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## A NOTE ON THE INTERVAL ESTIMATION OF $C_{pk}$ WITH ASYMMETRIC TOLERANCES

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Pearn and Chen (1998) proposed a generalization of the widely used process capability index (PCI)  $C_{pk}$  to handle processes with asymmetric tolerances. They investigated the sampling distribution and obtained the exact formulae for the expected value and variance of its natural estimator. Recently, Pearn and Lin (2000) considered a different estimator under different process condition, and investigated the statistical properties of the new estimator. However, their efforts focused on the small sample properties under the normality assumption. In this paper, we investigate the large sample properties of its natural estimator under the general condition. Based on the limiting distribution of the new estimator, we provide an approximate  $100(1 - \alpha)\%$  confidence interval of the considered PCI. The obtained confidence interval provides great benefit to quality engineers on monitoring the process and assessing process performance.

*Keywords:* Asymmetric tolerances; Process capability index

### 1 INTRODUCTION

Process capability index  $C_{pk}$  (Kane, 1986) has been widely used in the manufacturing industry to provide numerical measures of process potential and performance. As noted by many quality control researchers and practitioners,  $C_{pk}$  is yield-based and is independent of the target  $T$ , which fails to account for process centering with symmetric tolerances, has an even greater problem with asymmetric tolerances. To overcome the problem, Pearn and Chen (1998) considered a generalization of  $C_{pk}$ , referred to as  $C_{pk}^*$ , which is defined as:

$$C_{pk}^* = \frac{d^* - A^*}{3\sigma}, \quad (1)$$

where  $d^* = \min\{d_U, d_L\}$ ,  $A^* = \max\{d^*(\mu - T)/d_U, d^*(T - \mu)/d_L\}$ ,  $d_U = USL - T$  is the right hand side tolerance, and  $d_L = T - LSL$  is the left hand side tolerance. Clearly, if  $T = m$  (symmetric case), then  $d^* = d$ ,  $A^* = |\mu - m|$  and the generalization  $C_{pk}^*$  reduces to the original index  $C_{pk}$ . The factors  $d^*$  and  $A^*$  ensure that the generalization  $C_{pk}^*$  obtains its

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maximal value at  $T$  (process is on-target) regardless of whether the tolerances are symmetric or asymmetric.

**2 THE NATURAL ESTIMATOR OF  $C_{pk}^*$**

The natural estimator  $\hat{C}_{pk}^*$  of  $C_{pk}^*$  can be obtained by replacing  $\mu$  and  $\sigma$  by  $\bar{X} = \sum_{i=1}^n X_i/n$  and  $S_{n-1} = \{\sum_{i=1}^n (X_i - \bar{X})^2 / (n - 1)\}^{1/2}$  respectively in expression (1).

$$\hat{C}_{pk}^* = \frac{d^* - \hat{A}^*}{3S_{n-1}}, \tag{2}$$

where  $\hat{A}^* = \max\{d^*(\bar{X} - T)/d_U, d^*(T - \bar{X})/d_L\}$ . Under the normality assumption, Pearn and Chen (1998) showed that the  $r$ th moment (about 0) of  $\hat{C}_{pk}^*$  is:

$$E(\hat{C}_{pk}^{*r}) = \frac{\sqrt{n-1}}{3} E\left(\frac{(n-1)S_{n-1}^2}{\sigma^2}\right)^{-r/2} \sum_{j=0}^r \binom{r}{j} \left(\frac{d^*}{\sigma}\right)^j \left(-\frac{d^*}{\sqrt{n}}\right)^{r-j} \times E\left\{\max\left(\frac{Z}{d_U}, -\frac{Z}{d_L}\right)\right\}^{r-j}$$

