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Optimal 1-edge fault-tolerant designs for ladders ☆

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Abstract

A graph G^* is 1-edge fault-tolerant with respect to a graph G, denoted by 1-EFT(G), if every graph obtained by removing any edge from G^* contains G. A 1-EFT(G) graph is optimal if it contains the minimum number of edges among all 1-EFT(G) graphs. The kth ladder graph, L_k, is defined to be the cartesian product of the P_k and P_2 where P_n is the n-vertex path graph. In this paper, we present several 1-edge fault-tolerant graphs with respect to ladders. Some of these graphs are proven to be optimal.

Keywords: Cartesian product; Edge fault tolerance; Meshes; Ladders; Fault tolerance

1. Introduction and notations

In this paper, any *graph* means an undirected graph in which multiple edges are allowed. Let $G = (V, E)$ be a graph where V $(= V(G))$ is the vertex x of V, $\deg_G(x)$ denotes its degree in G. Let E' be a subset of E. We use $G - E'$ to denote the spanning subgraph of G with its edge set to be $E - E'$. For convenience, $G - e$ denotes $G - \{e\}$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *cartesian product* of G_1 and G_2, denoted by $G_1 \times G_2$, is the smallest graph with the vertex set $V_1 \times V_2$ such that the subgraph induced by $V_1 \times \{v_2\}$ is isomorphic to G_1 for every $v_2 \in V_2$, and the subgraph induced by $\{v_1\} \times V_2$ is isomorphic to G_2 for every $v_1 \in V_1$.

Motivated by the study of computers and communication networks that tolerate failure of their components, Harary and Hayes [6] have formulated the concept of edge fault tolerance in graphs. Given a *target graph* $G = (V, E)$, let $G^* = (V, E^*)$ be a spanning supergraph of G. G^* is said to be k-EFT(G), if $G^* - F$ contains a subgraph isomorphic to G, which is called a *reconfiguration* for k-edge fault F (or simply *reconfiguration*), for any $F \subset E^*$ and $|F| = k$. A reconfiguration can be viewed as a relabeling of vertices of G^* such that $G^* - F$ contains G. We sometimes write "G^* is a k-EFT(G) graph" as "G^* is a k-EFT(G)", for short. The graph G^* is said to be optimal if G^* contains the smallest number of edges among all k-EFT(G) graphs. We use $\text{eft}_k(G)$ to denote the difference between the number of edges in an opti-

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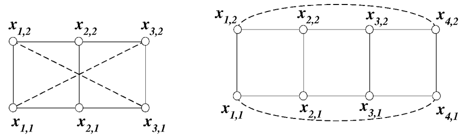
mal k-EFT(G) graph and that in G. Families of k-EFT graphs with respect to some graphs have been studied in literature [1,2,4,6–8,10–17].

