International Journal of Bifurcation and Chaos, Vol. 12, No. 12 (2002) 2957–2966

ON THE TEMPLATES CORRESPONDING TO CYCLE-SYMMETRIC CONNECTIVITY IN CELLULAR NEURAL NETWORKS

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Received July 12, 2001; Revised October 26, 2001

In the architecture of cellular neural networks (CNN), connections among cells are built on linear coupling laws. These laws are characterized by the so-called templates which express the local interaction weights among cells. Recently, the complete stability for CNN has been extended from symmetric connections to cycle-symmetric connections. In this presentation, we investigate a class of two-dimensional space-invariant templates. We find necessary and sufficient conditions for the class of templates to have cycle-symmetric connections. Complete stability for CNN with several interesting templates is thus concluded.

Keywords: Neural network; complete stability; cycle-symmetric matrix.

1. Introduction

This investigation aims to explore the structures of templates in cellular neural networks (CNN), which yield cycle-symmetric connectivity among cells, and hence complete stability for the networks. We shall illustrate our results in the CNN proposed by Chua and Yang [1988]. Assume that the model is formulated on a two-dimensional $n_1 \times n_2$ array $T_\mathbf{n} := \{(i, j) \in \mathbf{Z}^2 | 1 \le i \le n_1,\ 1 \le j \le n_2\}$. The circuit equation of a cell is given by

$$\frac{dx_{i,j}}{dt} = -x_{i,j} + \sum_{(k,\,\ell)\in N_r(i,\,j)} A(i,\,j;\,k,\,\ell) f_{k,\ell}(x_{k,\ell}) + b_{i,j}, \quad (i,\,j) \in T_\mathbf{n}\,, \tag{1}$$

where $N_r(i, j)$ represents the r-neighborhood of the cell at (i, j). That is,

$$N_r(i,\,j) = \{(k,\,\ell) | \max(|k-i|,\,|\ell-j|) \le r\}\,.$$

The feedback operator is represented by real numbers $A(i, j; k, \ell)$, $(i, j) \in T_\mathbf{n}$, $(k, \ell) \in N_r(i, j)$, and these real numbers constitute the template for CNN. This template describes the connection weights among cells. If $(k, \ell) \in N_r(i, j)$, for some $(i, j) \in T_\mathbf{n}$ and (k, ℓ) not in $T_\mathbf{n}$, then $x_{k,\ell}$ in (1) is determined by the imposed boundary condition, *cf.* [Thiran, 1993; Shih, 2000]. $b_{i,j}$ represent the terms from the control operator and threshold. $f_{k,l}$ is called output function. The standard output function is given by $f_{k,\ell}(\xi) = f(\xi) := 1/2(|\xi+1|+|\xi-1|)$, $(k, \ell) \in N_r(i, j)$, $(i, j) \in T_\mathbf{n}$. This piecewise-linear function f results in lack of smoothness for the

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Work partially supported by the National Science Council of Taiwan, R.O.C., the National Center for Theoretical Sciences, R.O.C., and the Lee and MTI Center for Networking Research.
†Work partially supported by The National Science Council of Taiwan, R.O.C.

vector field in (1). For details, please see [Chua & Yang, 1988; Chua, 1998; Lin & Shih, 1999].

Since there are finitely many cells, the indices $\{(i, j)\}$ can be rearranged into a one-dimensional setting. Such an arrangement can be expressed by a bijection between the corresponding index sets, namely, from $T_{\mathbf{n}}$ onto $\{i \in \mathbf{Z}^1 | 1 \le i \le n_1 \times n_2\}$. With this rearrangement, the coordinates $\{x_{i,j}\}$, $(i, j) \in T_{\mathbf{n}}$ become $\{x_i\}$, $1 \le i \le n$, $n := n_1 \times n_2$. Equation (1) is then recast into the following form.

$$\frac{d\mathbf{x}}{dt} = \mathcal{F}(\mathbf{x}) := -\mathbf{x} + \mathbf{A}\mathbf{y} + \mathbf{b}\,, \tag{2}$$

where $\mathbf{y} := (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$, $\mathbf{b}$ is a constant vector. In respecting (1), the $n \times n$ matrix $\mathbf{A}$ in (2) is generated from $A(i, j; k, \ell)$. We call $\mathbf{A}$ the matrix of connection weights or connection matrix. Notably, CNN with coupling (templates) of any dimension can be put into the form (2), as long as there are finitely many cells. The shortcoming for the expression (2) is that the templates features, that is, the local coupling relation among cells can no longer be read from the equation.

Equation (2) can also be interpreted as a matrix form of (1). Indeed, let each of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{b}$ be an $n_1 \times n_2$ matrix and let $\mathbf{A}$ be a linear operator on the space of $n_1 \times n_2$ matrices. That $\mathbf{A}$ acts on $\mathbf{y}$ yields an $n_1 \times n_2$ matrix $\mathbf{A}\mathbf{y}$, which is defined by $(\mathbf{A}\mathbf{y})_{ij} = \sum_{(k,\ell)\in T_{\mathbf{n}}} \mathbf{A}(i, j; k, \ell) y_{k,\ell} = \sum_{(k,\ell)\in T_{\mathbf{n}}} \mathbf{A}(i, j; k, \ell) f_{k,\ell}(x_{k,\ell})$, where

$$\mathbf{A}(i,j;k,\ell) = \begin{cases} A(i,j;k,\ell) & \text{if } \ (k,\ell) \in N_r(i,j), \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Indeed, $\mathbf{A}$ can also be regarded as an $n \times n$ matrix with each of its column and row indexed by elements of $T_{\mathbf{n}}$. For notational convenience, if $u = (i, j)$, $v = (k, \ell)$, we write $\mathbf{A}(u; v)$ for the (u, v)-entry of $\mathbf{A}$.

