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Fault-tolerant hamiltonian laceability of hypercubes ☆

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Abstract

It is known that every hypercube Q_n is a bipartite graph. Assume that $n \geqslant 2$ and F is a subset of edges with $|F| \leqslant n-2$. We prove that there exists a hamiltonian path in $Q_n - F$ between any two vertices of different partite sets. Moreover, there exists a path of length $2^n - 2$ between any two vertices of the same partite set. Assume that $n \geqslant 3$ and F is a subset of edges with $|F| \leqslant n-3$. We prove that there exists a hamiltonian path in $Q_n - \{v\} - F$ between any two vertices in the partite set without v. Furthermore, all bounds are tight. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Hamiltonian laceable; Hypercube; Fault tolerance

1. Introduction

In this paper, a network is represented as an undirected graph. For the graph definition and notation we follow [1]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an ordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices a and b are *adjacent* if $(a, b) \in E$. A path is a sequence of adjacent vertices, written as $\langle v_0, v_1, v_2, \ldots, v_m \rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except possibly $v_0 = v_m$. We also write the path $\langle v_0, P, v_m \rangle$, where $P = \langle v_1, \ldots, v_{m-1} \rangle$. A path is a *hamiltonian path* if its vertices are distinct and they span on V. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A cycle is a *hamiltonian cycle* if it traverses every vertex of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if $V(G)$ is the union of two disjoint sets V_0 and V_1 such that each edge consists of one vertex from each set.

As the hamiltonicity of a graph G is concerned, it is an important issue to investigate if G is hamiltonian or hamiltonian connected. However, any hamiltonian bipartite graph $G = (V_0 \cup V_1, E)$ satisfies $|V_0| = |V_1|$. Since the colors of the bipartite path alternates, all hamiltonian bipartite graphs are not hamiltonian-connected. Simmons [8] introduces the concept of hamiltonian laceability for those hamiltonian bipartite graphs. A hamiltonian bipartite graph $G = (V_0 \cup V_1, E)$ is *hamiltonian laceable* if there is a hamiltonian path between any two vertices x and y with $x \in V_0$ and $y \in V_1$. Hsieh et al. [3] further extend this concept into

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strongly hamiltonian laceable. A hamiltonian laceable graph $G = (V_0 \cup V_1, E)$ is *strongly* if there is a simple path of length $|V_0 \cup V_1| - 2$ between any two vertices of the same partite set. Lewinter et al. [7] also introduce the concept of hyper-hamiltonian laceable. A hamiltonian laceable graph $G = (V_0 \cup V_1, E)$ is *hyper-hamiltonian laceable* if for any vertex $v \in V_i$, $i = 0, 1$, there is a hamiltonian path of $G - v$ between any two vertices of V_{1-i}.

The edge fault-tolerant hamiltonicity proposed by Hsieh, Chen, and Ho [4], measures the performance of the hamiltonian property in the faulty networks. A hamiltonian graph G is *k edge fault-tolerant hamiltonian* if $G - F$ remains hamiltonian for every $F \subset E(G)$ with $|F| \leqslant k$. The *edge fault-tolerant hamiltonicity*, $\mathcal{H}_e(G)$, is defined to be the maximum integer k such that G is k edge fault-tolerant hamiltonian if G is hamiltonian, and undefined if otherwise. It is easy to see that $\mathcal{H}_e(G) \leqslant \delta(G) - 2$ for any hamiltonian graph G where