The n-dimensional mesh $M(m_1, m_2, \ldots, m_n)$ is defined to be the cartesian product $P_{m_1} \times P_{m_2} \times \cdots \times P_{m_n}$ of n paths. Mesh is a widely used graph model for computer networks [9]. Farrag [4] has presented families of 1-EFT graphs with respect to the n-dimensional meshes. In [6], the graph $C(m_1, m_2, \ldots, m_n) = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_n}$ was proposed as a 1-EFT graphs with respect to the n-dimensional mesh $M(m_1, m_2, \ldots, m_n)$. We call such graphs *multidimensional torus graphs* because their construction is similar to that of the torus for $n = 2$ [5]. Harary and Hayes [6] conjectured that these multidimensional torus graphs are optimal if $m_i \geqslant 3$ for every i. There is another 1-EFT graph for the n-dimensional meshes. We assume the vertices of $M(m_1, m_2, \ldots, m_n)$ are labeled canonically. Thus, $x_{i_1, i_2, \ldots, i_n}$ is a vertex of $M(m_1, m_2, \ldots, m_n)$ if and only if $1 \leqslant i_j \leqslant m_j$ for $1 \leqslant j \leqslant n$. Moreover, $x_{i_1, i_2, \ldots, i_n}$ is adjacent to another vertex $x_{j_1, j_2, \ldots, j_n}$ if there exist a index k such that $|i_k - j_k| = 1$ and $i_t = j_t$ for all indices $t \neq k$. Then, $V_p = \{x_{i_1, i_2, \ldots, i_n} \mid i_k = 1 \text{ or } m_k \text{ for some } 1 \leqslant k \leqslant n\}$ is the set of *peripheral vertices*. Let $x_{i_1, i_2, \ldots, i_n}$ be a vertex in V_p. The *antipodal vertex* of $x_{i_1, i_2, \ldots, i_n}$ is $x_{j_1, j_2, \ldots, j_n}$, with $j_k = m_k - i_k + 1$, which is another vertex in V_p. It is easy to check that every vertex in V_p has exactly one antipodal. In $M(m_1, m_2, \ldots, m_n)$, we add the edges joining each vertex in V_p to its antipodal counterpart to form a new graph $P(m_1, m_2, \ldots, m_n)$. We call these $P(m_1, m_2, \ldots, m_n)$ *projective-plane graphs* because their construction is similar to that of the projective plane when $n = 2$ [5]. It is proven in [3] that $P(m_1, m_2, \ldots, m_n)$ is also 1-EFT($M(m_1, m_2, \ldots, m_n)$) and it contains fewer edges than that of $C(m_1, m_2, \ldots, m_n)$. Thus, the conjecture posed in [6] is disproved with these projective-plane graphs.

The projective-plane graphs are optimal for some cases but not for all. Note that every n-dimensional hypercube can be viewed as the mesh $M(2, 2, \ldots, 2)$. Our $P(2, 2, \ldots, 2)$ is actually the same 1-EFT graph as that proposed in [1,6,7,13,16]. Thus, $P(2, 2, \ldots, 2)$ is an optimal 1-EFT graph. It is proved in [3] that the graph in Fig. 1(a) is a 1-EFT($M(3, 2)$) and the graph in Fig. 1(b) is a 1-EFT($M(4, 2)$). With these two examples, we know that the projective-plane graphs may not be optimal for some cases. Furthermore,



(a) (b)

Fig. 1. (a) A 1-EFT($M(3, 2)$), L_3^*; (b) a 1-EFT($M(4, 2)$), L_4^*.

the problem of finding the optimal 1-EFT for all n-dimensional meshes remains unsolved.

In this paper, we only aim at the 1-EFT graphs for $M(k, 2)$ with $k \geqslant 2$. For simplicity, the *kth ladder graph* L_k is defined to be $M(k, 2)$. Since the projective-plane graph $P(k, 2)$ is a 1-EFT(L_k) graph, we know that $\text{eft}_1(L_k) \leqslant k$. In this paper, we will prove by constructing a 1-EFT(L_k) graph L_k^* that $\text{eft}_1(L_k) \leqslant k - 1$ if k is odd and $k \geqslant 7$, and $\text{eft}_1(L_k) \leqslant k - 2$ if k is even and $k \geqslant 4$. Moreover, we prove that $\text{eft}_1(L_2) = \text{eft}_1(L_3) = \text{eft}_1(L_4) = 2$, and $\text{eft}_1(L_5) = 3$.

2. Some 1-EFT designs for ladders

The vertices of L_k can be labeled by $x_{i,j}$ with $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant 2$ canonically. The vertices $x_{1,1}$, $x_{k,1}$, $x_{1,2}$, and $x_{k,2}$ are called the *corner vertices* of L_k. We have the following theorem:

Theorem 1. *Let L_k^* be a* 1-EFT(L_k) *graph. Then we have*

(i) $\deg_{L_k^*}(x) \geqslant 3$ *for any vertex x of L_k^*, and*
(ii) $\text{eft}_1(L_k) \geqslant 2$.