By *complete stability*, we mean that every solution of the system tends to an equilibrium as time goes to infinity. The complete stability for (1) with standard output function was first studied in [Chua & Yang, 1988] when the model was proposed. [Lin & Shih, 1999] has overcome the nondifferentiability of the Lyapunov function and provided a rigorous proof for complete stability of CNN in the generic case, that is, the isolated equilibria case. The proof relies on the construction of a Lyapunov function and the use of LaSalle's invariance principle. The basic assumption in these investigations is the symmetry condition of the circuit parameters in (1):

$$A(i, j; k, \ell) = A(k, \ell; i, j), \quad \text{for all } \ (k, \ell), \ \ (i, j) \in T_{\mathbf{n}}\,. \tag{4}$$

With this assumption, if (1) is imposed with certain discrete-type boundary conditions, $\mathbf{A}$ is always symmetric, as (1) is reformulated into the form (2). Recently, the complete stability for (2) (hence (1)) has been extended to the so-called cycle-symmetric connections, cf. [Shih, 2001]. Cycle-symmetric matrices can be described as follows. Let $\mathbf{A} = [\mathbf{A}(u; v)]$ be an $n \times n$ matrix with either $\mathbf{A}(u; v) = 0$ or $\mathbf{A}(u; v)\mathbf{A}(v; u) \neq 0$ for $u, v \in T_{\mathbf{n}}$. There corresponds an undirected graph whose vertex v is joined to the vertex u by the edge $\overline{uv}$ if and only if $\mathbf{A}(u; v) \neq 0$ and $\mathbf{A}(v; u) \neq 0$. With the abuse of notation, we denote this graph also by $T_{\mathbf{n}}$. Let $u_1, \ldots, u_n$ be n distinct vertices. Then the sequence $u_1 u_2 \cdots u_\ell u_1$ is a *cycle* (of length ℓ) if any two consecutive vertices have an edge. Sometimes we treat the cycle as the edge set $\{\overline{u_1u_2}, \overline{u_2u_3}, \ldots, \overline{u_{\ell-1}u_\ell}, \overline{u_\ell u_1}\}$. An $n \times n$ matrix $\mathbf{A}$ is *cycle-symmetric* if $\mathbf{A}$ satisfies the following two conditions (H_1), (H_2).

$$(H_1) \quad \mathbf{A}(u; v)\mathbf{A}(v; u) > 0, \quad \text{if } \ \mathbf{A}(u; v) \neq 0\,, \tag{5}$$

$$(H_2) \qquad \prod_{\overline{uv}\in C} \mathbf{A}(u; v) = \prod_{\overline{uv}\in C} \mathbf{A}(v; u), \quad \text{for any cycle } \ C\,, \tag{6}$$

where Π denotes the product. $\mathbf{A}$ is called *sign-symmetric* if $\mathbf{A}$ satisfies (H_1). It is straightforward to verify that symmetric $\mathbf{A}$ satisfies (H_1) and (H_2).

Convergence of dynamics (complete stability herein) for several neural networks has been extended to connection matrices satisfying (H_1) and (H_2), see [Fiedler & Gedeon, 1998; Gedeon, 1999]. An interesting linear algebra theorem states that a square matrix $\mathbf{A}$ is similar to a symmetric matrix by a positive diagonal matrix (that is, there exists $D = \mathrm{diag}(d_1, \ldots, d_n)$, $d_i > 0$ such that $D\mathbf{A}D^{-1}$ is symmetric) if and only if $\mathbf{A}$ satisfies both (H_1) and (H_2). This theorem has simplified the verifications of convergence in the above-mentioned studies, *cf.* [Shih & Weng, 2000].

Notably, in addition to extending the complete stability to cycle-symmetric connection matrices, Shih [2001] has also addressed the complete stability for (2) with other sigmoidal output functions, and the case of nonisolated equilibria. As for the characterization and classification of equilibria and their output patterns for CNN, the readers are referred to [Juang & Lin, 1997, 2000; Shih, 1998; Ban *et al.*, 2001].

In this investigation, we shall explore the structures of space-invariant templates (see Sec. 2) which yield cycle-symmetric connection among cells. Restated, we plan to study the question: what kind of $A(i, j; k, \ell)$ in (1) would yield a cycle-symmetric $\mathbf{A}$ in (2)?

This presentation is organized as follows. A key lemma of this work is given in Sec. 2. The main result of this investigation, on two-dimensional templates corresponding to cycle-symmetric connectivity, is summarized in Theorem 3.2 in Sec. 3. We further extend our results to CNN with one-dimensional and three-dimensional templates and other two-dimensional templates in Sec. 4. We compare our results with earlier studies on the templates and connection matrices which yield complete stability or almost complete stability for CNN in Sec. 5.

2. Preliminaries

In this presentation, we plan to investigate the templates which have uniform local behaviors, that is, the connection among cells depends only on their relative positions. Restated, $A(i, j; k, \ell) = m(k-i; \ell-j)$. Such a kind of template is called space-invariant template. A 3×3 space-invariant template is characterized by a 3×3 matrix

$$\mathbf{M} = \left\{ \begin{matrix} m(-1;\,1) & m(0;\,1) & m(1;\,1) \\ m(-1;\,0) & m(0;\,0) & m(1;\,0) \\ m(-1;\,-1) & m(0;\,-1) & m(1;\,-1) \end{matrix} \right\}. \tag{7}$$

Accordingly,

$$\mathbf{A}(i, j; k, \ell) = \begin{cases} m(k-i;\, \ell-j) & \text{if } |k-i|\le 1 \text{ and } |\ell-j|\le 1\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{8}$$

Notably, the index of the entry of $\mathbf{M}$ is not standard. We use this index to make the representation of $\mathbf{A}$ in (8) more realizable. Observe that this is the case $r = 1$ in Eq. (1). Note that if $m(i; j) = m(-i; -j)$, then $\mathbf{A}$ is a symmetric matrix, that is, $\mathbf{A}(u; v) = \mathbf{A}(v; u)$, where $u = (i, j)$, $v = (k, \ell)$. Recall that

$$\mathbf{A}(u; v) \neq 0 \text{ iff } \overline{uv} \text{ is an edge in the graph } T_{\mathbf{n}}\,. \tag{9}$$

In this case, $m(v-u)\neq 0$ by (8), and we say that the edge $\overline{uv}$ has *type* "$v-u$", where the subtraction is the usual vector subtraction. Type "$u-v$" is called the *opposite* type of "$v-u$".