$$\delta(G) = \min\{\deg(v) \mid v \in V(G)\}.$$

We can further study other fault-hamiltonicity. A hamiltonian laceable graph G is *k edge fault-tolerant hamiltonian laceable* if $G - F$ remains hamiltonian laceable for every $F \subset E(G)$ with $|F| \leqslant k$. The *edge fault-tolerant hamiltonian laceability*, $\mathcal{H}_e^L(G)$, is defined to be the maximum integer k such that G is k edge fault-tolerant hamiltonian laceable, and undefined if otherwise. A strongly hamiltonian laceable graph G is *k edge fault-tolerant strongly hamiltonian laceable* if $G - F$ remains strongly hamiltonian laceable for every $F \subset E(G)$ with $|F| \leqslant k$. The *edge fault-tolerant strongly hamiltonian laceability*, $\mathcal{H}_e^{SL}(G)$, is defined to be the maximum integer k such that G is k edge fault-tolerant strongly hamiltonian laceable, and undefined if otherwise. A hyper-hamiltonian laceable graph G is *k edge fault-tolerant hyper-hamiltonian laceable* if $G - F$ remains hyper-hamiltonian laceable for every $F \subset E(G)$ with $|F| \leqslant k$. The *edge fault-tolerant hyper-hamiltonian laceability*, $\mathcal{H}_e^h(G)$, is defined to be the maximum integer k such that G is k edge fault-tolerant hyper-hamiltonian laceable, and undefined if otherwise.

Network topology is usually represented by a graph where nodes represent processors and edges represent links between processors. The binary hypercube, Q_n, is one of the most popular topologies [6]. Let $u = u_{n-1}u_{n-2}\ldots u_1u_0$ and $v = v_{n-1}v_{n-2}\ldots v_1v_0$ be two n-bit binary strings. The Hamming distance $h(u, v)$ between two vertices u and v is the number of different bits in the corresponding strings of both vertices. The *n-dimensional hypercube* consists of all n-bit binary strings as its vertices and two vertices u and v are adjacent if and only if $h(u, v) = 1$. Latifi et al. [5] proved that $\mathcal{H}_e(Q_n) = n - 2$ if $n \geqslant 2$. Harary et al. [2] proved that Q_n is strongly hamiltonian laceable if and only if $n \geqslant 2$. Lewinter et al. [7] proved that Q_n is hyper-hamiltonian laceable if and only if $n \geqslant 3$. In this paper, we prove that

$$\mathcal{H}_e^L(Q_n) = \mathcal{H}_e^{SL}(Q_n) = n - 2 \quad \text{if } n \geqslant 2$$

and

$$\mathcal{H}_e^h(Q_n) = n - 3 \quad \text{if } n \geqslant 3.$$

Using our approach, we can easily prove that $\mathcal{H}_e(Q_n) = n - 2$ if $n \geqslant 2$.

2. Fault-tolerant hamiltonian laceability of hypercubes

Let Q_n be the n-dimensional hypercube. In this section, we will prove that $\mathcal{H}_e^{SL}(Q_n) \geqslant n - 2$ and $\mathcal{H}_e^h(Q_n) \geqslant n - 3$. It is clear that $\mathcal{H}_e^{SL}(G) \leqslant n - 2$ and $\mathcal{H}_e^h(G) \leqslant n - 3$, for any n-regular bipartite graph G. So $\mathcal{H}_e^{SL}(Q_n) = n - 2$, $\mathcal{H}_e^h(Q_n) = n - 3$, and these results are optimal. Q_n can be divided into two copies of Q_{n-1}, denoted by Q_{n-1}^0 and Q_{n-1}^1. Let E_c be the set of crossing edges, i.e.,

$$E_c = \{(u, u') \mid (u, u') \in E(Q_n),\ u \in V(Q_{n-1}^0) \text{ and } u' \in V(Q_{n-1}^1)\}.$$

Let F be the set of faulty edges of Q_n, $F_0 = F \cap E(Q_{n-1}^0)$, $F_1 = F \cap E(Q_{n-1}^1)$, and $F_c = F \cap E_c$. Also let $f_0 = |F_0|$, $f_1 = |F_1|$, and $f_c = |F_c|$. Since Q_n is edge symmetric, it suffices to consider only the case that $f_c \geqslant 1$. That means, Q_n can be split into Q_{n-1}^0 and Q_{n-1}^1 using any dimension d, where $1 \leqslant d \leqslant n$. Therefore, given a faulty edge set F, Q_n can be split into Q_{n-1}^0 and Q_{n-1}^1 such that F_c is not an empty set.

