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Modeling self-consistent multi-class dynamic traffic flow

Hsun-Jung Cho, Shih-Ching Lo *

Department of Transportation Technology and Management, National Chiao Tung University, Hsinchu 30050, Taiwan

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Abstract

In this study, we present a systematic self-consistent multiclass multilane traffic model derived from the vehicular Boltzmann equation and the traffic dispersion model. The multilane domain is considered as a two-dimensional space and the interaction among vehicles in the domain is described by a dispersion model. The reason we consider a multilane domain as a two-dimensional space is that the driving behavior of road users may not be restricted by lanes, especially motorcyclists. The dispersion model, which is a nonlinear Poisson equation, is derived from the car-following theory and the equilibrium assumption. Under the concept that all kinds of users share the finite section, the density is distributed on a road by the dispersion model. In addition, the dynamic evolution of the traffic flow is determined by the systematic gas-kinetic model derived from the Boltzmann equation. Multiplying Boltzmann equation by the zeroth, first- and second-order moment functions, integrating both side of the equation and using chain rules, we can derive continuity, motion and variance equation, respectively. However, the second-order moment function, which is the square of the individual velocity, is employed by previous researches does not have physical meaning in traffic flow. Although the second-order expansion results in the velocity variance equation, additional terms may be generated. The velocity variance equation we propose is derived from multiplying Boltzmann equation by the individual velocity variance. It modifies the previous model and presents a new gas-kinetic traffic flow model. By coupling the gas-kinetic model and the dispersion model, a self-consistent system is presented. © 2002 Elsevier Science B.V. All rights reserved.

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* Corresponding author. No. 8, Lane 282, Wenhua St., Pingjen City, 324, Taiwan. Tel.: +886-931-040227; fax: +886-945-866383.

E-mail address: shihchinglo.tem84g@nctu.edu.tw (S.-C. Lo).

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1. Introduction

During the recent five decades by developing kinetic traffic flow model, it is possible to model more realistic traffic phenomena for traffic scientists in laboratories. Kinetic traffic flow models describe and forecast the time variant traffic variables, such as density, traffic volume and velocity. In addition, the performance of the traffic-control alternatives and the network design can be evaluated by traffic simulation.

Since Lighthill and Whitham [1] and Richards [2] first proposed their kinetic model, the related subjects are broadly researched and debated. The LWR model was extended to second-order model, which includes the continuity equation and a phenomenological velocity equation. The second-order model was named PW model [3–14]. However, this kind of models has a lot of arguments, so families of gas-kinematic models [15–40] are presented. The development of gas-kinetic models includes the discussion of multilane, multiclass users and overtaking, lane-changing, relaxation and interaction maneuvers. As the review of Boltzmann equation, Boltzmann equation is a phase-plane distribution. The macroscopic quantities are derived as follows. The first step is multiplying Boltzmann equation by the moment functions. The second step is integrating both sides of the equations and using the chain rules. At last, the macroscopic quantities are obtained. Therefore, the resulting macroscopic quantity and the moment function must have physical meanings. From the previous researches, the second-order moment function multiplied to Boltzmann equation is the square of individual velocity [18,19,22,30–40]. Nevertheless, the square of individual velocity, which denotes the individual kinetic energy in physics, is meaningless in traffic. Although the second-order expansion results in the velocity variance equation, additional terms may be generated. For this reason, we multiply Boltzmann equation by the individual velocity variance in order to modify the derivation of velocity variance equation herein.

A complete dynamic system should include motion equations and state equations. The state equation considered in this study is the vehicular dispersion model [41]. The model is derived from the car-following theory and the equilibrium assumption. Under a specific macroscopic situation, the most possible microscopic combination is defined as the equilibrium state. And the system is assumed to tend toward the equilibrium state. According to the dispersion model, density is distributed on the road. By coupling the dispersion model to the kinetic system, a self-consistent system is obtained. Furthermore, we consider the multilane model in a two-dimensional space because the driving behavior of road users may not be restricted to drive one by one, especially motorcyclists [41,42].

The rest of this paper is organized as follows. Section 2 presents the historical evolution from LWR model to gas-kinetic traffic flow models briefly. Section 3 introduces the concept of traffic field and the derivation from Boltzmann equation to macroscopic systems. In Section 4, the model of the multiclass users is presented. After that, the paper concludes with some perspectives in Section 5.

2. Historical evolution of dynamic macroscopic traffic flow models

LWR model and the extended models are the most popular macroscopic dynamic traffic flow models. LWR model describes that traffic is like fluid continuum flowing on highways. Researches based on the model extend the discussion to shock wave analyses, higher order effect, traffic control applications, multilane traffic, and so on. As the purpose of this study is developing a dynamic macroscopic traffic flow system, the historical evolution of dynamic macroscopic traffic flow models is mentioned first in this section.

2.1. The LWR-like models

Lighthill and Whitham [1] and Richards [2] are the first persons who presented the macroscopic kinetic traffic flow model. They used kinematical concepts to describe waves in traffic. The basic premises of their model are that traffic is conversed and that there exists a one-to-one relationship between velocity and density. The LWR model can be viewed as a good and basic approximation. Mathematically, LWR model states that the density k and flow Q satisfy

$$\frac{\partial k(x,t)}{\partial t} + \nabla \cdot Q(x,t) = 0\,, \tag{1}$$

where t denotes time and x denotes position. Eq. (1) expresses the conservation of vehicles. In addition, Q, k and velocity u are assumed to satisfy $Q = ku(k)$. From these assumption, Eq. (1) has the solution $k = F(x - ct)$, where F is an arbitrary function (the initial condition), c is the wave speed and $c = \mathrm{d}Q/\mathrm{d}k$. Eq. (1) implies that inhomogeneities, such as, changes in density of cars, propagate along a stream of cars with constant wave speed c with respect to a stationary observer.

The LWR model is a simple but sufficient traffic theory if the size and end location of a queue is the things that one only cares about; such as, the time-space trajectory of a shock. Unfortunately, traffic flow phenomena are very complex, and some important phenomena that we are also interested in elude the LWR model. One such phenomenon is the stop–start waves in long queues often observed on congested freeways. Therefore, the development of continuum models extends the LWR model by replacing the instantaneous flux function with a dynamic one. This is referred to higher order models, which is also named the PW model.

