective value is -246.

Example 4 This example is an optimal design problem introduced in Shin *et al.* [20]. This is a three-bar truss design problem as depicted in Figure 2. The indeterminate three bar truss is subject to vertical and horizontal forces. The weight is to be minimized under the constraint that the stress in all members should be smaller than the allowable stress. The problem can be stated as follows.

$$
\begin{aligned}
\text{Minimize} \quad & 2x_1 + x_2 + \sqrt{2}x_3 \\
\text{Subject to} \quad & -1 + \frac{\sqrt{3}x_2 + 1.932x_3}{1.5x_1x_2 + \sqrt{2}x_2x_3 + 1.319x_1x_3} \le 0 \\
& -1 + \frac{0.634x_1 + 2.828x_3}{1.5x_1x_2 + \sqrt{2}x_2x_3 + 1.319x_1x_3} \le 0 \\
& -1 + \frac{0.5x_1 - 2x_2}{1.5x_1x_2 + \sqrt{2}x_2x_3 + 1.319x_1x_3} \le 0 \\
& -1 - \frac{0.5x_1 - 2x_2}{1.5x_1x_2 + \sqrt{2}x_2x_3 + 1.319x_1x_3} \le 0
\end{aligned}
$$

where x_i are discrete variables, $x_i \in \{0.1, 0.2, 0.3, 0.5, 0.8, 1.0, 1.2\}$, $\quad i = 1, 2, 3$.

This problem is nonconvex because of the constraints. The nonconvex terms $-x_1x_2$, $-x_1x_3$, and $-x_2x_3$ can be treated by Rule 4. The problem is then transformed into an equivalent convex integer program as follows.

$$
\begin{aligned}
\text{Minimize} \quad & 2x_1 + x_2 + \sqrt{2}x_3 \\
\text{Subject to} \quad & \sqrt{3}x_2 + 1.932x_3 - 1.5X_1^{0.5}X_2^{0.5} - \sqrt{2}X_2^{0.5}X_3^{0.5} - 1.319X_1^{0.5}X_3^{0.5} \le 0 \\
& 0.634x_1 + 2.828x_3 - 1.5X_1^{0.5}X_2^{0.5} - \sqrt{2}X_2^{0.5}X_3^{0.5} - 1.319X_1^{0.5}X_3^{0.5} \le 0 \\
& 0.5x_1 - 2x_2 - 1.5X_1^{0.5}X_2^{0.5} - \sqrt{2}X_2^{0.5}X_3^{0.5} - 1.319X_1^{0.5}X_3^{0.5} \le 0 \\
& -0.5x_1 + 2x_2 - 1.5X_1^{0.5}X_2^{0.5} - \sqrt{2}X_2^{0.5}X_3^{0.5} - 1.319X_1^{0.5}X_3^{0.5} \le 0 \\
& x_1 = 0.1 + 0.1u_{11} + 0.2u_{12} + 0.4u_{13} + 0.7u_{14} + 0.9u_{15} + 1.1u_{16} \\
& x_2 = 0.1 + 0.1u_{21} + 0.2u_{22} + 0.4u_{23} + 0.7u_{24} + 0.9u_{25} + 1.1u_{26} \\
& x_3 = 0.1 + 0.1u_{31} + 0.2u_{32} + 0.4u_{33} + 0.7u_{34} + 0.9u_{35} + 1.1u_{36} \\
& X_1 = 0.01 + 0.03u_{11} + 0.08u_{12} + 0.24u_{13} + 0.63u_{14} + 0.99u_{15} + 1.43u_{16} \\
& X_2 = 0.01 + 0.03u_{21} + 0.08u_{22} + 0.24u_{23} + 0.63u_{24} + 0.99u_{25} + 1.43u_{26} \\
& X_3 = 0.01 + 0.03u_{31} + 0.08u_{32} + 0.24u_{33} + 0.63u_{34} + 0.99u_{35} + 1.43u_{36} \\
& u_{11} + u_{12} + u_{13} + u_{14} + u_{15} + u_{16} \le 1 \\
& u_{21} + u_{22} + u_{23} + u_{24} + u_{25} + u_{26} \le 1 \\
& u_{31} + u_{32} + u_{33} + u_{34} + u_{35} + u_{36} \le 1
\end{aligned}
$$

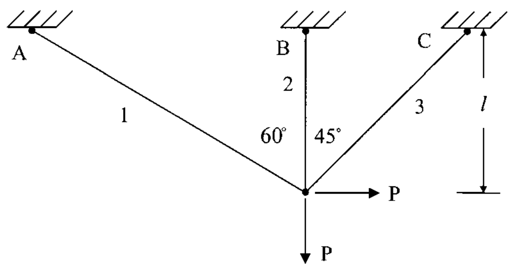

FIGURE 2 Three bar truss for Example 4 (Shin *et al.* [20]).

where $u_{ij} \in \{0, 1\}$, x_i are discrete variables, $x_i \in \{0.1, 0.2, 0.3, 0.5, 0.8, 1.0, 1.2\}$, $i = 1, 2, 3$, and $j = 1, 2, \ldots, 6$.

Solving this convex integer program by LINGO 7.0 [13] gives the global optimal solution $(x_1, x_2, x_3) = (1.2, 0.5, 0.1)$ and the objective value 3.0414. Shin *et al.* [20] and Li and Chou [12] solved this problem and got the same solution. Their methods, however, cannot claim the solution found is a global optimum.

5 CONCLUSIONS

This study proposes global optimization techniques to obtain the global optimal solutions of several types of SDP problems. Different convexification techniques for SDP problems were presented. The transformation methods are general and practical for many kinds of nonconvex global optimization problems. The numerical examples chosen from the literature demonstrate that the proposed methods can obtain the global solutions effectively.

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