Lemma 1. *Q_3 is 1 edge fault-tolerant hamiltonian laceable,* $\mathcal{H}_e^L(Q_3) = 1$.

Proof. Let e be a faulty edge in Q_3. Q_3 can be split into Q_2^0 and Q_2^1 such that e is neither in $E(Q_2^0)$ nor in $E(Q_2^1)$. Suppose that x and y are with different colors.

Case 1: $x, y \in V(Q_2^0)$ or $x, y \in V(Q_2^1)$. Without loss of generality, we may assume that $x, y \in V(Q_2^0)$. Q_2 is a cycle of length four, so x and y are adjacent. Let $P = \langle x, x_0, x_1, y\rangle$ be a path in Q_2^0. Since $|F| = 1$, there exists an edge, denoted by (u, v), in this path such that (u, u') and (v, v') are fault-free and $u', v' \in Q_2^1$. Obviously, u' and v' are adjacent. Let $R = \langle u', w, z, v'\rangle$ be a path in Q_2^1. Therefore, $E(P) \cup E(R) - \{(u, v)\}$ forms a hamiltonian path in Q_3 joining x and y.

Case 2: $x \in V(Q_2^0)$ and $y \in V(Q_2^1)$, or $x \in V(Q_2^1)$ and $y \in V(Q_2^0)$. Without loss of generality, we may assume that $x \in V(Q_2^0)$ and $y \in V(Q_2^1)$. Since there are two vertices in Q_2^0 adjacent to x, we may choose a fault-free edge (u, v) such that $u \in V(Q_2^0)$, u is adjacent to x and $v \in V(Q_2^1)$. Obviously, v and y are adjacent. Let $\langle x, x_0, x_1, u\rangle$ be a path in Q_2^0 and $\langle y, y_0, y_1, v\rangle$ be a path in Q_2^1, respectively. Combining these two paths, we have a hamiltonian path in Q_3 joining x and y. □

Lemma 2. *Q_3 is 1 edge fault-tolerant strongly hamiltonian laceable, i.e.,* $\mathcal{H}_e^{SL}(Q_3) = 1$.

Proof. Let e be a faulty edge in Q_3. Q_3 can be split into Q_2^0 and Q_2^1 such that e is neither in $E(Q_2^0)$ nor in $E(Q_2^1)$. Suppose that x and y are with the same color. In order to prove this lemma, we will construct a fault-free path of length 6 joining x and y.

Case 1: $x, y \in V(Q_2^0)$ or $x, y \in V(Q_2^1)$. Without loss of generality, we may assume that $x, y \in V(Q_2^0)$. Let $\langle x, u, y\rangle$ be a path in Q_2^0 such that (u, u') is fault-free and $u' \in Q_2^1$. There is a vertex v in $\{x, y\}$ such that (v, v') is fault free and $v' \in Q_2^1$. Without loss of generality, we assume that $v = x$. Obviously, u' and v' are adjacent. Let $\langle u', w, z, v'\rangle$ be a path in Q_2^1. Therefore, $\langle x, v', z, w, u', u, y\rangle$ forms a path of length 6 in Q_3 joining x and y.

Case 2: $x \in V(Q_2^0)$ and $y \in V(Q_2^1)$, or $x \in V(Q_2^1)$ and $y \in V(Q_2^0)$. Without loss of generality, we may assume that $x \in V(Q_2^0)$ and $y \in V(Q_2^1)$. Let u be another vertex in $V(Q_2^0)$ having the same color as x. Also let $v \in V(Q_2^1)$, v and y have the same color. So these four vertices x, y, u, and v are all with the same color. And at least one of u and v is not an endpoint of the faulty edge e. We may assume that (u, u') is a fault-free edge such that $u' \in V(Q_2^1)$. Obviously, u' and y are adjacent. Let $\langle y, y_0, v, u'\rangle$ be a path in Q_2^1 and $\langle x, x_0, u\rangle$ be a path in Q_2^0, respectively. Therefore, $\langle x, x_0, u, u', v, y_0, y\rangle$ forms a path of length 6 joining x and y. □

Lemma 3. *The hypercube Q_n, $n \geqslant 2$, is $(n-2)$ edge fault-tolerant hamiltonian laceable, i.e.,* $\mathcal{H}_e^L(Q_n) = n - 2$.