2.2. The PW-like models

The assumption of $u{=}u(k)$ is a steady state assumption of velocity, which means that velocity changes instantaneously as density changes. It is certainly not valid in some traffic flow situations. To overcome the steady state assumption of velocity, Payne [4] used a motion equation to obtain time variant speed.

$$\frac{\partial u}{\partial t} + u(\nabla \cdot u) = -\frac{1}{k}\nabla \cdot (P_e(k)) + \frac{1}{\tau}(u_e(k) - u)\,, \tag{2}$$

where $u_e(k)$ is an equilibrium speed–density relation. The motion equation (or the so-called momentum conservation) coupled with the continuity equation is referred to PW model since the concept of employing a higher order function in LWR model is proposed by Whitham first. The term $-\nabla \cdot (P_e(k))/k = -[P_e'(k)(\nabla \cdot k)]/k$ is an anticipation term, which takes it into account that drivers beware of the preceding traffic condition, where $P_e(k)$ is an equilibrium traffic pressure. Payne used an anticipation term determined by $P_e'(k) = (1/2\tau)|u_e'(k)|$. In general, we have $P_e(k) = k\theta_e(k)$, where $\theta_e(k)$ denotes the equilibrium velocity variance. Papageorgiou et al. [5] substituted an Euler-like discrete form for the anticipation term and then Michalopoulos et al. [6] developed a semi-viscous model. Zhang [7] also proposed a new additional model to a long list of existing momentum equation. Also, different models are determined by different assumptions of $u_e(k)$ and $\theta_e(k)$. For example, Kühne [8], and Kerner and Kohnhäuser [9,10] suggested that $\theta_e(k)$ be a constant value c_0^2, whereas Phillips [11] suggested that $\theta_e(k)$ be a linear relation $\theta_e(k) = \theta_m(1 - k/k_m)$. The explicit functions of $u_e(k)$ and $\theta_e(k)$ can be derived from the equilibrium distribution function of kinetic theory, respectively.

Daganzo [12] mentioned that although the result of higher order model is a little better than first-order model, it needs more computation. He also pointed out higher order models bring the wrong result; that is, in some cases, vehicle speed will be negative. Aw and Rascle [13] explained that the phenomena is caused by the wrong assumption of traffic pressure. Recently, Günther et al. [14] presented a modeling procedure to ensure that the PW-like models describe all situations correctly.

2.3. The gas-kinetic models

Gas-kinetic theory is a further modeling methodology. This kind of models is first employed to describe traffic flow by Prigogine and his colleagues [15–17] and is referred to the Boltzmann-like model. Hoogendoorn and Bovy [18–20] classified the gas-kinetic models mesoscopic models. Boltzmann equation is widely applied in applied science, such as, gas dynamics, population analysis, traffic flow, semiconductor and so on [21]. Generally, Boltzmann equation is used to describe properties of a fluid in the large domain by examining the statistics of motion of constituent particles. Prigogine described a traffic fluid with a probability density for the velocity (v) of an individual car, $f(x, v, t)$, which may vary with a function of time t and the coordinate x along the highway. This density is assumed to satisfy the equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial t}\right)_{relaxation} + \left(\frac{\partial f}{\partial t}\right)_{interaction} . \tag{3}$$

The first term of the right-hand element of Eq. (3) stems from the fraction that $f(x, v, t)$ differs from some desired velocity distribution $f^0(v)$. The second term describes that a fast car will slow down owing to the influence of a slow car. The interaction term has been criticized. It has been argued that the collision term is only valid in the situation, which describes the incoming vehicle passes each

single car in the queue independently. Therefore, Paveri-Fontana [22] proposed an improved model to overcome the shortcoming of Prigogine's approach. He generalized the phase-plane distribution as $\tilde{f}(x,v,v^0,t)$, where v^0 is the individual desired velocity. $f(x,v,t)=\int \tilde{f}(x,v,v^0,t)\,\mathrm{d}v^0$. Nevertheless, his model still considered that the maneuver on the lanes of a multilane road is the same and his model does not take queueing effects into account. Transitory and stationary solutions and numerical simulations are proposed by researches [24–29] under different interaction, relaxation, and adjustment terms.

The nature of the equilibrium solutions of a Boltzmann-like equation is likely to be reflected in the nature of the associated hydrodynamic models [18,19,22,30–40]. Helbing [30] presents a gas-kinetic model for traffic operations. In contrast to the model of [22], the model of [30] considers additional terms. The first is a velocity diffusion term, which takes the individual fluctuations of the velocity into account due to imperfect driving. The other one is the rate of vehicles entering and leaving the roadway. The macroscopic systematic equation Helbing [30] obtained from the mesoscopic equation has one more equation than PW-like model. The equation is the velocity variance equation. The previous models handle the velocity variance as an equilibrium quantity. However, in nonequilibrium situations, the velocity variance may be better treated as a dynamic variable with a further equation describing its evolution. In particular, to predict traffic jams, an increase of the variance appears to be a very important indicator. The velocity variance equation is shown as

$$\frac{\partial\theta}{\partial t} + u(\nabla\cdot\theta) = -2\theta(\nabla\cdot u) + 2\,\frac{\mu}{k}(\nabla\cdot u)^2 + \frac{2}{\tau}(\theta_e(k)-\theta) + \frac{\mu}{k}\,\nabla\cdot(\nabla u) + \frac{\kappa}{k}\,\nabla\cdot(\nabla\theta)\,, \tag{4}$$

where θ is velocity variance, μ,κ are coefficients, τ is relaxation time and θ_e is equilibrium velocity variance. θ_e is assumed to depend on k only as it is assumed in the PW-like models. Helbing [30] employed a numerical simulation and obtained following results:

(a) The section of high density induces low speed and small speed variance.
(b) The section of low density induces high speed and large speed variance.
(c) The largest speed variance takes place at the highest speed behind a platoon.