We shall determine all the templates $\mathbf{M}$ in (7) that ensure $\mathbf{A}$ to be cycle-symmetric in the next section. The complete stability for CNN with these templates is thus concluded. We need the following technical lemma to prove our main result. This lemma states that a cycle C of $T_{\mathbf{n}}$ of length at least 5 can be decomposed into two cycles by a path of $T_{\mathbf{n}}$, *cf*. Figs. 1 and 2. We omit the proof since it is not particularly illuminating.

Lemma 2.1. *Let $C : u_1u_2\cdots u_su_1$ be a cycle of length $s \ge 5$ in $T_{\mathbf{n}}$. Then there is a path $P : v_1\cdots v_\tau$*

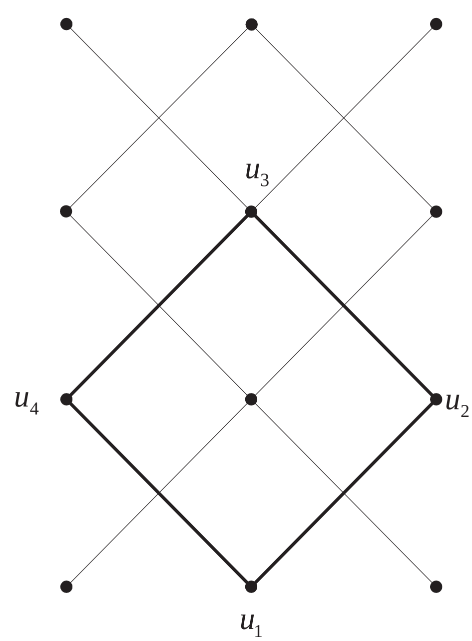

Fig. 1. The cycle $u_1u_2u_3u_4u_1$ cannot be decomposed into two cycles.

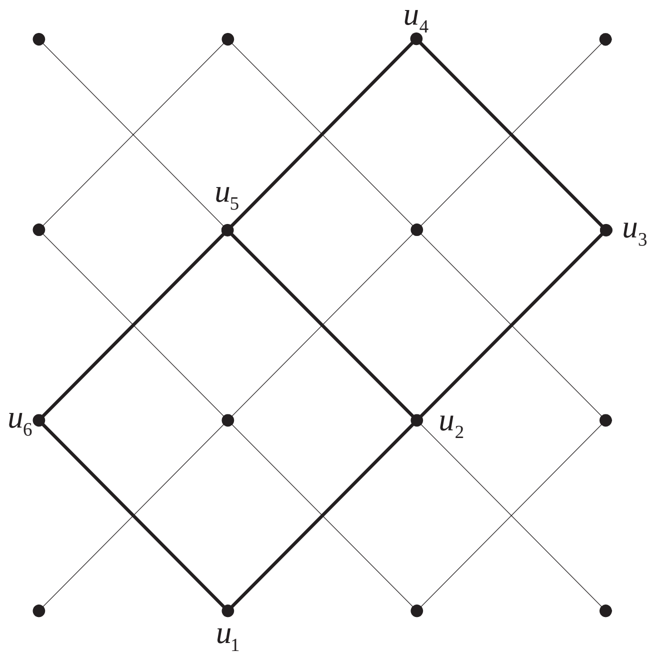

Fig. 2. The cycle $u_1u_2u_3u_4u_5u_6u_1$ is decomposed into two cycles by the edge $\overline{u_2u_5}$.

in $T_{\mathbf{n}}$ such that the following (a) and (b) hold:

(a) $v_1 = u_p$ *and* $v_\tau = u_q$ *for some* p, q $(1 \le p \le q-1 \le s-1)$.
(b) *If* $\tau = 2$, *then edge* $\overline{v_1v_2} \ne$ *edge* $\overline{u_1u_s}$. *If* $\tau > 2$, *then* $v_2, \ldots, v_{\tau-1}$ *belong to the bounded region inside of* C.

In fact, the path P in Lemma 2.1 can be chosen such that each of its edge has the same or the opposite type of some edge in the cycle C.

3. The Main Result

Throughout this section, we assume that $\mathbf{A}$ is defined by (8), and

$$m(i;\, j) = 0 \ \text{ if and only if } \ m(-i;\, -j) = 0 \quad (i,\, j = -1,\, 0,\, 1)\,. \tag{10}$$