Hence, the first two moments of  $\hat{C}_{pk}^*$  are (Pearn and Chen, 1998):

$$E\hat{C}_{pk}^* = \frac{1}{b_{n-1}} \left\{ C_{pk}^* - \frac{1}{6} \left( \frac{d^*}{d_U} + \frac{d^*}{d_L} \right) \sqrt{\frac{2}{n\pi}} \exp\left(-\frac{\delta^2}{2}\right) - \left( \frac{\Phi(-|\delta|)}{3} \right) \left[ \left( \frac{d^*}{d_U} - \frac{d^*}{d_L} \right) \left( \frac{\delta}{\sqrt{n}} \right) - \left( \frac{2A^*}{\sigma} \right) \right] \right\},$$

$$E(\hat{C}_{pk}^*)^2 = \left( \frac{n-3}{n-1} \right) \left\{ (C_{pk}^*)^2 + \left( \frac{\Phi(-|\delta|)}{3} \right) \left( \frac{d^*}{3\sigma} \right) \left( \frac{4A^*}{\sigma} \right) + \frac{1}{18n} \left[ \left( \frac{d^*}{d_U} \right)^2 + \left( \frac{d^*}{d_L} \right)^2 \right] - \frac{1}{3} \left( \frac{d^*}{3\sigma} \right) \left( \frac{d^*}{d_U} + \frac{d^*}{d_L} \right) \sqrt{\frac{2}{n\pi}} \exp\left(-\frac{\delta^2}{2}\right) + \left( \frac{\Phi(-|\delta|)}{3} \right) \left( \frac{\delta^2}{3n} \right) \times \left[ \left( \frac{d^*}{d_U} \right)^2 + \left( \frac{d^*}{d_L} \right)^2 \right] - \left( \frac{\Phi(-|\delta|)}{3} \right) \left( \frac{d^*}{3\sigma} \right) \left( \frac{2\delta}{n} \right) \left( \frac{d^*}{d_U} - \frac{d^*}{d_L} \right) - \left( \frac{2}{3} \right) \left( \frac{\Phi(-|\delta|)}{3} \right) \left( \frac{A^*}{\sigma} \right)^2 + \frac{1}{9n} \left[ \left( \frac{d^*}{d_U} \right)^2 - \left( \frac{d^*}{d_L} \right)^2 \right] \times \left[ \left( \frac{\delta}{\sqrt{2\pi}} \right) \exp\left(-\frac{\delta^2}{2}\right) + \frac{\delta}{2|\delta|} (1 - 2\Phi(-|\delta|)) \right] \right\},$$

where  $\delta = \sqrt{n}(\mu - T)/\sigma$ ,  $b_{n-1} = \sqrt{2/(n-1)}\{\Gamma[(n-1)/2]/\Gamma[(n-2)/2]\}$ .

### 3 LARGE SAMPLE PROPERTIES OF $\hat{C}_{pk}^*$

Pearn and Chen (1998) succeeded in obtaining the moments of  $\hat{C}_{pk}^*$ . Their investigation focused on processes with normal distribution. Under general conditions, the asymptotic behavior of  $\hat{C}_{pk}^*$  is desirable. Let  $X_1, X_2, \dots, X_n$  be a random sample of measurements from a process which has distribution  $G$  with mean  $\mu$  and variance  $\sigma^2$ . We note that  $\hat{C}_{pk}^*$  is a continuous function of  $(\bar{X}, S_{n-1}^2)$ , and  $(\bar{X}, S_{n-1}^2)$  converges to  $(\mu, \sigma^2)$  in probability. Therefore,  $\hat{C}_{pk}^*$  is a consistent estimator of  $C_{pk}^*$ .

**THEOREM 1** *Let  $X_1, X_2, \dots, X_n$  be a random sample of measurements from a process whose fourth central moment  $\mu_4$  exists and  $LSL \leq \mu \leq USL$ , then as  $n \rightarrow \infty$ ,  $\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*)$  converges to the following in distribution.*