Empirical studies were also employed by Helbing [34,36] to validate the model. Multilane traffic can also be extended by the model [37,40]. Besides, Hoogendoorn and Bovy [18–20] derived a multiple user-classes traffic flow model from Boltzmann equation by a similar approach. From the works of Paveri-Fontana, Helbing, Hoogendoorn and Bovy, macroscopic models can be derived from Boltzmann-like model, which is a microscopic model. Boltzmann-like models can be developed with behavioral analysis. Therefore, the macroscopic models derived from them improve the lack of behavioral analyses of macroscopic models.

3. Derivation from Boltzmann equation to macroscopic models

Since macroscopic models derived from Boltzmann equation can aggregate the microscopic behavior to be group behavior, this study proposes a Boltzmann equation and derives it to macroscopic models. There are three main differences between this study and previous studies. The first one is that a multilane road is considered as a two-dimensional domain. The second one is that the acceleration effect of Boltzmann equation is considered as the influence of traffic field. The last one is that the second-order moment function considered herein is individual velocity variance. Traffic field is derived from car-following theory, which describes the interaction between vehicles. The detail of derivation of traffic field is illustrated in Section 3.2. The concept of traffic field not only describes the interaction between vehicles, but also makes the macroscopic system consistent. Before introducing traffic field, definitions of variables and the relations among variables should be mentioned first.

3.1. Definitions

By reason of some driving behavior of road users cannot be restricted in one lane or even they derive in one lane they still not be restricted to drive one after one, such as, driving behavior of motorcycles. Therefore, a multilane highway is regarded as a two-dimensional space in this study. We assume that there exists a phase-plane distribution function $f(\mathbf{x},\mathbf{v},t)$ at a given time and at a given point, where $\mathbf{x}=(x,y)$ denotes position, $\mathbf{v}=(v_x,v_y)$ denotes individual velocity and t denotes time. Since, $\mathbf{v}$ denotes individual velocity, it is impossible to restrict a specific velocity at a specific time and place. Thus, $\mathbf{v}$ is independent to position $\mathbf{x}$ and t, i.e., $\nabla_{\mathbf{x}}\mathbf{v}=0$, $\nabla_{\mathbf{x}}\cdot\mathbf{v}=0$, $\partial\mathbf{v}/\partial t=0$.

In addition, $\mathrm{d}\mathbf{x}/\mathrm{d}t=\mathbf{v}$ and $\mathrm{d}\mathbf{v}/\mathrm{d}t=e\mathbf{E}$, where $\mathbf{E}$ denotes traffic field and is going to derive in detail in Section 3.2. By Taylor's expansion or total derivation of f, the changing of f is shown by

$$\begin{aligned}\frac{\mathrm{d}f(\mathbf{x},\mathbf{v},t)}{\mathrm{d}t} &= \frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{x}}f(\mathbf{x},\mathbf{v},t)+e\mathbf{E}\cdot\nabla_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)\\ &= \left(\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\right)_{coll},\end{aligned} \tag{5}$$

$$f(\mathbf{x},\mathbf{v},t)|_{\partial\Omega_{\mathbf{v}}}=0\,, \tag{6}$$

where f is defined on Ω and $\partial\Omega$ is the boundary of $\Omega\cdot\partial\Omega_{\mathbf{v}}$ is the boundary of individual velocity. Since f is a distribution function, it is reasonable to assume that f is equal to 0 at the extreme value (i.e., boundary $\partial\Omega_{\mathbf{v}}$). Thus, density is given by

$$k(\mathbf{x},t)=\int_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v} \tag{7}$$

and flow density, which is defined by Cho and Lo [14], is given by

$$\mathbf{q}(\mathbf{x},t)=\int_{\mathbf{v}}\mathbf{v}f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}=k(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\,, \tag{8}$$

where $\mathbf{u}(\mathbf{x},t)$ denotes average velocity (or the so-called group velocity), which is defined as

$$\mathbf{u}(\mathbf{x},t)=\frac{\int\mathbf{v}f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}}{\int_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}}\,. \tag{9}$$

Flow is denoted by $Q(x,t)=\int_y\mathbf{q}(\mathbf{x},t)\,\mathrm{d}y$, where y is the width of the road.

Next, three kinds of velocity variance are defined. The first one is total velocity variance, which is velocity variance between individual velocity and equilibrium velocity. Total velocity variance is defined by

$$\Theta(\mathbf{x},t)=\frac{\int_{\mathbf{v}}\|\mathbf{v}-\mathbf{u}_e\|^2f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}}{\int_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)}\,. \tag{10}$$

The second one is individual velocity variance, which is the velocity variance between individual velocity and group velocity. Individual velocity variance is given by

$$\theta_a(\mathbf{x},t)=\frac{\int_{\mathbf{v}}\|\mathbf{v}-\mathbf{u}(\mathbf{x},t)\|^2f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}}{\int_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)}\,. \tag{11}$$

The last one is group velocity variance, which is the velocity variance between group velocity and equilibrium velocity. Group velocity variance is given by

$$\theta_e(\mathbf{x},t)=\frac{\int_{\mathbf{v}}\|\mathbf{u}(\mathbf{x},t)-\mathbf{u}_e\|^2f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}}{\int_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)}=\frac{\|\mathbf{u}(\mathbf{x},t)-\mathbf{u}_e\|^2\int_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}}{\int_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)}\,. \tag{12}$$

If traffic flow is uniform, individual velocity is equal to average velocity $\theta_a(\mathbf{x},t)=0$. If traffic flow is equilibrium, average velocity is equal to equilibrium velocity $\theta_e(\mathbf{x},t)=0$. The relationship among three different velocity variances is

$$\begin{aligned}\Theta(\mathbf{x},t)&=\frac{\int_{\mathbf{v}}\|\mathbf{v}-\mathbf{u}(\mathbf{x},t)+\mathbf{u}(\mathbf{x},t)-\mathbf{u}_e\|^2f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}}{\int_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)}\\&=\theta_a(\mathbf{x},t)+\theta_e(\mathbf{x},t)\,,\end{aligned}$$

since

$$\int_{\mathbf{v}}[2(\mathbf{v}-\mathbf{u}(\mathbf{x},t))\cdot(\mathbf{u}(\mathbf{x},t)-\mathbf{u}_e(k))]f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}=0\,. \tag{13}$$

That implies: (a) total velocity variance is equal to the summation of individual and group velocity variance, (b) $\Theta\geqslant\theta_a\geqslant 0$ and $\Theta\geqslant\theta_e\geqslant 0$, (c) $\Theta=0$ implies $\theta_a=\theta_e=0$.