Proof. Suppose some vertex x with $\deg_{L_k^*}(x) = 2$. Let e be any edge incident with x. Obviously, $\deg_{L_k^* - e}(x) = 1$. Since $\deg_{L_k}(x) \geqslant 2$ for any vertex x of L_k, L_k is not a subgraph of $L_k^* - e$. We obtain a contradiction that L_k^* is a 1-EFT(L_k) graph. Hence, $\deg_{L_k^*}(x) \geqslant 3$. Since there are exactly four corner vertices in every L_k, we have $\text{eft}_1(L_k) \geqslant 2$. □

Corollary 1. $\text{eft}_1(L_k) > 2$ *if $k > 4$.*

Proof. It is observed that there are exactly three different ways of joining the four corner vertices in L_k with two edges, namely $\{(x_{1,1}, x_{1,2}), (x_{k,1}, x_{k,2})\}$, $\{(x_{1,1},$

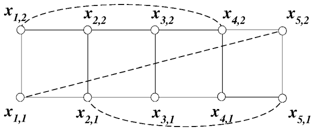


Fig. 2. A 1-EFT($M(5,2)$), L_5^*.

$x_{k,1})$, $(x_{1,2}, x_{k,2})\}$, and $\{(x_{1,1}, x_{k,2}), (x_{1,2}, x_{k,1})\}$. It is observed that none of the graphs obtained by joining two edges to the corner vertices of L_k with $k > 4$ is 1-EFT(L_k). Hence $\text{eft}_1(L_k) > 2$ if $k > 4$. □

2.1. *Optimal* 1-EFT(L_2), 1-EFT(L_3), 1-EFT(L_4) *graphs*

Let L_2^* (L_3^*, and L_4^*, respectively) be the graph $P(2,2)$ (the graph in Figs. 1(a) and 1(b), respectively). From the above discussion, L_k^* is 1-EFT(L_k) for $k = 2$, 3, and 4. Since there are exactly 2 edges added to L_k with $k = 2$, 3, and 4, by Theorem 1 these graphs are optimal. It can be verified that the optimal 1-EFT(L_k) is unique for $k = 2$, 3, and 4 by checking all the three cases joining two edges to the corner vertices of L_k. We obtain the following theorem:

Theorem 2. $\text{eft}_1(L_k) = 2$ *for* $k = 2, 3,$ *and* 4.

2.2. *An optimal* 1-EFT(L_5) *graph*

Consider the spanning supergraph L_5^* of L_5 given by $E(L_5^*) = E(L_5) \cup \{(x_{1,1}, x_{5,2}), (x_{1,2}, x_{4,2}), (x_{2,1}, x_{5,1})\}$ as shown in Fig. 2.

Edges of L_5 can be divided into the following 7 classes: namely,

$$A = \{(x_{1,1}, x_{1,2})\},$$
$$B = \{(x_{i,1}, x_{i,2}) \mid 2 \leqslant i \leqslant 4\},$$
$$C = \{(x_{5,1}, x_{5,2})\},$$
$$D = \{(x_{1,1}, x_{2,1}), (x_{1,2}, x_{2,2})\},$$
$$E = \{(x_{2,1}, x_{3,1}), (x_{2,2}, x_{3,2})\},$$
$$F = \{(x_{3,1}, x_{4,1}), (x_{3,2}, x_{4,2})\},$$
$$G = \{(x_{4,1}, x_{5,1}), (x_{4,2}, x_{5,2})\}.$$

We can reconfigure L_5 in L_5^* for any faulty edge e in A (B, C, D, E, F, and G, respectively) as shown in Figs. 3(a), (3(b), 3(c), 3(d), 3(e), 3(f), and 3(g), respectively). Hence L_5^* is 1-EFT(L_5). The following theorem follows from Corollary 1.

Theorem 3. $\text{eft}_1(L_5) = 3$.

2.3. 1-EFT(L_k) *for graphs where* $k \geqslant 4$ *and even*

In this subsection, we are going to construct 1-EFT(L_k) graphs where k is an even integer with $k \geqslant 4$. Let the spanning supergraph L_k^* of L_k be the graph that adds $E' = \{(x_{i,j}, x_{k-i+1,j}) \mid 1 \leqslant i < k/2,\ j = 1, 2\}$ to $E(L_k)$ as shown in Fig. 4(a). The graph in Fig. 4(a) is actually isomorphic to $M(k/2, 2, 2)$ as shown in Fig. 4(b). We can reconfigure L_k in L_k^* as shown in Fig. 4(c) for any faulty edge of the form $(x_{i,1}, x_{i,2})$ or as shown in Fig. 4(d) for any faulty edge of the form $(x_{i,1}, x_{i+1,1})$ or $(x_{i,2}, x_{i+1,2})$. Hence, $M(k/2, 2, 2)$ is a 1-EFT(L_k). We obtain the following theorem:

Theorem 4. $\text{eft}_1(L_k) \leqslant k - 2$ *where* k *is an even integer with* $k \geqslant 4$.