- (a)  $N(0, \sigma_{pk1}^{*2})$ , if  $\mu > T$ ,
- (b)  $N(0, \sigma_{pk2}^{*2})$ , if  $\mu < T$ , and
- (c)  $-|V|/(3\sigma) - [d^*/(6\sigma^3)]W$ , if  $\mu = T$ ,

where

$$\begin{aligned} \sigma_{pk1}^{*2} &= \frac{1}{9} + \frac{\mu_3}{3\sigma^3} \left\{ \frac{d_L - (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d_L - (\mu - T)}{3\sigma} \right\}^2, & \text{if } USL - T > T - LSL, \\ \sigma_{pk1}^{*2} &= \frac{1}{9} + \frac{\mu_3}{3\sigma^3} \left\{ \frac{d - (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d - (\mu - T)}{3\sigma} \right\}^2, & \text{if } USL - T = T - LSL, \\ \sigma_{pk1}^{*2} &= \frac{1}{9} + \frac{\mu_3}{3\sigma^3} \left\{ \frac{d_U - (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d_U - (\mu - T)}{3\sigma} \right\}^2, & \text{if } USL - T < T - LSL, \\ \sigma_{pk2}^{*2} &= \frac{1}{9} - \frac{\mu_3}{3\sigma^3} \left\{ \frac{d_L + (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d_L + (\mu - T)}{3\sigma} \right\}^2, & \text{if } USL - T > T - LSL, \\ \sigma_{pk2}^{*2} &= \frac{1}{9} - \frac{\mu_3}{3\sigma^3} \left\{ \frac{d + (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d + (\mu - T)}{3\sigma} \right\}^2, & \text{if } USL - T = T - LSL, \\ \sigma_{pk2}^{*2} &= \frac{1}{9} - \frac{\mu_3}{3\sigma^3} \left\{ \frac{d_U + (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d_U + (\mu - T)}{3\sigma} \right\}^2, & \text{if } USL - T < T - LSL, \end{aligned}$$

$(V, W) \sim N((0, 0), \Sigma)$  with variance-covariance matrix  $\Sigma = \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}$ .

*Proof* See Appendix.

**THEOREM 2**  $\hat{C}_{pk}^*$  is asymptotically unbiased.

*Proof* From Theorem 1, we know that as  $n \rightarrow \infty$ ,  $E\{\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*)\} \rightarrow 0$ , if  $\mu \neq T$ , and  $E\{\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*)\} \rightarrow -\sqrt{2/\pi}/3$ , if  $\mu = T$ , since  $E\{|V|\} = \sigma\sqrt{(2/\pi)}$ . Therefore, as  $n \rightarrow \infty$ , then  $E(\hat{C}_{pk}^* - C_{pk}^*) = (1/\sqrt{n})E\{\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*)\} \rightarrow 0$  implies that the estimator  $\hat{C}_{pk}^*$  is asymptotically unbiased.

**THEOREM 3** *If the process characteristic follows a normal distribution  $N(\mu, \sigma^2)$ , then  $\hat{C}_{pk}^*$  is asymptotically efficient, if  $\mu \neq T$ .*

*Proof* From Theorem 1, we know that if  $\mu \neq T$ , then as  $n \rightarrow \infty$ ,  $\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*)$  converges in distribution to  $N(0, \sigma_{pk1}^{*2})$ , if  $\mu > T$ , and  $N(0, \sigma_{pk2}^{*2})$ , if  $\mu < T$ . Under normality assumption,

$\mu_3 = 0, \mu_4 = 3\sigma^4$  implies that  $\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*)$  converges in distribution to  $N(0, \sigma_{pkn}^{*2})$ , where  $\sigma_{pkn}^{*2} = (1/9) + (C_{pk}^{*2}/2)$ . The information matrix is

$$\mathbf{I}(\theta) = \mathbf{I}(\mu, \sigma) = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{bmatrix}. \text{ Since the Cramer-Rao lower bound}$$

$$\begin{bmatrix} \frac{\partial C_{pk}^*}{\partial \mu} & \frac{\partial C_{pk}^*}{\partial \sigma^2} \end{bmatrix} \frac{\mathbf{I}^{-1}(\theta)}{n} \begin{bmatrix} \frac{\partial C_{pk}^*}{\partial \mu} \\ \frac{\partial C_{pk}^*}{\partial \sigma^2} \end{bmatrix} = \frac{\sigma_{pkn}^{*2}}{n}$$

is achieved, therefore  $\hat{C}_{pk}^*$  is asymptotically efficient.