Furthermore, equilibrium average velocity and equilibrium average variance are denoted by $\mathbf{u}_e(k,\mathbf{u},\Theta)$ and $\Theta_e(k,\mathbf{u},\Theta)$, respectively. An equilibrium state is defined as the most possible microscopic state under a specific macroscopic state. Since $\mathbf{u}_e$ depends upon k, $\mathbf{u}$ and Θ only, $\nabla_{\mathbf{x}}\mathbf{u}_e=0$, $\nabla_{\mathbf{x}}\cdot\mathbf{u}_e=0$, $\partial\mathbf{u}_e/\partial t=0$.

Another variable, which appears in the derivation, is skewness. It is defined by

$$\mathbf{\Gamma}(\mathbf{x},t) = \frac{\int_{\mathbf{v}} (\mathbf{v}-\mathbf{u}(k))\|\mathbf{v}-\mathbf{u}(k)\|^2 f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}}{\int_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)}\,, \tag{14}$$

which is the bias of the distribution. From the empirical study by Helbing [34,36], $\mathbf{\Gamma}(\mathbf{x},t) \approx \mathbf{0}$. Since the influences of the third or higher order moment functions are negligible, we do not have to expand the higher order conservation laws. The basic variables k, Q, Θ, θ_a and θ_e are scalar and $\mathbf{q}, \mathbf{u}, \mathbf{\Gamma}(\mathbf{x},t)$ are vectors.

The basic idea of deriving macroscopic models from Boltzmann equation is the same as finding the expectation of a random variable. Therefore, finding the individual variables that are meaningful and multiplying them to distribution f will obtain macroscopic variables (average or group quantities). The individual variables are named as moment functions and denoted as $\chi(\mathbf{x},\mathbf{v},t)$. The moment functions chosen by related researches are 1, $\mathbf{v}$, and $\mathbf{v}^2$. However, $\mathbf{v}^2$ does not have physical meaning in traffic. For this reason, the moment functions chosen herein are 1, $\mathbf{v}$, and $\|\mathbf{v}-\mathbf{u}_e\|^2$, where $\|\mathbf{v}-\mathbf{u}_e\|^2$ is individual velocity variance. Thus, multiplying Eq. (16) by $\chi(\mathbf{x},\mathbf{v},t)$, we have

$$\begin{aligned}&\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\chi(\mathbf{x},\mathbf{v},t) + [\mathbf{v}\cdot\nabla_{\mathbf{x}} f(\mathbf{x},\mathbf{v},t)]\chi(\mathbf{x},\mathbf{v},t) + [e\mathbf{E}\cdot\nabla_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)]\chi(\mathbf{x},\mathbf{v},t)\\ &\quad = \left(\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\right)_{coll}\chi(\mathbf{x},\mathbf{v},t)\,.\end{aligned} \tag{15}$$

The integration form of Eq. (15) is illustrated as

$$\begin{aligned}&\int_{\mathbf{v}} \frac{\mathrm{d}f(\mathbf{x},\mathbf{v},t)}{\mathrm{d}t}\chi(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}\\ &\quad = \int_{\mathbf{v}} \frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\chi(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v} + \int_{\mathbf{v}} [\mathbf{v}\cdot\nabla_{\mathbf{x}} f(\mathbf{x},\mathbf{v},t)]\chi(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}\\ &\qquad + \int_{\mathbf{v}} [e\mathbf{E}\cdot\nabla_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)]\chi(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}\\ &\quad = \int_{\mathbf{v}} \left(\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\right)_{coll}\chi(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}\,.\end{aligned} \tag{16}$$

By substituting 1, $\mathbf{v}$, and $\|\mathbf{v}-\mathbf{u}_e\|^2$ for $\chi(\mathbf{x},\mathbf{v},t)$ in Eq. (16), the macroscopic system will be obtained. The derivation is shown from Sections 3.3–3.5. Before deriving the macroscopic model from Boltzmann equation, the concept of traffic field [41] should be mentioned in brief first.

3.2. Traffic field

Traffic field is employed to describe the traffic pressure and the accelerated effect in this study. Since the traffic field distributes density on a road, the relation between the traffic field and the density is named as the dispersion model. The derivation of the

dispersion model starts with the discussion of the interaction between a single vehicle and other vehicles by car-following theory [43–46]. Two assumptions are made. The first one is that the influence of cars in the same lane is M times larger than that in the adjacent lanes, where M is a scalar. The second one is that the traffic field is independent of velocity. For the sake of safety, one vehicle on a road adjusts its velocity and spacing according to the relative position between other vehicles so as to avoid the accident. It is assumed that each vehicle has its own field. Vehicles exclude each other by their own field. Thus, the interaction (in terms of traffic force or traffic pressure, which is denoted by $\mathbf{F}$), which is produced by the traffic field ($\tilde{\mathbf{E}}$), among vehicles is a resistance. To simplify the complication of the problem, $\tilde{\mathbf{E}}$ is assumed to depend on spacing and to satisfy the inverse-square law (the gravity model), which means the influence of other vehicles is larger when the spacing is smaller. If we consider the interaction between two vehicles (vehicle 0 and 1), the traffic field produced by vehicle 1 (leader) will act on vehicle 0 (follower). The traffic field acting on vehicle 0 can be formulated as

$$\tilde{\mathbf{E}}_{01} = \frac{e}{\varepsilon_0}\left(\frac{\tilde{x}_0 - \tilde{x}_1}{|\tilde{x}_0 - \tilde{x}_1|^3}\,\mathbf{i} + \frac{\tilde{y}_0 - \tilde{y}_1}{M^2|\tilde{y}_0 - \tilde{y}_1|^3}\,\mathbf{j}\right), \tag{17}$$