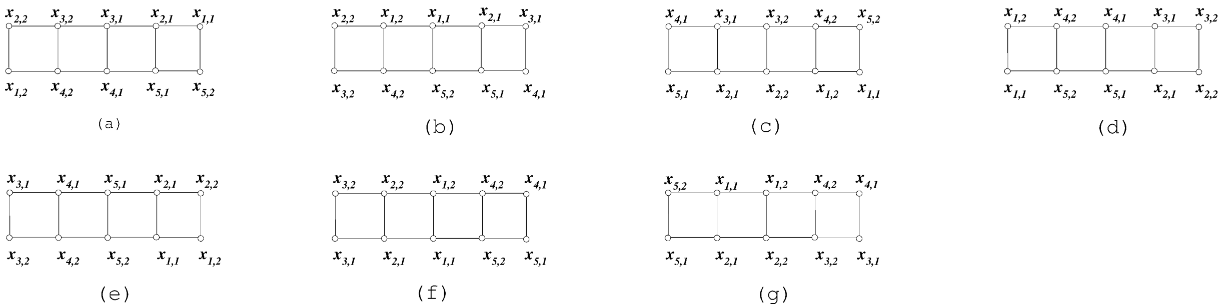


Fig. 3. A 1-EFT($M(5,2)$), L_5^*.