Since  $\hat{C}_{pk}^*$  is asymptotically efficient for  $\mu \neq T$ , defining  $M_3 = nm_3[(n-1)(n-2)]^{-1}$ , and  $M_4 = [n(n^2 - 2n + 3)m_4 - 3n(2n - 3)m_2^2] [(n-1)(n-2)(n-3)]^{-1}$ , where  $m_k = \sum_{j=1}^n (X_j - \bar{X})^k/n, k = 2, 3, 4$ , we can show that  $M_3, M_4$  are unbiased estimators of  $\mu_3, \mu_4$ , respectively. If  $\mu > T$ , an estimator for  $\sigma_{pk1}^{*2}$  is obtained as:

$$\hat{\sigma}_{pk1}^{*2} = \frac{1}{9} + \left( \frac{M_3}{3S_{n-1}^3} \right) \hat{C}_{pk1}^* + \left( \frac{M_4 - S_{n-1}^4}{4S_{n-1}^4} \right) \hat{C}_{pk1}^{*2}, \tag{3}$$

where  $\hat{C}_{pk1}^* = [d^* - (\bar{X} - T)]/(3S_{n-1})$ . Using this estimator, an approximate  $100(1 - \alpha)\%$  one-sided confidence interval of  $C_{pk}^*$  can be constructed as:

$$\left( \hat{C}_{pk1}^* - \frac{\hat{\sigma}_{pk1}^*}{\sqrt{n}} z_\alpha, \infty \right), \tag{4}$$

where  $z_\alpha$  represents the upper  $\alpha$ th quantile of the standard normal distribution.

Similarly, if  $\mu < T$ , an estimator for  $\sigma_{pk2}^{*2}$  is obtained as:

$$\hat{\sigma}_{pk2}^{*2} = \frac{1}{9} - \left( \frac{M_3}{3S_{n-1}^3} \right) \hat{C}_{pk2}^* + \left( \frac{M_4 - S_{n-1}^4}{4S_{n-1}^4} \right) \hat{C}_{pk2}^{*2}, \tag{5}$$

where  $\hat{C}_{pk2}^* = [d^* + (\bar{X} - T)]/(3S_{n-1})$ . Using the estimator and its limiting distribution, an approximate  $100(1 - \alpha)\%$  one-sided confidence interval of  $C_{pk}^*$  can be constructed as:

$$\left( \hat{C}_{pk2}^* - \frac{\hat{\sigma}_{pk2}^*}{\sqrt{n}} z_\alpha, \infty \right). \tag{6}$$

#### 4 AN EXAMPLE

Consider the following example taken from a manufacturer and supplier in Taiwan exporting high-end audio speaker components including rubber edge, Pulus edge, Kevlar cone, honeycomb and many others. The production specifications for a particular model of Pulus edge are  $(LSL, T, USL) = (5.650, 5.835, 5.950)$ . A total of 90 observations were collected which are displayed in Table I. If the true  $C_{pk}^*$  value fell into expressions (4) or (6), we conclude that the process is capable, otherwise, the process is incapable under the given confidence level.

TABLE I Collected Sample Data (90 Observations).

|      |      |      |      |      |      |      |      |      |      |
|------|------|------|------|------|------|------|------|------|------|
| 5.88 | 5.83 | 5.84 | 5.80 | 5.89 | 5.81 | 5.84 | 5.83 | 5.82 | 5.83 |
| 5.81 | 5.82 | 5.85 | 5.81 | 5.81 | 5.81 | 5.84 | 5.82 | 5.80 | 5.84 |
| 5.86 | 5.87 | 5.82 | 5.87 | 5.80 | 5.81 | 5.85 | 5.84 | 5.83 | 5.86 |
| 5.81 | 5.81 | 5.82 | 5.83 | 5.85 | 5.80 | 5.86 | 5.82 | 5.86 | 5.83 |
| 5.80 | 5.77 | 5.82 | 5.85 | 5.84 | 5.82 | 5.85 | 5.81 | 5.86 | 5.79 |
| 5.84 | 5.83 | 5.80 | 5.83 | 5.81 | 5.83 | 5.81 | 5.85 | 5.83 | 5.88 |
| 5.82 | 5.87 | 5.80 | 5.82 | 5.83 | 5.81 | 5.84 | 5.79 | 5.85 | 5.85 |
| 5.84 | 5.84 | 5.80 | 5.82 | 5.84 | 5.85 | 5.86 | 5.81 | 5.81 | 5.85 |
| 5.86 | 5.81 | 5.81 | 5.83 | 5.85 | 5.85 | 5.82 | 5.83 | 5.86 | 5.81 |