where e is the passenger car equivalent, ε_0 is the interacting parameter and $(\tilde{x}_0, \tilde{y}_0)$ and $(\tilde{x}_1, \tilde{y}_1)$ are the position of vehicle 0 and 1, respectively. The influence between two vehicles is larger as the distance between them decreases. Therefore, the assumption of the inverse-square law is reasonable herein. For convenience, we transform the domain from $\tilde{\Omega}$ to Ω, that is, let $x = \tilde{x}$, $y = M\tilde{y}$ and traffic field acting on vehicle 0 in Ω is denoted by

$$\mathbf{E} = \sum_i^N (e_i \mathbf{X}_i/\varepsilon_i \|\mathbf{X}_i\|^3), \tag{18}$$

where N is the number of vehicles on the road, $\mathbf{X}_i$ denotes the spacing. In the continuous space, Eq. (19) can be represented as

$$\mathbf{E} = \frac{e}{\varepsilon}\int_\Omega ((k - k_s)/\|\mathbf{X}\|^2)\,\mathrm{d}\Omega, \tag{19}$$

where e denotes the passenger car equivalent and ε denotes the interacting parameter, if vehicles and driving behavior on the road are the same. k is the actual density and k_s is the unrestrained density that is the density which vehicles do not interfere with each other. The transformed traffic field is a conservative field. Then, a potential function ϕ exists by the potential theory. The potential function ϕ satisfies $\mathbf{E} = -\nabla_{\mathbf{x}}\phi$. Thus, the magnitude of traffic field is illustrated as

$$\mathrm{d}iv\,\mathbf{E} = -\Delta\phi = e(k - k_s)/\varepsilon + K_a, \tag{20}$$

where $\mathrm{d}iv\,\mathbf{E}$ denotes the magnitude of traffic field, $K_a = K_a(\mathbf{x})$, which depends on the position $\mathbf{x}$, is the adjust term of the road condition if the road condition is ideal

$$K_a = 0.$$

3.3. Continuity equation (conservation of vehicle numbers)

After introducing traffic field and Poisson equation, derivation from Boltzmann equation to the macroscopic system is presented. Firstly, let $\chi = 1$, so Eq. (16) becomes Eq. (21), which is expectation of density.

$$\int_{\mathbf{v}} \frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\,\mathrm{d}\mathbf{v} + \int_{\mathbf{v}} [\mathbf{v}\cdot\nabla_{\mathbf{x}} f(\mathbf{x},\mathbf{v},t)]\,\mathrm{d}\mathbf{v} + \int_{\mathbf{v}} [e\mathbf{E}\cdot\nabla_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)]\,\mathrm{d}\mathbf{v}$$
$$= \int_{\mathbf{v}} \left(\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\right)_{coll} \mathrm{d}\mathbf{v}\,. \tag{21}$$

Each term in Eq. (21) is discussed separately; the first term of left-hand side (LHS) can be obtained from the definition (Eq. (7)):

$$\int_{\mathbf{v}} \frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\,\mathrm{d}\mathbf{v} = \frac{\partial}{\partial t}\left[\int_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}\right] = \frac{\partial k(\mathbf{x},t)}{\partial t}\,. \tag{22}$$

Since individual velocity is independent of position and Eq. (8), the second term of LHS is given by

$$\int_{\mathbf{v}} \mathbf{v}\cdot\nabla_{\mathbf{x}} f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v} = \int_{\mathbf{v}} [\nabla_{\mathbf{x}}\cdot(f\mathbf{v}) - f\cdot\nabla_{\mathbf{x}}\mathbf{v}]\,\mathrm{d}\mathbf{v} = \nabla_{\mathbf{x}}\cdot\mathbf{q}(\mathbf{x},t)\,. \tag{23}$$

From Eq. (6), the third term of LHS is represented as

$$\begin{aligned}\int_{\mathbf{v}} [e\mathbf{E}\cdot\nabla_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)]\,\mathrm{d}\mathbf{v} &= e\mathbf{E}\cdot\left[\int_{\mathbf{v}} \nabla\mathbf{v} f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}\right]\\ &= e\mathbf{E}\cdot[f(\mathbf{x},\mathbf{v},t)|_{\partial\Omega_{\mathbf{v}}}]\\ &= 0\,.\end{aligned} \tag{24}$$

At last, the collision term is assumed to be equal to zero. Thus right-hand side (RHS) of Eq. (21) is given by

$$\int_{\mathbf{v}} \left(\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\right)_{\mathrm{coll}} \mathrm{d}\mathbf{v} = 0\,. \tag{25}$$

Therefore, Eqs. (22)–(25) give the first conservation law; that is, conservation of vehicle numbers.

$$\frac{\partial k(\mathbf{x},t)}{\partial t} + \nabla_{\mathbf{x}}\cdot\mathbf{q}(\mathbf{x},t) = 0\,. \tag{26}$$

3.4. Motion equation (conservation of momentum)

The motion equation is derived by substituting the first order moment function, $\mathbf{v}$, for χ in Eq. (16). Then the expectation function of velocity is illustrated as

$$\int_{\mathbf{v}} \frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\,\mathbf{v}\,\mathrm{d}\boldsymbol{v} + \int_{\mathbf{v}} [\mathbf{v}\cdot\nabla_{\mathbf{x}} f(\mathbf{x},\mathbf{v},t)]\mathbf{v}\,\mathrm{d}\boldsymbol{v} + \int_{\mathbf{v}} [e\mathbf{E}\cdot\nabla_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)]\mathbf{v}\,\mathrm{d}\boldsymbol{v}$$

$$= \int_{\mathbf{v}} \left(\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\right)_{coll} \mathbf{v}\,d\boldsymbol{v}\,. \tag{27}$$