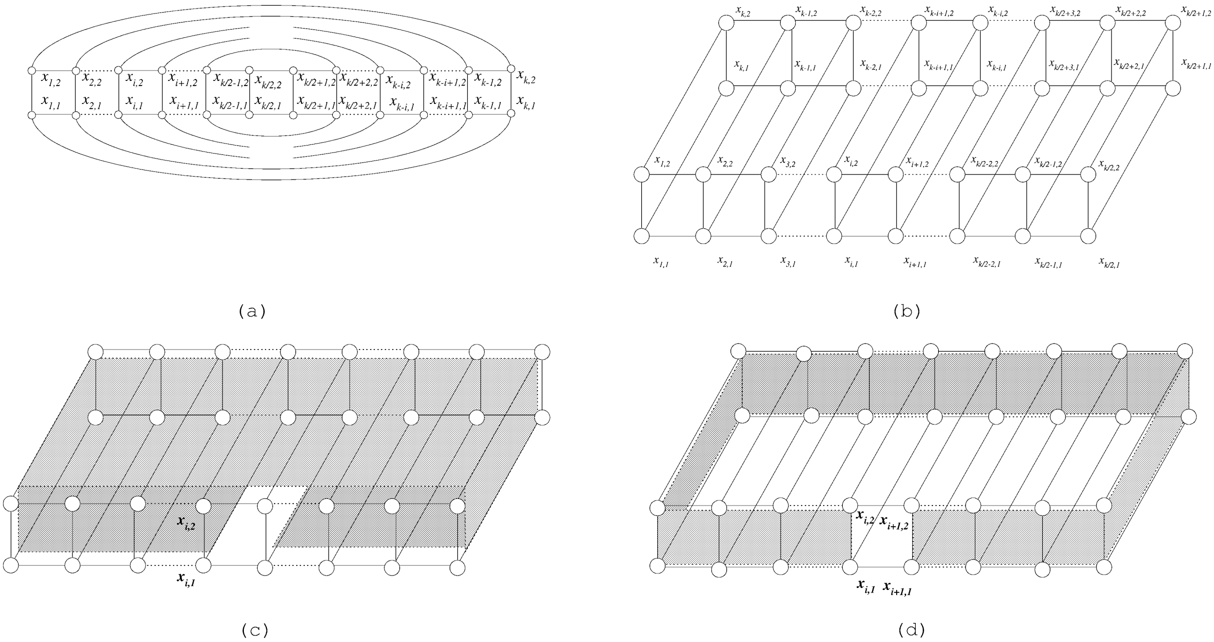


Fig. 4. (a) L_k^*, a 1-EFT(L_k) where k is even and $k \geqslant 4$; (b) the 3-dimensional mesh $M(k/2, 2, 2)$; (c) reconfigure L_k for any faulty edge of the form $(x_{i,1}, x_{i,2})$; and (d) reconfigure L_k for any faulty edge of the form $(x_{i,1}, x_{i+1,1})$, or $(x_{i,2}, x_{i+1,2})$.

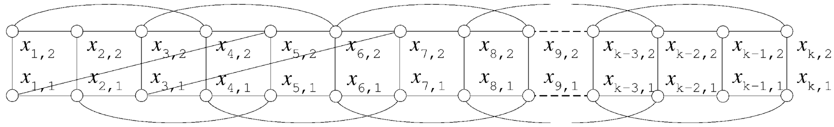


Fig. 5. A 1-EFT(L_k) where k is odd and $k \geqslant 7$.

2.4. 1-EFT(L_k) *graphs for* $k \geqslant 7$ *and odd*

Assume k is an odd integer with $k \geqslant 7$. Construct the spanning supergraph L_k^* of L_k by adding $E' = \{(x_{1,2}, x_{4,2}), (x_{3,2}, x_{6,2}), (x_{2,1}, x_{5,1}), (x_{4,1}, x_{7,1}), (x_{1,1}, x_{5,2}), (x_{3,1}, x_{7,2})\} \cup \{(x_{2i,j}, x_{2i+3,j}) \mid 3 \leqslant i \leqslant (k-3)/2,\ j = 1, 2\}$ as shown in Fig. 5.

Edges of L_k can be divided into the following 7 classes:

$$A = \big\{(x_{i,1}, x_{i,2}) \mid i = 1, 2\big\} \cup \big\{(x_{2i,j}, x_{2i+1,j}) \mid 4 \leqslant i \leqslant (k-3)/2,\ j = 1, 2\big\};$$

$$B = \big\{(x_{i,1}, x_{i,2}) \mid i = 3, 4\big\} \cup \big\{(x_{2i-1,j}, x_{2i,j}) \mid 4 \leqslant i \leqslant (k-1)/2,\ j = 1, 2\big\};$$

$$C = \big\{(x_{5,1}, x_{5,2})\big\} \cup \big\{(x_{i,1}, x_{i,2}) \mid 4 \leqslant i \leqslant (k-1)/2\big\};$$

$$D = \big\{(x_{i,1}, x_{i,2}) \mid i = 6, 7\big\} \cup \big\{(x_{i,1}, x_{i,2}) \mid i = k, k-1\big\};$$

$$E = \big\{(x_{1,j}, x_{2,j}) \mid j = 1, 2\big\} \cup \big\{(x_{3,j}, x_{4,j}) \mid j = 1, 2\big\};$$

$$F = \big\{(x_{2,j}, x_{3,j}) \mid j = 1, 2\big\} \cup \big\{(x_{5,j}, x_{6,j}) \mid j = 1, 2\big\};$$

$$G = \big\{(x_{4,j}, x_{5,j}) \mid j = 1, 2\big\} \cup \big\{(x_{6,j}, x_{7,j}) \mid j = 1, 2\big\} \cup \big\{(x_{k-1,j}, x_{k,j}) \mid j = 1, 2\big\}.$$

We can reconfigure L_k in L_k^* for any faulty edge e in A, B, C, D, E, F, and G respectively as shown in Figs. 6(a), 6(b), 6(c), 6(d), 6(e), 6(f), and 6(g), respectively. Hence L_k^* is 1-EFT(L_k). We obtain the following theorem:

Theorem 5. $\text{eft}_1(L_k) \leqslant k-1$ *where* k *is an odd integer with* $k \geqslant 7$.