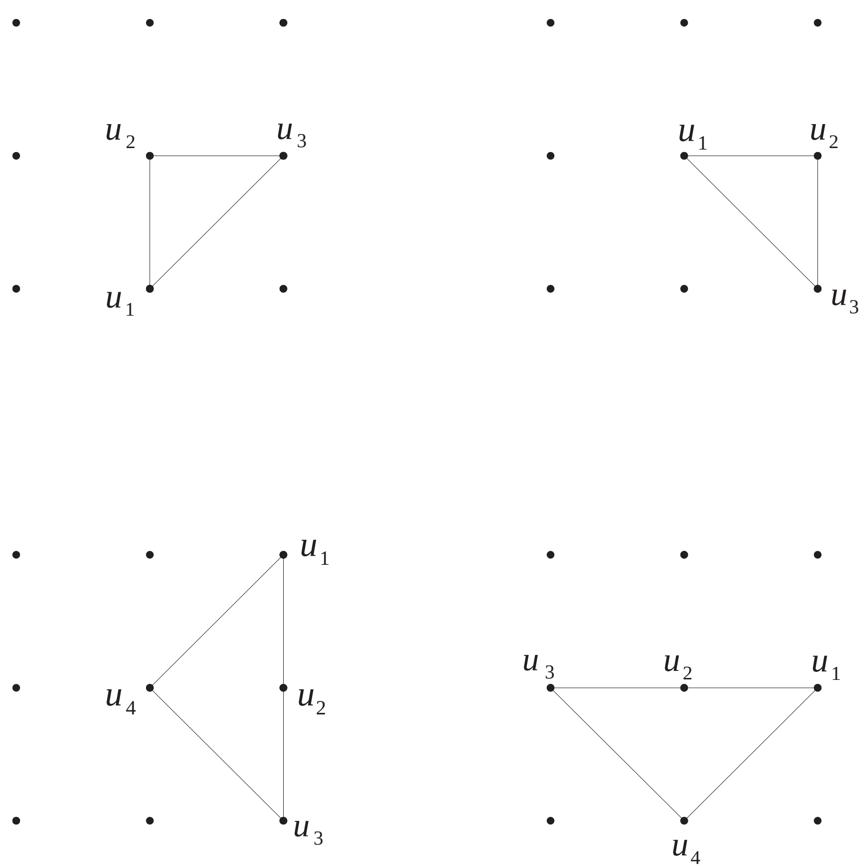

Fig. 3. Only four basic cycles C need to be checked in (6), if $\mathbf{A}$ is generated from a 3×3 space-invariant template.

The assumption (10) on $\mathbf{M}$ ensures the condition $\mathbf{A}(i, j; k, \ell) = 0$ if and only if $\mathbf{A}(k, \ell; i, j) = 0$. The following lemma is immediate from (5), (8) and (10).

Lemma 3.1. $\mathbf{A}$ *is sign-symmetric if and only if* $m(i; j)m(-i; -j) \geq 0$, $i, j = -1, 0, 1$, *and (10) holds.*

Suppose $\mathbf{A}$ is sign-symmetric. We shall prove that the following conditions (i)–(iv) are equivalent to $\mathbf{A}$ being cycle-symmetric.

Conditions

(i) $m(0; 1)m(1; 0)m(-1; -1) = m(1; 1)m(-1; 0)m(0; -1)$;

(ii) $m(1; 0)m(0; -1)m(-1; 1) = m(1; -1)m(0; 1)m(-1; 0)$;

(iii) $m(0; -1)^2m(-1; 1)m(1; 1) = m(-1; -1)m(1; -1)m(0; 1)^2$;

(iv) $m(-1; 0)^2m(1; -1)m(1; 1) = m(-1; -1)m(-1; 1)m(1; 0)^2$.

Notably, conditions (i)–(iv) say that to determine whether $\mathbf{A}$ is cycle-symmetric, one only needs to examine (6) for the four basic cycles shown in Fig. 3. In addition, in (i), $m(0; 1)m(1; 0)$ $m(-1; -1) = m(1; 1)m(-1; 0)m(0; -1)$ is exactly the condition that $\mathbf{M}$ is cycle-symmetric, and in (ii), $m(1; 0)m(0; -1)m(-1; 1) = m(1; -1)$ $m(0; 1)m(-1; 0)$ is the condition that

$$\mathbf{M}^t \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is cycle-symmetric, where $\mathbf{M}^t$ is the transpose of $\mathbf{M}$. See (6) and (7).

Theorem 3.2. *Assume that* $m(i; j)m(-i; -j) \geq 0$, $i, j = -1, 0, 1$, *and (10) holds. Then* $\mathbf{A}$ *defined by (8) is cycle-symmetric if and only if Conditions (i)–(iv) hold.*

Proof. ($\Rightarrow$) Suppose $\mathbf{A}$ is cycle-symmetric. To prove (i), set $u_1 = (1, 1)$, $u_2 = (1, 2)$, $u_3 = (2, 2)$, and apply (8) to (6) with the cycle $u_1u_2u_3u_1$. Note that both sides of (i) are zero if $u_1u_2u_3u_1$ is not a cycle. Similarly, setting $u_1 = (1, 2)$, $u_2 = (2, 2)$, $u_3 = (2, 1)$, and considering the cycle $u_1u_2u_3u_1$ prove (ii). To prove (iii), set $u_1 = (2, 3)$, $u_2 = (2, 2)$, $u_3 = (2, 1)$, $u_4 = (1, 2)$, and apply (8) to (6) with the cycle $u_1u_2u_3u_4u_1$. Similarly, (iv) can be proved by setting $u_1 = (3, 2)$, $u_2 = (2, 2)$, $u_3 = (1, 2)$, $u_4 = (2, 1)$, and considering the cycle $u_1u_2u_3u_4u_1$.