Each term in Eq. (27) is also discussed separately; the first term of LHS is obtained

$$\int_{\mathbf{v}} \frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\,\mathbf{v}(\mathbf{v})\,\mathrm{d}\mathbf{v} = \int_{\mathbf{v}} \frac{\partial}{\partial t}[f(\mathbf{x},\mathbf{v},t)\mathbf{v}(\mathbf{v})]\,\mathrm{d}\mathbf{v} - \int_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)\frac{\partial}{\partial t}[\mathbf{v}(\mathbf{v})]\,\mathrm{d}\mathbf{v}$$

$$= \frac{\partial(k\mathbf{u})}{\partial t}\,. \tag{28}$$

By vector analysis, Eqs. (29) and (30) are true,

$$[\mathbf{v}\cdot\nabla f(\mathbf{x},\mathbf{v},t)] = \nabla\cdot(f\mathbf{v}) - (f\cdot\nabla\mathbf{v}) = \nabla\cdot(f\mathbf{v}), \tag{29}$$

$$[\mathbf{v}\cdot\nabla f(\mathbf{x},\mathbf{v},t)]\mathbf{v}(\mathbf{v}) = \nabla\cdot(f\mathbf{v})\mathbf{v} = \nabla\cdot(f\mathbf{v}\mathbf{v}) - f\mathbf{v}\cdot\nabla\mathbf{v}$$

$$= \nabla\cdot(f\mathbf{v}\mathbf{v}). \tag{30}$$

Thus,

$$\int_{\mathbf{v}} [\mathbf{v}\cdot\nabla_{\mathbf{x}} f(\mathbf{x},\mathbf{v},t)]\mathbf{v}(\mathbf{v})\,\mathrm{d}\mathbf{v} = \int_{\mathbf{v}} \nabla_{\mathbf{x}}\cdot(f\mathbf{v}\mathbf{v})\,\mathrm{d}\mathbf{v}$$

$$= \nabla_{\mathbf{x}}\cdot\left(\int_{\mathbf{v}} f\mathbf{v}\mathbf{v}\,\mathrm{d}\boldsymbol{v}\right)\,, \tag{31}$$

where

$$\mathbf{v}\mathbf{v} = (\mathbf{v}-\mathbf{u}+\mathbf{u}-\mathbf{u}_e+\mathbf{u}_e)(\mathbf{v}-\mathbf{u}+\mathbf{u}-\mathbf{u}_e+\mathbf{u}_e)$$

$$= [(\mathbf{v}-\mathbf{u})(\mathbf{u}-\mathbf{u}_e) + (\mathbf{v}-\mathbf{u})\mathbf{u}_e + (\mathbf{u}-\mathbf{u}_e)(\mathbf{v}-\mathbf{u}) + \mathbf{u}_e(\mathbf{v}-\mathbf{u})]$$

$$+ [\mathbf{u}\mathbf{u}] + (\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u}) \tag{32}$$

is a tensor. With $\int_{\mathbf{v}} f(\mathbf{v}-\mathbf{u})(\mathbf{u}-\mathbf{u}_e)\,\mathrm{d}\mathbf{v} = k(\mathbf{u}-\mathbf{u})(\mathbf{u}-\mathbf{u}_e) = 0$ and $\int_{\mathbf{v}} f(\mathbf{v}-\mathbf{u})\mathbf{u}_e\,\mathrm{d}\mathbf{v} = k(\mathbf{u}-\mathbf{u})\mathbf{u}_e = 0$. Eq. (31) becomes

$$\int_{\mathbf{v}} [\mathbf{v}\cdot\nabla_{\mathbf{x}} f(\mathbf{x},\mathbf{v},t)]\mathbf{v}(\mathbf{v})\,\mathrm{d}\mathbf{v} = \nabla_{\mathbf{x}}\cdot\left(\int_{\mathbf{v}} f\mathbf{v}\mathbf{v}\,\mathrm{d}\boldsymbol{v}\right)$$

$$= \nabla_{\mathbf{x}}\cdot\left[\int_{\mathbf{v}} f(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})\,\mathrm{d}\mathbf{v} + \int_{\mathbf{v}} f\mathbf{u}\mathbf{u}\,\mathrm{d}\boldsymbol{v}\right]\,. \tag{33}$$

Assume the influence of velocity is dominated by the component of the same direction, i.e., $(\mathbf{v}_x - \mathbf{u}_x)^2\mathbf{ii} \gg (\mathbf{v}_x - \mathbf{u}_x)(\mathbf{v}_y - \mathbf{u}_y)\mathbf{ij}$ and $(\mathbf{v}_y - \mathbf{u}_y)^2\mathbf{jj} \gg (\mathbf{v}_x - \mathbf{u}_x)(\mathbf{v}_y - \mathbf{u}_y)\mathbf{ji}$.

Define $\int_{\mathbf{v}} f(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})\,\mathrm{d}\mathbf{v} \approx \int_{\mathbf{v}} f(\mathbf{vv}-\mathbf{uu})\mathbf{I}\,\mathrm{d}\boldsymbol{v} \equiv k\boldsymbol{\theta}_a$ where $\mathbf{I}$ is the 2×2 identity matrix. The summation of diagonal of $\boldsymbol{\theta}_a$ equals to θ_a, i.e., $\mathrm{tr}(\boldsymbol{\theta}_a) = \theta_a$.

Since

$$\int_{\mathbf{v}} f\mathbf{uu}\,\mathrm{d}\boldsymbol{v} = \mathbf{uu}\int_{\mathbf{v}} f\,\mathrm{d}\mathbf{v} = k\mathbf{uu}\,,$$

the second term of LHS becomes

$$\int_{\mathbf{v}} [\mathbf{v}\cdot\nabla_{\mathbf{x}} f(\mathbf{x},\mathbf{v},t)]\mathbf{v}(\mathbf{v})\,\mathrm{d}\mathbf{v} = \nabla_{\mathbf{x}}\cdot(k\boldsymbol{\theta}_a + k\mathbf{uu})\,. \tag{34}$$