We note that the mid-point  $m = (\text{USL} + \text{LSL})/2 = 5.800$  and the half length of the specification interval  $d = (\text{USL} - \text{LSL})/2 = 0.150$ ,  $d^* = \min(\text{USL} - T, T - \text{LSL}) = 0.125$ . The sample mean  $\bar{x} = \sum_{i=1}^{90} x_i/90 = 5.83033$ , sample standard deviation  $s_{n-1} = \{\sum_{i=1}^{90} (x_i - \bar{x})^2 / (90 - 1)\}^{1/2} = 0.02334$ ,  $\hat{A}^* = 0.00290$ ,  $m_2 = 5.39 \times 10^{-4}$ ,  $m_3 = 2.49 \times 10^{-6}$ ,  $m_4 = 7.64 \times 10^{-7}$ ,  $M_2 = 5.45 \times 10^{-4}$ ,  $M_3 = 2.87 \times 10^{-8}$  and  $M_4 = 7.79 \times 10^{-7}$ . Since the sample mean is less than the target value,  $\hat{C}_{pk2}^* = [(\text{USL} - T) + (\bar{x} - T)] / (3s_{n-1}) = 1.57563$ . Apply the expression (5), we obtained  $\hat{\sigma}_{pk2}^* = 1.11739$ . From the expression (6), an approximate 95% one-sided confidence bound of  $C_{pk}^*$  is  $[1.38, \infty)$ . A process with  $C_{pk}^* < 1.00$  is called “inadequate”, a process with  $1.00 \leq C_{pk}^* < 1.33$  is called “capable”, a process with  $1.33 \leq C_{pk}^* < 1.50$  is called “satisfactory”, a process with  $1.50 \leq C_{pk}^* < 2.00$  is called “excellent”. In this example, we have 95% confidence to claim that the process is satisfactory.

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## APPENDIX

### Proof of Theorem 1

(I) If  $\text{USL} - T > T - \text{LSL}$ , then  $C_{pk}^* = (d_L - |\mu - T|) / (3\sigma)$  and  $\hat{C}_{pk}^* = (d_L - |\bar{X} - T|) / (3S_{n-1})$ .

Case (a) We first consider the case with  $\mu > T$ . We define the function  $g_{L1}(x, y) = (d_L - (x - T)) / (3\sqrt{y})$ , where  $x > T$  and  $y > 0$ . Note that  $g_{L1}$  is a real-valued and differentiable function for  $x > T$  and  $y > 0$ , with

$$\left. \frac{\partial g_{L1}}{\partial x} \right|_{(\mu, \sigma^2)} = -\frac{1}{3\sigma}, \quad \text{and} \quad \left. \frac{\partial g_{L1}}{\partial y} \right|_{(\mu, \sigma^2)} = -\frac{1}{2\sigma^2} \left\{ \frac{d_L - (\mu - T)}{3\sigma} \right\}.$$

Define  $D_{L1} = \left( \frac{\partial g_{L1}}{\partial x} \Big|_{(\mu, \sigma^2)}, \frac{\partial g_{L1}}{\partial y} \Big|_{(\mu, \sigma^2)} \right)$ . Note that  $D_{L1} \neq (0, 0)$ . Then, we have

$$\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*) = \sqrt{n}\{g_{L1}(\bar{X}, S_{n-1}^2) - g_{L1}(\mu, \sigma^2)\}$$

converges to  $N(0, \sigma_{pk1}^{*2})$  in distribution (Serfling, 1980), where

$$\sigma_{pk1}^{*2} = D_{L1}\Sigma D_{L1} = \frac{1}{9} + \frac{\mu_3}{3\sigma^3} \left\{ \frac{d_L - (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d_L - (\mu - T)}{3\sigma} \right\}^2.$$