($\Leftarrow$) Suppose (i)–(iv) hold. We shall prove (6) for any cycle C: $u_1 \cdots u_su_1$ of $T_{\mathbf{n}}$. Note that the area of enclosed region of C is $V = k/2$ for some positive integer k. We prove by induction on k. Suppose $k = 1$. Observe that this is equivalent to $s = 3$, and $u_1u_2u_3u_1$ is a triangle. There are at most eight possible triangles of area $1/2$ starting with u_1. For each triangle, we find Conditions (i)–(ii) are enough to obtain (6) for the cycle C. To prove the case $k = 2$, we prove a more general case that $s = 4$. There are essentially three types of cycles of length 4: a square of area 1, a square of area 2, and a triangle of area 1 with its base of length 2. For squares, (6) is a trivial equality. For a triangle, Conditions (iii) and (iv) are enough to obtain (6) for the cycle C. Now suppose $k > 2$ and $s \geq 5$. Let $P = v_1 \cdots v_\tau$ be a path satisfying (a) and (b) in Lemma 2.1. Define the following three paths P_1, P_2, P_3.

$$\begin{aligned} &P_1 : u_1u_2 \cdots u_p \\ &P_2 : u_pu_{p+1} \cdots u_q \\ &P_3 : u_qu_{q+1} \cdots u_su_1\,, \end{aligned} \tag{11}$$

see Fig. 4. Let P^{-1} be the reversed path of P. Note that $P_1 \cup P \cup P_3$ and $P^{-1} \cup P_2$ both are cycles with smaller enclosed areas. By induction,

$$\prod_{\overline{u_ju_{j+1}} \in P_1 \cup P \cup P_3} \mathbf{A}(u_j; u_{j+1}) = \prod_{\overline{u_ju_{j+1}} \in P_1 \cup P \cup P_3} \mathbf{A}(u_{j+1}; u_j)\,, \tag{12}$$

and

$$\prod_{\overline{u_ju_{j+1}} \in P^{-1} \cup P_2} \mathbf{A}(u_j; u_{j+1}) = \prod_{\overline{u_ju_{j+1}} \in P^{-1} \cup P_2} \mathbf{A}(u_{j+1}; u_j)\,. \tag{13}$$

Multiplying the same sides of (12) and (13) together, we have

$$\prod_{\overline{u_ju_{j+1}} \in P_1 \cup P_2 \cup P_3 \cup P \cup P^{-1}} \mathbf{A}(u_j; u_{j+1}) = \prod_{\overline{u_ju_{j+1}} \in P_1 \cup P_2 \cup P_3 \cup P \cup P^{-1}} \mathbf{A}(u_{j+1}; u_j)\,. \tag{14}$$

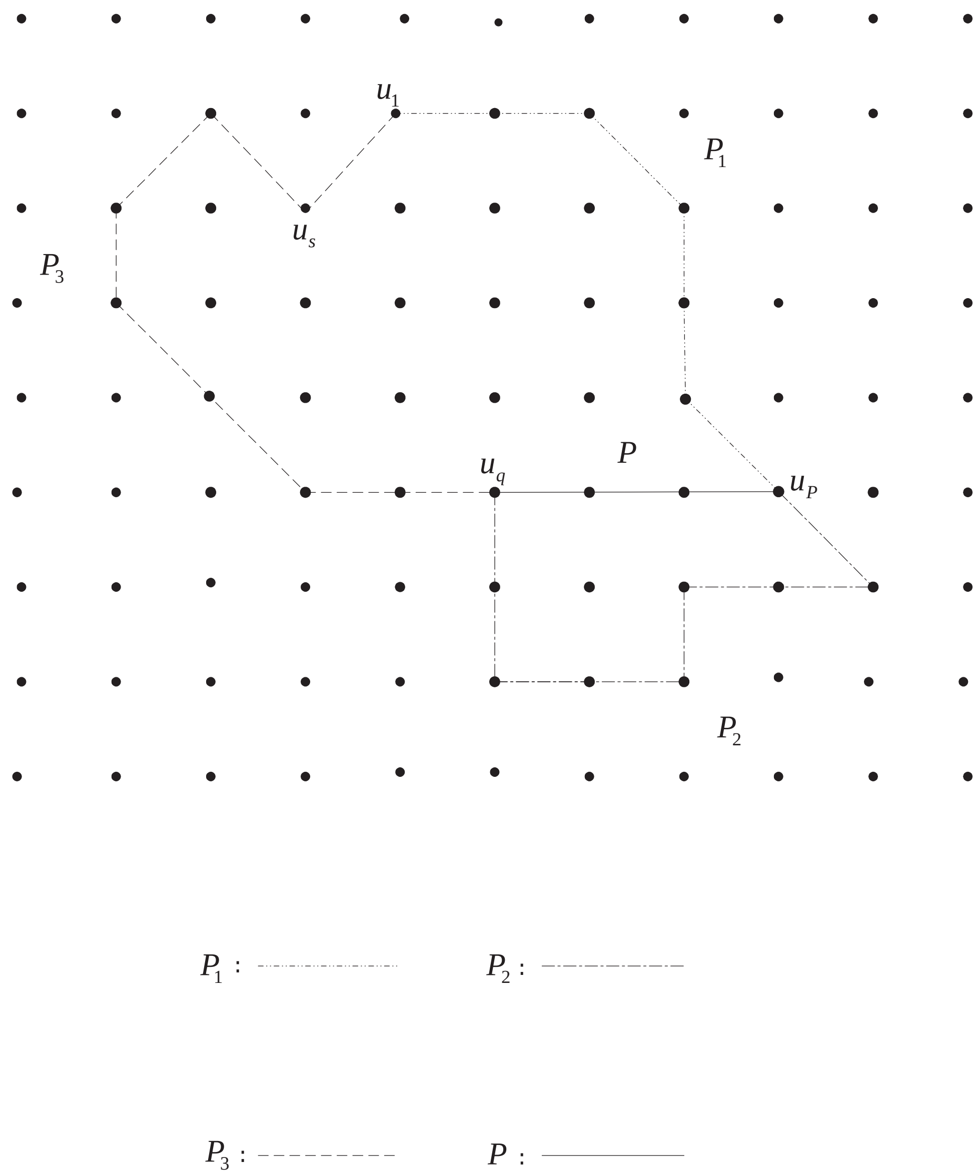

Fig. 4. A cycle $u_1 u_2 \cdots u_s u_1$ decomposed into two cycles.