The third term of LHS is computed by

$$\begin{aligned}\int_{\mathbf{v}} [e\mathbf{E}\cdot\nabla_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)]\mathbf{v}\,\mathrm{d}\boldsymbol{v} &= e\mathbf{E}\cdot\int_{\mathbf{v}}\nabla_{\mathbf{v}}(f\mathbf{v})\,\mathrm{d}\mathbf{v} - e\mathbf{E}\cdot\int_{\mathbf{v}} f\nabla_{\mathbf{v}}\mathbf{v}\,\mathrm{d}\boldsymbol{v}\\ &= e\mathbf{E}\cdot[(f\mathbf{v})|_{\partial\Omega_{\mathbf{v}}}] - e\mathbf{E}\cdot\int_{\mathbf{v}} f\,\mathrm{d}\mathbf{v}\\ &= -ek\mathbf{E}\,.\end{aligned} \tag{35}$$

The RHS of Eq. (27) is assumed to satisfy the relaxation time approximation, i.e.,

$$\int_{\mathbf{v}}\left(\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\right)_{coll}\mathbf{v}\,d\boldsymbol{v} = \left[\frac{\partial}{\partial t}(k\mathbf{u})\right]_{coll} = -\frac{k\mathbf{u}-k\mathbf{u}_e}{\tau_m}\,, \tag{36}$$

where τ_m is the velocity relaxation time. Eq. (36) means that if the average velocity does not equal the equilibrium velocity, it will become the equilibrium velocity gradually after a period of time $\tau_m(k)$. Therefore, Eq. (27) becomes

$$\frac{\partial(k\mathbf{u})}{\partial t} + \nabla_{\mathbf{x}}\cdot(k\boldsymbol{\theta}_a + k\mathbf{uu}) = ek\mathbf{E} - \frac{k\mathbf{u}-k\mathbf{u}_e}{\tau_m}\,. \tag{37}$$

From Eq. (26), Eq. (37) is represented as

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\nabla_{\mathbf{x}}\cdot\mathbf{u} = -\frac{1}{k}[\nabla_{\mathbf{x}}\cdot(k\boldsymbol{\theta}_a)] + e\mathbf{E} - \frac{\mathbf{u}-\mathbf{u}_e}{\tau_m}\,, \tag{38}$$

if $k \neq 0$. Thus, the conservation of momentum is obtained, which describes the changing of group velocity. Therefore, Eq. (38) also is known as a motion equation.

From Eq. (37), the explicit form of flow density can be derived as follows:

$$\mathbf{q} = k\mathbf{u} = -\tau_m k\frac{\partial\mathbf{u}}{\partial t} - \tau_m k\mathbf{u}\nabla_{\mathbf{x}}\cdot\mathbf{u} - \tau_m[\nabla_{\mathbf{x}}\cdot(k\boldsymbol{\theta}_a)] + \tau_m e\mathbf{E} + k\mathbf{u}_e\,. \tag{39}$$

Under steady state and homogeneous velocity assumption, it's reasonable to assume that $\mathbf{u} \propto \mathbf{u}_e$. Let $k\mathbf{u} - k\mathbf{u}_e = \varsigma k\mathbf{u}$ and $\mu = \tau_m/\varsigma$. Eq. (39) becomes

$$\mathbf{q} = -\mu[\nabla_{\mathbf{x}}\cdot(k\boldsymbol{\theta}_a)] + ek\mu\mathbf{E} = ek\mu\mathbf{E} - v\cdot\nabla_{\mathbf{x}}k\,, \tag{40}$$

where average velocity variance matrix equals to the equilibrium velocity variance ($\boldsymbol{\theta}_a = \boldsymbol{\Theta}_e$), and $v = \mu\boldsymbol{\Theta}_e$. The first term of Eq. (40) is called the drift term, which is induced by velocity (or field). The second term of Eq. (40) is called the diffusion term, which is induced by distribution of density. Eq. (40) is the same as the fundamental diagram with diffusion effect $\mathbf{q} = ek\mathbf{u}_{drift} - v \cdot \nabla_{\mathbf{x}} k$, where $\mathbf{u}_{drift} = \mu\mathbf{E} = \tau_m \mathbf{E}/\varsigma$. The result can also be derived from Fick's law. Eq. (40) can be employed when velocity and velocity variance are homogeneous and stationary. Generally, Eq. (38) is employed as the explicit function of flow density.

3.5. Variance equation (conservation of energy)

The last equation considered herein is the variance equation, which is obtained by substituting the second moment function, $\|\mathbf{v} - \mathbf{u}_e\|^2$, for χ in Eq. (16). Then the expectation function of velocity variance is illustrated as

$$\int_{\mathbf{v}} \frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t} \|\mathbf{v} - \mathbf{u}_e\|^2 \, \mathrm{d}\mathbf{v} + \int_{\mathbf{v}} [\mathbf{v} \cdot \nabla_{\mathbf{x}} f(\mathbf{x},\mathbf{v},t)] \|\mathbf{v} - \mathbf{u}_e\|^2 \, \mathrm{d}\mathbf{v}$$

$$+ \int_{\mathbf{v}} [e\mathbf{E} \cdot \nabla_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)] \|\mathbf{v} - \mathbf{u}_e\|^2 \, \mathrm{d}\mathbf{v}$$

$$= \int_{\mathbf{v}} \left(\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t} \right)_{coll} \|\mathbf{v} - \mathbf{u}_e\|^2 \, \mathrm{d}\mathbf{v} \,. \tag{41}$$