Proof. We prove this lemma by induction on n. First, we observe that the lemma holds for $n = 2$. By Lemma 1, the lemma holds if $n = 3$. For $n \geqslant 4$, we assume that the lemma is true for every integer $k < n$. Let x and y be two vertices with different colors in Q_n, i.e., x and y are in different partite sets. In order to prove that $\mathcal{H}_e^L(Q_n) \geqslant n - 2$, we must construct a fault-free hamiltonian path joining x and y for any given faulty edge set F with $|F| = n-2$. Since $f_c \geqslant 1$, $f_0 \leqslant n - 3$ and $f_1 \leqslant n - 3$.

Case 1: $x, y \in V(Q_{n-1}^0)$ or $x, y \in V(Q_{n-1}^1)$. (See Fig. 1(a).) Without loss of generality, we may assume that x and y are in Q_{n-1}^0. By induction hypothesis, $\mathcal{H}_e^L(Q_{n-1}^0) = n - 3$, there exists a hamiltonian path $\langle x, P_0, y\rangle$ with $2^{n-1} - 1$ edges. We claim that there exists an edge $(u, v) \in E(\langle x, P_0, y\rangle)$ such that both crossing edges (u, u') and (v, v') are fault-free. Since $|E(\langle x, P_0, y\rangle)| = 2^{n-1} - 1$, we have $2^{n-1} - 1$ choices. If none of the edges of $\langle x, P_0, y\rangle$ meets the requirements of such an edge (u, v), then there are at least $\lceil (2^{n-1} - 1)/2\rceil$ faults in F_c. (Because a single fault in F_c eliminates 2 edges of $\langle x, P_0, y\rangle$.) And $\lceil (2^{n-1} - 1)/2\rceil > n-2$ for $n \geqslant 4$, this contradicts with the fact that $|F| \leqslant n - 2$. Therefore, we can always find such an edge (u, v). Obviously, u' and v' are with different colors. Since $\mathcal{H}_e^L(Q_{n-1}^1) = n - 3$, there exists a fault-free hamiltonian path $\langle u', P_1, v'\rangle$ in Q_{n-1}^1. Therefore,

$$E\big(\langle x, P_0, y\rangle\big) \cup E\big(\langle u', P_1, v'\rangle\big) \cup \big\{(u, u'), (v, v')\big\} - \big\{(u, v)\big\}$$

forms a hamiltonian path in $Q_n - F$ joining x and y.

Case 2: $x \in V(Q_{n-1}^0)$ and $y \in V(Q_{n-1}^1)$, or $x \in V(Q_{n-1}^1)$ and $y \in V(Q_{n-1}^0)$. (See Fig. 1(b).) Without

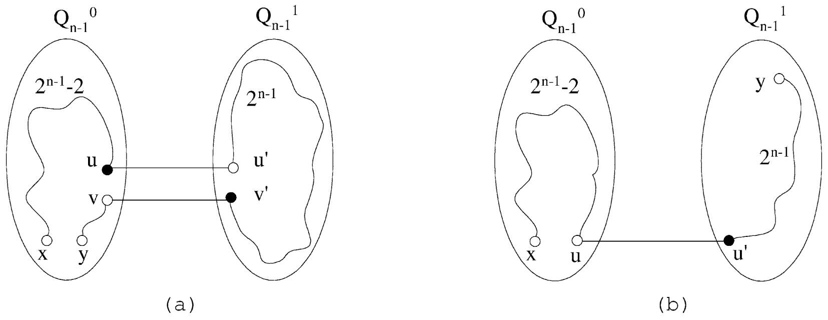

Fig. 1. (a) Case 1: $x, y \in V(Q^0_{n-1})$. (b) Case 2: $x \in V(Q^0_{n-1})$ and $y \in V(Q^1_{n-1})$.