Deleting the common terms, it follows that

$$\prod_{\overline{u_j u_{j+1}} \in P \cup P^{-1}} \mathbf{A}(u_j;\, u_{j+1}) = \prod_{\overline{u_j u_{j+1}} \in P \cup P^{-1}} \mathbf{A}(u_{j+1};\, u_j)\,. \qquad (15)$$

which is nonzero ensured by (9), we have (6). Notably, **A** is sign-symmetric by Lemma 3.1. This completes the proof. ■

Corollary 3.3. *In addition to the assumptions of Theorem 3.2, suppose* $m(0;\, 1)m(-1;\, 0) \neq 0$ *holds. Then* **A** *is cycle-symmetric if and only if Conditions (i) and (ii) hold.*

Proof. **A** is sign-symmetric by Lemma 3.1. Multiply the same sides of the equalities in Conditions (i) and (ii), we obtain

$$m(1;\, 0)^2 m(-1;\, -1)m(-1;\, 1)m(0;\, 1)m(0;\, -1) = m(-1;\, 0)^2 m(1;\, 1)m(1;\, -1)m(0;\, 1)m(0;\, -1)\,. \qquad (16)$$

Observe that $m(0;\, 1)m(0;\, -1) \neq 0$, by the assumptions. Dividing both sides of (16) by $m(0;\, 1)$ $m(0;\, -1)$, we have (iv). Multiply the opposite sides of equalities in Conditions (i) and (ii), we obtain

$$m(0;\,1)^2 m(-1;\,-1)m(1;\,-1)m(-1;\,0)m(1;\,0)$$
$$= m(0;\,-1)^2 m(-1;\,1)m(1;\,1)m(-1;\,0)m(1;\,0)\,. \quad (17)$$

Observe $m(-1;\,0)m(1;\,0) \neq 0$. Dividing both sides of (17) by $m(0;\,-1)m(0;\,1)$, we obtain (iii). ■

Combining Theorem 3.2 with the theorem in [Shih, 2001], we have the following conclusion.

Corollary 3.4. *The complete stability of CNN holds for any sign-symmetric templates which satisfy Conditions (i)–(iv). In particular, the complete stability of CNN holds for the following sign-symmetric square-crossed and diagonal-crossed templates*:

$$\mathbf{M} = \begin{pmatrix} 0 & m(0;\,1) & 0 \\ m(-1;\,0) & m(0;\,0) & m(1;\,0) \\ 0 & m(0;\,-1) & 0 \end{pmatrix},$$

$$\mathbf{M} = \begin{pmatrix} m(-1;\,1) & 0 & m(1;\,1) \\ 0 & m(0;\,0) & 0 \\ m(-1;\,-1) & 0 & m(1;\,-1) \end{pmatrix},$$

where $m(i;\,j)m(-i;\,-j) \geq 0$, *and* $m(-i;\,-j) = 0$ *whenever* $m(i;\,j) = 0$, $i,\,j = -1,\,0,\,1$.

Remarks

(a) For the case of symmetric template, that is, $m(i;\,j) = m(-i;\,-j)$, $-1 \leq i,\,j \leq 1$, (10) and (i)–(iv) hold obviously. Thus, cycle-symmetric connectivity indeed generalizes symmetric connectivity.

(b)
$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & m(0;\,0) & 1 \\ 2 & 0 & 2 \end{pmatrix}$$
satisfies Conditions (i)–(iii), but does not satisfy Condition (iv). This shows (iv) is necessary in Theorem 3.2. Similarly, (iii) is necessary.

4. Other Templates Corresponding to Cycle-Symmetric Connection

In this section, we would like to explore more templates corresponding to cycle-symmetric connection in CNN. We shall consider 1×3 and 1×5 one-dimensional templates, 5×5 square-crossed and diagonal-crossed two-dimensional templates. Our results can even be extended to CNN with three-dimensional templates.

The evolution equation of CNN with one-dimensional templates is given by

$$\frac{dx_i}{dt} = -x_i + \sum_{k \in N_r(i)} A(i;k) f_k(x_k) + b_i, \quad i \in T_n\,, \quad (18)$$

where $T_n := \{i \in \mathbf{Z}^1 | 1 \leq i \leq n\}$. As $r = 1$, we consider the space-invariant template denoted by $\mathbf{M}_{1\times 3} = [m_{-1} \;\; m_0 \;\; m_1]$. Restated, $A(i;\,k) = m_{k-i}$, for $i \in T_n$ and $k \in N_1(i)$. Accordingly, for each $i \in T_n$, we define $\mathbf{A}(i;\,k) = A(i;\,k)$ if $k \in N_1(i)$, and zero if $k \in T_n \setminus N_1(i)$. It is obvious that $\mathbf{A}$ is a tridiagonal $n \times n$ matrix. Assume that $m_{-1}m_1 > 0$, then the graph induced by $\mathbf{A}$ can only have cycles of length two and (H_2) in (6) is trivially satisfied.

For the case $r = 2$, we consider the following space-invariant template

$$\mathbf{M}_{1\times 5} = [m_{-2} \;\; m_{-1} \;\; m_0 \;\; m_1 \;\; m_2]\,.$$

Theorem 4.1. *CNN with space-invariant template* $\mathbf{M}_{1\times 3}$ *or* $\mathbf{M}_{1\times 5}$ *is completely stable if* $\mathbf{M}_{1\times 3}$ *satisfies* $m_{-1}m_1 > 0$, *and* $\mathbf{M}_{1\times 5}$ *satisfies* $m_{-i}m_i > 0$, $i = 1,\,2$ *and* $m_2 m_{-1}^2 = m_{-2} m_1^2$.