*Case (b)* For the case with  $\mu < T$ , we define the function

$$g_{L2}(x, y) = \frac{d_L + (x - T)}{3\sqrt{y}},$$

where  $x < T$  and  $y > 0$ . Note that  $g_{L2}$  is also a real-valued and differentiable function for all  $x < T$  and  $y > 0$ , with

$$\frac{\partial g_{L2}}{\partial x} \Big|_{(\mu, \sigma^2)} = \frac{1}{3\sigma}, \quad \text{and} \quad \frac{\partial g_{L2}}{\partial y} \Big|_{(\mu, \sigma^2)} = -\frac{1}{2\sigma^2} \left\{ \frac{d_L + (\mu - T)}{3\sigma} \right\}.$$

Define  $D_{L2} = \left( \frac{\partial g_{L2}}{\partial x} \Big|_{(\mu, \sigma^2)}, \frac{\partial g_{L2}}{\partial y} \Big|_{(\mu, \sigma^2)} \right)$ . Note that  $D_{L2} \neq (0, 0)$ . Then, we have

$$\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*) = \sqrt{n}\{g_{L2}(\bar{X}, S_{n-1}^2) - g_{L2}(\mu, \sigma^2)\} \text{ converges to } N(0, \sigma_{pk2}^{*2})$$

in distribution (Serfling, 1980), where

$$\sigma_{pk2}^{*2} = D_{L2}\Sigma D_{L2} = \frac{1}{9} - \frac{\mu_3}{3\sigma^3} \left\{ \frac{d_L + (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d_L + (\mu - T)}{3\sigma} \right\}^2.$$

*Case (c)* For the case with  $\mu = T$ ,

$$\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*) = -\frac{\sqrt{n}|\bar{X} - \mu|}{3S_{n-1}} - \frac{d_L}{3\sigma(\sigma + S_{n-1})} \left\{ \frac{\sqrt{n}(S_{n-1}^2 - \sigma^2)}{S_{n-1}} \right\}.$$

Since  $\sqrt{n}(\bar{X} - \mu, S_{n-1}^2 - \sigma^2)$  converges to  $(V, W) \sim N((0, 0), \Sigma)$  in distribution (Serfling, 1980), and  $(-1/(3S_{n-1}), -d_L/(3\sigma(\sigma + S_{n-1})S_{n-1}))$  converges to  $(-1/(3\sigma), -d_L/(6\sigma^3))$  in probability (Serfling, 1980), then, we have  $\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*)$  converges to  $W_L^* = -|V|/(3\sigma) - (d_L/(6\sigma^3))W$  in distribution (Serfling, 1980), where  $(V, W) \sim N((0, 0), \Sigma)$  with variance-covariance matrix  $\Sigma = \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}$ .

Therefore, if  $USL - T > T - LSL$ , then  $\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*)$  converges to the following in distribution.

- (a)  $N(0, \sigma_{pk1}^{*2})$ , if  $\mu > T$ ,
- (b)  $N(0, \sigma_{pk2}^{*2})$ , if  $\mu < T$ ,
- (c)  $-|V|/(3\sigma) - (d_L/(6\sigma^3))W$ , if  $\mu = T$ ,

where

$$\sigma_{pk1}^{*2} = \frac{1}{9} + \frac{\mu_3}{3\sigma^3} \left\{ \frac{d_L - (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d_L - (\mu - T)}{3\sigma} \right\}^2,$$

$$\sigma_{pk2}^{*2} = \frac{1}{9} - \frac{\mu_3}{3\sigma^3} \left\{ \frac{d_L + (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d_L + (\mu - T)}{3\sigma} \right\}^2,$$

$(V, W) \sim N((0, 0), \Sigma)$  with variance-covariance matrix  $\Sigma = \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}$ .

(II) If  $USL - T < T - LSL$ , then  $C_{pk}^* = (d_U - |\mu - T|)/(3\sigma)$  and  $\hat{C}_{pk}^* = (d_U - |\bar{X} - T|)/(3S_{n-1})$ .