loss of generality, we assume that $x \in V(Q^0_{n-1})$ and $y \in V(Q^1_{n-1})$. We claim that we can choose a vertex u in Q^0_{n-1} such that u and x are with different colors, and the crossing edge (u, u') is fault-free, where $u' \in V(Q^1_{n-1})$. Since $|V(Q^0_{n-1})| = 2^{n-1}$, we have 2^{n-2} choices. (Because there are 2^{n-2} vertices in Q^0_{n-1} which have different colors from x.) If none of the vertices in Q^0_{n-1} meets the requirements of such vertex u, then there are at least 2^{n-2} faults in F_c. This contradicts with the fact that $|F| \leqslant n-2$ for $n \geqslant 2$. Therefore, we can always find such a vertex u. Then, u' and y are with different colors. Since $\mathcal{H}^L_e(Q^0_{n-1}) = \mathcal{H}^L_e(Q^1_{n-1}) = n-3$, there exists a hamiltonian path $\langle x, P_0, u\rangle$ and $\langle u', P_1, y\rangle$ in Q^0_{n-1} and in Q^1_{n-1}, respectively. Therefore,

$$E\big(\langle x, P_0, u\rangle\big) \cup E\big(\langle u', P_1, y\rangle\big) \cup \big\{(u, u')\big\}$$

forms a hamiltonian path in $Q_n - F$ joining x and y. This completes the induction. □

Given any faulty edge set F with $|F| = n-2$, we can chose an edge (u, v) in $Q_n - F$. By the proof above, there exists a hamiltonian path $\langle u, P, v\rangle$ in $Q_n - F$ joining u and v. So it is easy to see that

$$E\big(\langle u, P, v\rangle\big) \cup \big\{(u, v)\big\}$$

forms a hamiltonian cycle in $Q_n - F$. Hence $\mathcal{H}_e(Q_n) = n-2$ if $n \geqslant 2$.

Theorem 1. *The hypercube Q_n, $n \geqslant 2$, is $(n-2)$ edge fault-tolerant strongly hamiltonian laceable, i.e., $\mathcal{H}^{SL}_e(Q_n) = n-2$.*

Proof. By Lemma 3, we have $\mathcal{H}^L_e(Q_n) = n-2$. So all we have to show is the following. Let x and y be two vertices with the same color in Q_n. We must find a fault-free path $\langle x, P, y\rangle$ of length $2^n - 2$ for any given faulty edge set F with $|F| = n-2$. This theorem can be proved using by the same way of Lemma 3 and hence the detail proof is omitted. □

Theorem 2. *The hypercube Q_n, $n \geqslant 3$, is $(n-3)$ edge fault-tolerant hyper-hamiltonian laceable, i.e., $\mathcal{H}^h_e(Q_n) = n-3$.*

Proof. The proof is again by using induction on n. Lewinter et al. [7] proved that Q_n, $n \geqslant 3$, is hyper-hamiltonian laceable. So the lemma holds for the case $n = 3$. For $n \geqslant 4$, we assume the theorem is true for every integer $k < n$. By induction hypothesis, $\mathcal{H}^h_e(Q_{n-1}) = n-4$. Now, we consider Q_n. By Lemma 3, we have $\mathcal{H}^l_e(Q_n) = n-2$. Obviously, Q_n is hamiltonian laceable after removing $n-3$ edges. In order to prove that $\mathcal{H}^h_e(Q_n) \geqslant n-3$, it suffices to show the following. After deleting a given vertex w from Q_n, let x and y be any two vertices in the larger partite set of Q_n. We must construct a hamiltonian path of $Q_n - F - w$ joining x and y for any given faulty edge set F with $|F| = n-3$. Without loss of generality, we may assume that $w \in V(Q^0_{n-1})$. Since $f_c \geqslant 1$, $f_0 \leqslant n-4$ and $f_1 \leqslant n-4$.