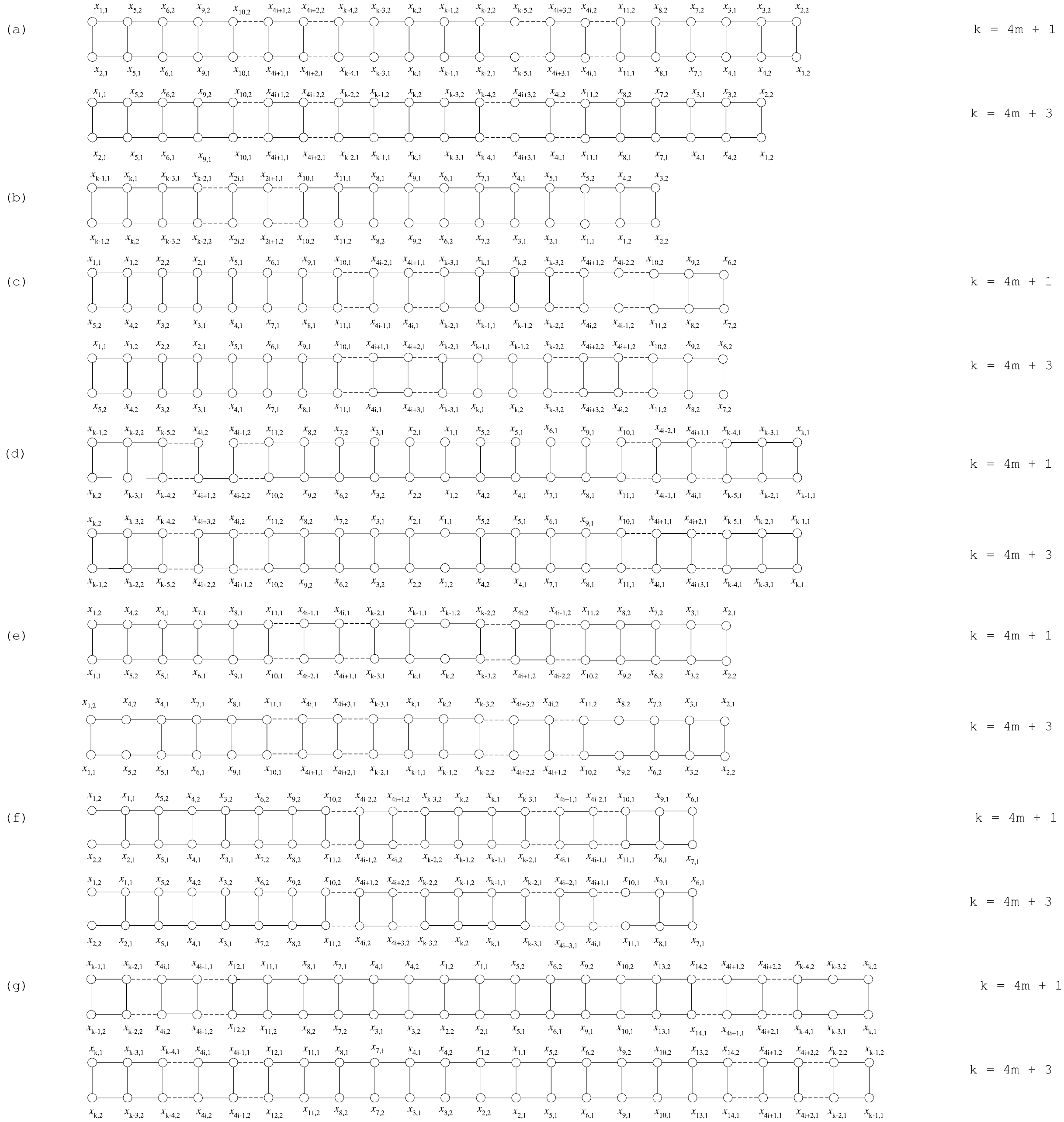


Fig. 6. Reconfigures of L_k in L_k^* where k is odd and $k \geqslant 7$ for any faulty edge in A, B, C, D, E, F, and G, respectively.

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