Applying the same techniques used in (I) with

$$g_{U1}(x, y) = \frac{d_U - (x - T)}{3\sqrt{y}}, \quad \text{for } \mu > T, \quad g_{U2}(x, y) = \frac{d_U + (x - T)}{3\sqrt{y}}, \quad \text{for } \mu < T,$$

and

$$W_U^* = -\frac{|V|}{3\sigma} - \left( \frac{d_U}{6\sigma^3} \right) W, \quad \text{for } \mu = T.$$

As  $n \rightarrow \infty$ ,  $\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*)$  converges to  $\begin{cases} N(0, \sigma_{pk1}^{*2}) & \text{for } \mu > T. \\ N(0, \sigma_{pk2}^{*2}) & \text{for } \mu < T. \end{cases}$  in distribution.

Therefore, if  $USL - T < T - LSL$ , then as  $n \rightarrow \infty$ ,  $\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*)$  converges to the following in distribution.

- $N(0, \sigma_{pk1}^{*2})$ , if  $\mu > T$ ,
- $N(0, \sigma_{pk2}^{*2})$ , if  $\mu < T$ ,
- $-|V|/(3\sigma) - (d_U/6\sigma^3)W$ , if  $\mu = T$ ,

where

$$\sigma_{pk1}^{*2} = \frac{1}{9} + \frac{\mu_3}{3\sigma^3} \left\{ \frac{d_U - (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d_U - (\mu - T)}{3\sigma} \right\}^2,$$

$$\sigma_{pk2}^{*2} = \frac{1}{9} - \frac{\mu_3}{3\sigma^3} \left\{ \frac{d_U + (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d_U + (\mu - T)}{3\sigma} \right\}^2,$$

$(V, W) \sim N((0, 0), \Sigma)$  with variance-covariance matrix  $\Sigma = \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}$ .

(III) If  $USL - T = T - LSL$ , then  $C_{pk}^* = (d - |\mu - m|)/(3\sigma)$ , and  $\hat{C}_{pk}^* = (d - |\bar{X} - m|)/(3S_{n-1})$ .

Again, we apply the same technique with

$$g_{M1}(x, y) = \frac{d - (x - m)}{3\sqrt{y}}, \quad \text{for } \mu > m, \quad g_{M2}(x, y) = \frac{d + (x - m)}{3\sqrt{y}}, \quad \text{for } \mu < m,$$



and

$$W_M^* = -\frac{|V|}{(3\sigma)} - \left(\frac{d}{6\sigma^3}\right)W, \text{ for } \mu = m.$$

Then as  $n \rightarrow \infty$ ,  $\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*)$  converges to  $\begin{cases} N(0, \sigma_{pk1}^{*2}) & \text{for } \mu > m. \\ N(0, \sigma_{pk2}^{*2}) & \text{for } \mu < m. \end{cases}$  in distribution.

Therefore, if  $USL - T = T - LSL$ , then as  $n \rightarrow \infty$ ,  $\sqrt{n}(\hat{C}_{pk}^* - C_{pk}^*)$  converges to the following in distribution.

- (a)  $N(0, \sigma_{pk1}^{*2})$ , if  $\mu > T$ ,
- (b)  $N(0, \sigma_{pk2}^{*2})$ , if  $\mu < T$ ,
- (c)  $-|V|/3\sigma - (d/6\sigma^3)W$ , if  $\mu = T$ ,

where

$$\sigma_{pk1}^{*2} = \frac{1}{9} + \frac{\mu_3}{3\sigma^3} \left\{ \frac{d - (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d - (\mu - T)}{3\sigma} \right\}^2,$$

$$\sigma_{pk2}^{*2} = \frac{1}{9} - \frac{\mu_3}{3\sigma^3} \left\{ \frac{d + (\mu - T)}{3\sigma} \right\} + \frac{\mu_4 - \sigma^4}{4\sigma^4} \left\{ \frac{d + (\mu - T)}{3\sigma} \right\}^2,$$

$(V, W) \sim N((0, 0), \Sigma)$  with variance-covariance matrix  $\Sigma = \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}$ .