Case 1: $x, y \in V(Q^0_{n-1})$. (See Fig. 2(a).) By induction hypothesis, $\mathcal{H}^h_e(Q^0_{n-1}) = n-4$, there exists a fault-free hamiltonian path $\langle x, P_0, y\rangle$ with $2^{n-1}-2$ edges in $Q^0_{n-1} - w$. We now show that there exists an edge $(u, v) \in E(\langle x, P_0, y\rangle)$ such that both crossing edges (u, u') and (v, v') are fault-free. Since $|E(\langle x, P_0, y\rangle)| = 2^{n-1} - 2$, we have $2^{n-1} - 2$ choices. If none of the edges of $\langle x, P_0, y\rangle$ meets the requirements of (u, v), then there are at least $(2^{n-1} - 2)/2 =$

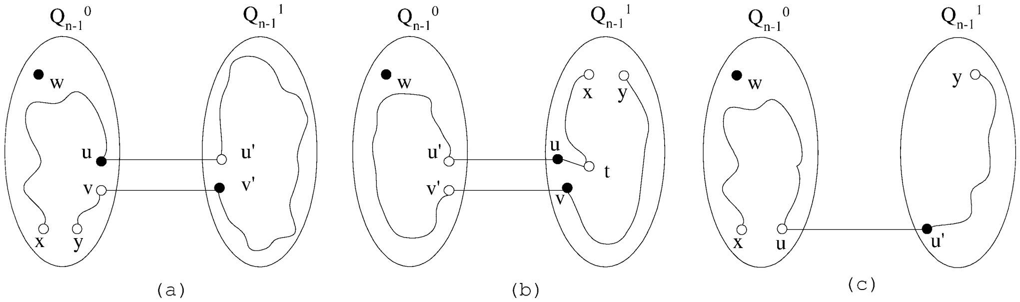

Fig. 2. (a) Case 1: $x, y \in V(Q^0_{n-1})$. (b) Case 2: $x, y \in V(Q^1_{n-1})$. (c) Case 3: $x \in V(Q^0_{n-1})$ and $y \in V(Q^1_{n-1})$.

$2^{n-2} - 1$ faults in F_c. (Because a single fault in F_c eliminates 2 edges of $\langle x, P_0, y\rangle$.) And $2^{n-2} - 1 > n - 3$ for $n \geqslant 4$, this contradicts with the fact that $|F| \leqslant n - 3$. Therefore, we can always find such an edge (u, v). Obviously, u' and v' are with different colors. Since $\mathcal{H}^L_e(Q^1_{n-1}) = n - 3$, there exists a fault-free hamiltonian path $\langle u', P_1, v'\rangle$ in Q^1_{n-1}. Therefore,

$$E\big(\langle x, P_0, y\rangle\big) \cup E\big(\langle u', P_1, v'\rangle\big) \cup \big\{(u, u'), (v, v')\big\} - \big\{(u, v)\big\}$$

forms a hamiltonian path in $Q_n - F - w$ joining x and y.