Proof. The case with template $\mathbf{M}_{1\times 3}$ has already been illustrated. For the template $\mathbf{M}_{1\times 5}$, consider a cycle C in T_n. When moving along the cycle C in a fixed direction, there are four types of edges: "1", "2", "−1", "−2". They represent the coordinate difference between two nodes connected by an edge. More precisely, if $\overline{u_1 u_2}$ is an edge of type "1" (resp. "2"), then u_2 is in the east of u_1 with coordinate distance 1 (resp. 2). Similarly, the edge $\overline{u_1 u_2}$ of types "−1" or "−2" means that u_2 is in the west of u_1 with coordinate distance 1 or 2, respectively. Suppose the edge of type "i" appears n_i times in C. Then necessarily,

$$n_1 + 2n_2 - n_{-1} - 2n_{-2} = 0\,, \quad (19)$$

since C is a cycle. To verify (6), we need to show

$$m_1^{n_1} m_2^{n_2} m_{-1}^{n_{-1}} m_{-2}^{n_{-2}} = m_1^{n_{-1}} m_2^{n_{-2}} m_{-1}^{n_1} m_{-2}^{n_2}\,. \quad (20)$$

It is clear that (20) holds from (19) if we assume that $m_2 m_{-1}^2 = m_{-2} m_1^2$. This completes the proof. ■

The result in Theorem 4.1 can be easily generalized to larger two-dimensional templates. Consider the following square-crossed 5×5 template

$$\mathbf{M}^{+}_{5\times 5} = \begin{pmatrix} 0 & 0 & m(0;\,2) & 0 & 0 \\ 0 & 0 & m(0;\,1) & 0 & 0 \\ m(-2;\,0) & m(-1;\,0) & m(0;\,0) & m(1;\,0) & m(2;\,0) \\ 0 & 0 & m(0;\,-1) & 0 & 0 \\ 0 & 0 & m(0;\,-2) & 0 & 0 \end{pmatrix}. \tag{21}$$

Theorem 4.2. *CNN with space-invariant template* $\mathbf{M}^{+}_{5\times 5}$ *is completely stable if* $m(-i;\,0)m(i;\,0) > 0$ *and* $m(0;\,-i)m(0;\,i) > 0$, $i = 1, 2$, *and* $m(2;\,0)m(-1;\,0)^2 = m(-2;\,0)m(1;\,0)^2$, $m(0;\,2)\times m(0;\,-1)^2 = m(0;\,-2)m(0;\,1)^2$.

The idea of the proof is rather straightforward. Pick any node u of a cycle C in the graph induced by the matrix $\mathbf{A}$ defined from $\mathbf{M}^{+}_{5\times 5}$. Since $\mathbf{M}^{+}_{5\times 5}$ is square-crossed, u can go east, west, north and south directions. Starting from u, in order to come back to itself, the total length that u travels to the east should be the same as that traveled by u to the west, while the total length that u travels to the south should coincide with the length that u travels to the north. The verification of the theorem then follows from similar argument as in the proof of Theorem 4.1.

It is not difficult to see that the above result also holds for the diagonal-crossed templates. Furthermore, we can extend the above theorems to CNN with three-dimensional templates. Notice that there are six neighboring cells connected to each cell at $(i_1, i_2, i_3) \in \mathbf{Z}^3$, as the orthogonal template $\mathbf{M}^{\perp}_{3\times 3\times 3}$ in Fig. 5 is considered. The conditions in Theorem 4.2 can easily be extended to a three-dimensional version to conclude the completely stability for CNN. Restated, if $m(1;\,0;\,0)m(-1;\,0;\,0) > 0$, $m(0;\,1;\,0)\, m(0;\,-1;\,0) > 0$, $m(0;\,0;\,1)m(0;\,0;\,-1) > 0$, then $\mathbf{M}^{\perp}_{3\times 3\times 3}$ is a template which yields complete stability for CNN. One can also consider a two-dimensional template with six connecting neighbors for each cell. For example, $\mathbf{M}$ in (7) with $m(-1;\,1) = m(1;\,-1) = 0$. In addition to $m(i;\,j)m(-i;\,-j) > 0$, the Condition (i) in Sec. 3 still needs to be satisfied for complete stability of CNN. This comparison indicates that three-dimensional template $\mathbf{M}^{\perp}_{3\times 3\times 3}$ requires weaker circuit parameter condition in concluding the complete stability of CNN.

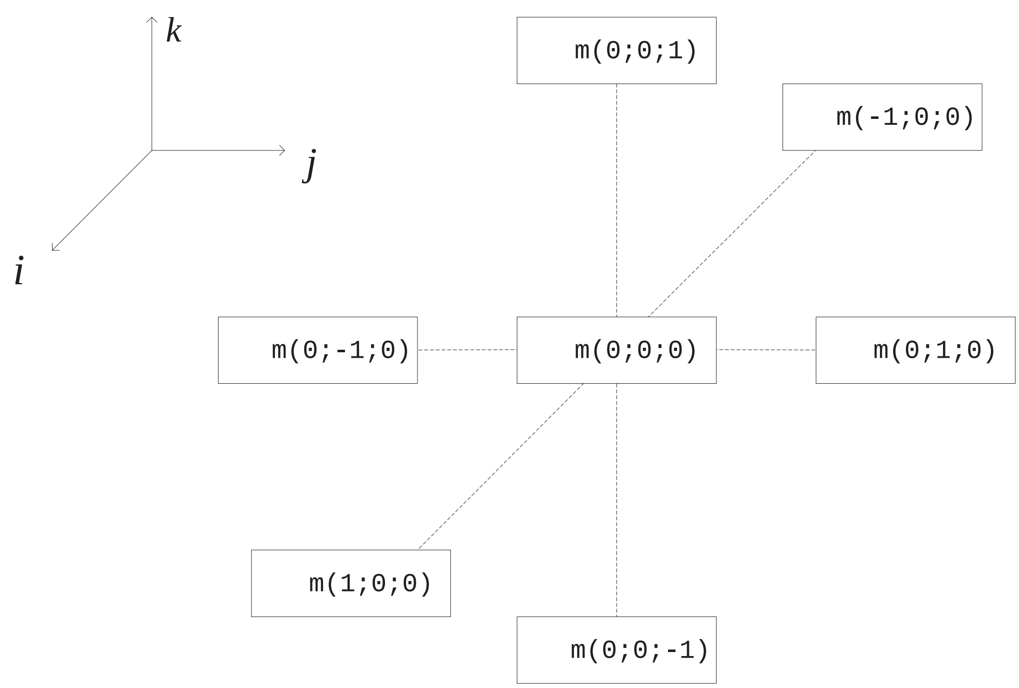

Fig. 5. An orthogonal three-dimensional template $\mathbf{M}^{\perp}_{3\times 3\times 3}$.