Case 2: $x, y \in V(Q^1_{n-1})$. (See Fig. 2(b).) First, we will choose a vertex u in Q^1_{n-1} such that u and x are with different colors, and the crossing edge (u, u') is fault-free, where $u' \in V(Q^0_{n-1})$. Since $|V(Q^1_{n-1})| = 2^{n-1}$, we have 2^{n-2} choices. (Because there are 2^{n-2} vertices in Q^1_{n-1} which have different colors from x.) If none of the vertices in Q^1_{n-1} meets the requirements of such vertex u, then there are at least 2^{n-2} faults in F_c. This contradicts with the fact that $|F| \leqslant n - 3$ for $n \geqslant 2$. Therefore, we can always find such a vertex u. Since $\mathcal{H}^h_e(Q^1_{n-1}) = n-4$, there exists a fault-free hamiltonian path $\langle x, P, y\rangle$ in $Q^1_{n-1} - u$ joining x and y. We claim that there exists a vertex t such that the edge (u, t) is fault-free in Q^1_{n-1}, the edge $(t, v) \in E(\langle x, P, y\rangle)$, and the crossing edge (v, v') is fault-free in Q_n where $v' \in Q^0_{n-1}$. Since the number of neighboring vertices of u in Q^1_{n-1} is $n - 1$, we have $n - 1$ choices. If none of the vertices in Q^1_{n-1} meets the requirements of such a vertex t, then there are at least $n - 1$ faults in F_c, making it contradictory to the fact that $|F| \leqslant n - 3$. Therefore, we can always find such a vertex t.

We then divide the path $\langle x, P, y\rangle$ into two sections $\langle x, P_0, v\rangle$ and $\langle t, P_1, y\rangle$, or $\langle x, P_0, t\rangle$ and $\langle v, P_1, y\rangle$. Without loss of generality, we assume the case $\langle x, P_0, t\rangle$ and $\langle v, P_1, y\rangle$. Thus, we have two sections as $\langle x, P_0, u\rangle$ and $\langle v, P_1, y\rangle$. Let (u, u') and (v, v') be two crossing edges incident to vertices u and v, respectively. Then, u' and v' are with the same color in Q^0_{n-1}. Since u and x are with different colors, u' and v' are in the larger partite set of $Q^0_{n-1} - w$. By induction hypothesis, $\mathcal{H}^L_e(Q^0_{n-1}) = n - 4$, there exists a fault-free hamiltonian path $\langle u', R, v'\rangle$ in Q^0_{n-1}. Therefore,

$$E\big(\langle x, P_0, t\rangle\big) \cup E\big(\langle u', R, v'\rangle\big) \cup E\big(\langle v, P_1, y\rangle\big) \cup \big\{(u, u'), (v, v'), (t, u)\big\}$$

forms a hamiltonian path in $Q_n - F - w$ joining x and y.

Case 3: $x \in V(Q^0_{n-1})$ and $y \in V(Q^1_{n-1})$, or $x \in V(Q^1_{n-1})$ and $y \in V(Q^0_{n-1})$. (See Fig. 2(c).) Without loss of generality, we assume that $x \in V(Q^0_{n-1})$ and $y \in V(Q^1_{n-1})$. First, we will choose a vertex u, $u \neq x$, in Q^0_{n-1} such that u and x are with the same color, and the crossing edge (u, u') is fault-free, where $u' \in V(Q^1_{n-1})$. Since $|V(Q^0_{n-1})| = 2^{n-1}$, we have $2^{n-2} - 1$ choices. (Because there are $2^{n-2} - 1$ fault-free vertices in Q^0_{n-1} which have the same color as x.) If none of the vertices in Q^0_{n-1} meets the requirements of such vertex u, then there are at least $2^{n-2} - 1$ faults in F_c. This contradicts with the fact that $|F| \leqslant n - 3$ for $n \geqslant 3$. Therefore, we can always find such a vertex u. Then, u' and y are with different colors. By induction hypothesis, $\mathcal{H}^h_e(Q^0_{n-1}) = n - 4$, there exists a fault-free hamiltonian path $\langle x, P_0, u\rangle$ in Q^0_{n-1} joining x and u. Also, since $\mathcal{H}^L_e(Q^1_{n-1}) = n - 3$, there exists a fault-free hamiltonian path $\langle u', P_1, y\rangle$ in Q^1_{n-1} joining u' and y. Therefore,

$$E\big(\langle x, P_0, u\rangle\big) \cup E\big(\langle u', P_1, y\rangle\big) \cup \big\{(u, u')\big\}$$

forms a hamiltonian path in $Q_n - F - w$ joining x and y. The proof is complete. □

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