5. Discussions and Conclusions

There are several previous works addressing the matrices of connection weights which yield complete stability for CNN. Gilli [1994] has studied a class of connection matrices. A matrix in this class can be transformed to a symmetric matrix by multiplying a positive diagonal matrix from its left. Restated, let $\mathbf{A}$ be an $n \times n$ matrix. If there exists a positive diagonal matrix Λ such that $\Lambda\mathbf{A}$ is symmetric, then the CNN (2) with connection matrix $\mathbf{A}$ was proved to be completely stable. However, in addition to that some arguments therein remain to be rigorously justified, proving that $\lim_{t\to\infty} \dot{x}(t) = 0$ does not imply complete stability. Nevertheless, there is a linear algebra theorem which relates this class of $n \times n$ matrices to our cycle-symmetric matrices.

Gedeon [1999] studied the neural networks of Lotka–Volterra type and Grossberg's models. In deriving the convergence (complete stability herein) for the systems, he proposed the same class of matrices of connection weights. Let us describe the setting. A square matrix $\mathbf{A} = [a_{ij}]$ is called combinatorially symmetric if $a_{ij} \neq 0$ implies $a_{ji} \neq 0$. We quote Theorem 3.1 in [Gedeon, 1999] as follows.

Let $\mathbf{A}$ be a real, combinatorially symmetric, irreducible matrix. There is a positive diagonal matrix Λ such that $\Lambda\mathbf{A}$ is a symmetric matrix if and only if

(a) there is a spanning tree T of the graph $G(\mathbf{A})$ such that for every edge $e_{ij} \in T$, we have $a_{ij}a_{ji} > 0$, and
(b) if C_T is a chord cycle in the graph $G(\mathbf{A})$ with respect to tree T, then $\Pi_{C_T} a_{ij} = \Pi_{C_T} a_{ji}$.

Though (a) is a little weaker than our condition (H_1), (a) and (b) together are equivalent to (H_1) with (H_2) in (5), (6). Furthermore, the assertion that there exists a positive diagonal matrix Λ such that $\Lambda\mathbf{A}$ is a symmetric matrix is equivalent to that there exists a positive diagonal matrix D such that $D\mathbf{A}D^{-1}$ is symmetric. Indeed, if $D\mathbf{A}D^{-1}$ is symmetric with D being positive diagonal, then $DD\mathbf{A}D^{-1}D$ is symmetric, that is, $D^2\mathbf{A}$ is symmetric and $\Lambda = D^2$ is positive diagonal. On the other hand, if there is a positive diagonal matrix Λ such that $\Lambda\mathbf{A}$ is symmetric, then $(\sqrt{\Lambda})^{-1}\Lambda\mathbf{A}(\sqrt{\Lambda})^{-1} = \sqrt{\Lambda}\mathbf{A}(\sqrt{\Lambda})^{-1}$ is symmetric and $D = \sqrt{\Lambda}$ is positive diagonal. With these structures recognized, it is no surprise that some of the templates discussed in [Gilli, 1994] coincide with ours. However, our Conditions (i)–(iv) in Sec. 3 have provided more complete descriptions on the 3×3 templates which yield this class of connection matrices, hence complete stability for CNN.

As continuously differentiable increasing output functions are considered, Chua and Wu [1992] derived a class of templates for almost completely stable CNN, by applying the theory in cooperative systems [Hirsch, 1985]. The result is stated as follows. Let J be a diagonal matrix with entries either 1 or -1. Then CNN with $n\times n$ connection matrix $\mathbf{A}$ is stable almost everywhere if the equilibria are isolated and $J\mathbf{A}J = J\mathbf{A}J^{-1}$ is non-negative. With the same kind of output functions or standard output function (piecewise-linear), CNN with some of the templates discussed therein, such as sign-symmetric square-crossed or diagonal-crossed templates, are in fact completely stable, instead of merely almost stable, according to our Corollary 3.4. In [Wu & Chua, 1997], assuming continuously differentiable increasing output functions again, CNN with connection matrix $\mathbf{A}$ is proved to be completely stable if there exist diagonal matrices $D = \mathrm{diag}(d_1, \ldots, d_n)$ and $T = \mathrm{diag}(t_1, \ldots, t_n)$ with $d_i t_i > 0$ such that $D\mathbf{A}T$ is symmetric. It is not difficult to verify that such $n \times n$ matrix $\mathbf{A}$ also satisfies our (H_1) and (H_2) and vice versa. Though these two descriptions are equivalent, the conditions (H_1) and (H_2) are more concrete and explicit. Examining (H_1) and (H_2) is more straightforward than finding these diagonals D and T such that $D\mathbf{A}T$ is symmetric. Moreover, all the results in this presentation hold for CNN with standard output function as well as continuously differentiable increasing output functions, as indicated in [Shih, 2001].

As a conclusion, the present investigation has illustrated that generalizing symmetric connectivity to cycle-symmetric connectivity among cells is not merely a process of changing variables to the equation or an application of a linear algebra theorem. It indeed provides important information on the structures of space-invariant templates which yield complete stability for CNN.

Acknowledgment

The authors would like to thank the referee for calling their attention to the works by Gilli and Chua and Wu.

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