Also, the derivation is done by each term of Eq. (41). The first term of LHS is

$$\int_{\mathbf{v}} \frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t} \|\mathbf{v} - \mathbf{u}_e\|^2 \, \mathrm{d}\mathbf{v} = \int_{\mathbf{v}} \frac{\partial}{\partial t} (f \|\mathbf{v} - \mathbf{u}_e\|^2) \, \mathrm{d}\mathbf{v} - \int_{\mathbf{v}} f \, \frac{\partial}{\partial t} (\|\mathbf{v} - \mathbf{u}_e\|^2) \, \mathrm{d}\mathbf{v}$$

$$= \frac{\partial (k\Theta)}{\partial t} \,. \tag{42}$$

The second term of LHS is

$$\int_{\mathbf{v}} [\mathbf{v} \cdot \nabla_{\mathbf{x}} f] \|\mathbf{v} - \mathbf{u}_e\|^2 \, \mathrm{d}\mathbf{v}$$

$$= \int_{\mathbf{v}} \nabla_{\mathbf{x}} \cdot \lfloor f\mathbf{v}(\|\mathbf{v} - \mathbf{u}\|^2 + 2(\mathbf{v} - \mathbf{u}) \cdot (\mathbf{u} - \mathbf{u}_e) + \|\mathbf{u} - \mathbf{u}_e\|^2) \rfloor \, \mathrm{d}\mathbf{v}$$

$$- \int_{\mathbf{v}} f\mathbf{v} \cdot \nabla_{\mathbf{x}} (\|\mathbf{v} - \mathbf{u}_e\|^2) \, \mathrm{d}\mathbf{v}$$

$$= \nabla_{\mathbf{x}} \cdot [(k\boldsymbol{\Gamma}) + (k\theta_a \mathbf{u}) + 2k(\mathbf{u} - \mathbf{u}_e)\boldsymbol{\theta}_a + (k\theta_e \mathbf{u})]$$

$$\approx \nabla_{\mathbf{x}} \cdot [(k\Theta\mathbf{u}) + 2k\boldsymbol{\theta}_a \cdot (\mathbf{u} - \mathbf{u}_e)] \,, \tag{43}$$

since,

$$\int_{\mathbf{v}} (f\mathbf{u}(\mathbf{v} - \mathbf{u}) \cdot (\mathbf{u} - \mathbf{u}_e)) \, \mathrm{d}\mathbf{v} = \left[\int_{\mathbf{v}} (f(\mathbf{v} - \mathbf{u})) \, \mathrm{d}\mathbf{v} \cdot (\mathbf{u} - \mathbf{u}_e) \right] \mathbf{u} = 0, \quad \boldsymbol{\Gamma} \approx 0 \,,$$

and

$$\int_{\mathbf{v}} f\mathbf{v}\cdot\nabla_{\mathbf{x}}(\|\mathbf{v}-\mathbf{u}_e\|^2)\,\mathrm{d}\mathbf{v}=0$$

as mentioned before.

The third term of LHS is

$$\begin{aligned}\int_{\mathbf{v}}[e\mathbf{E}\cdot\nabla_{\mathbf{v}}f(\mathbf{x},\mathbf{v},t)]\|\mathbf{v}-\mathbf{u}_e\|^2\,\mathrm{d}\mathbf{v}&=e\mathbf{E}\cdot\int_{\mathbf{v}}\nabla_{\mathbf{v}}(f\|\mathbf{v}-\mathbf{u}_e\|^2)\,\mathrm{d}\mathbf{v}\\&\quad-e\mathbf{E}\cdot\int_{\mathbf{v}}f\nabla_{\mathbf{v}}\|\mathbf{v}-\mathbf{u}_e\|^2\,\mathrm{d}\mathbf{v}\\&=e\mathbf{E}\cdot[(f\|\mathbf{v}-\mathbf{u}_e\|^2)|_{\partial\Omega_{\mathbf{v}}}]\\&\quad-e\mathbf{E}\cdot\int_{\mathbf{v}}f(2\mathbf{v}-2\mathbf{u}_e)\,\mathrm{d}\mathbf{v}\\&=-2e\mathbf{E}\cdot k(\mathbf{u}-\mathbf{u}_e)\,.\end{aligned}\tag{44}$$

The last term is the RHS of Eq. (41), which is also assumed to satisfy the relaxation time approximation, i.e.,

$$\begin{aligned}\int_{\mathbf{v}}\left(\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t}\right)_{coll}\|\mathbf{v}-\mathbf{u}_e\|^2\,\mathrm{d}\mathbf{v}&=\left[\frac{\partial}{\partial t}(k\Theta)\right]_{coll}-\int_{\mathbf{v}}(f(\mathbf{x},\mathbf{v},t))_{coll}\frac{\partial}{\partial t}\|\mathbf{v}-\mathbf{u}_e\|^2\,\mathrm{d}\mathbf{v}\\&=-\frac{k\Theta-k\Theta_e}{\tau_e}\,,\end{aligned}\tag{45}$$

where τ_e is the relaxation time of velocity variance. Therefore,

$$\frac{\partial(k\Theta)}{\partial t}+\nabla_{\mathbf{x}}\cdot[(k\Theta\mathbf{u})+2k(\mathbf{u}-\mathbf{u}_e)\boldsymbol{\theta}_a]=-2ek\mathbf{E}\cdot(\mathbf{u}-\mathbf{u}_e)-\frac{k\Theta-k\Theta_e}{\tau_e}\tag{46}$$

is obtained. From Eq. (26), Eq. (46) is represented as

$$\frac{\partial\Theta}{\partial t}+\mathbf{u}\cdot\nabla_{\mathbf{x}}\Theta=-2e\mathbf{E}\cdot(\mathbf{u}-\mathbf{u}_e)-\frac{\Theta-\Theta_e}{\tau_e}-\frac{2}{k}\nabla_{\mathbf{x}}\cdot[k(\mathbf{u}-\mathbf{u}_e)\boldsymbol{\theta}_a]\,,\tag{47}$$

if $k\neq 0$. Thus, the conservation of energy is obtained, which describes the changing of group velocity variance.

Other conservation laws can be derived by multiplying higher order moment functions to Boltzmann equation as long as the moment functions are meaningful and the macroscopic quantities obtained are significant. As mentioned by Helbing [34,36], Boltzmann equation multiplied by the third-order moment function produces the expectation function of skewness, which is near zero. Therefore, higher order moment functions are not discussed in this study.

The system equations developed above includes three conservation laws derived from Boltzmann equation and Poisson equation derived from traffic field. These four equations are transient equations, which describe the changing of variables. However, a complete dynamic system not only needs transient equations, but also needs state equations, which describe the state of variables. A state equation is needed so as to make

the system self-consistent. In this study, the dynamic system is assumed to become the equilibrium state gradually. Therefore, the equilibrium distribution is employed to be the state equation in the system. The derivation of the equilibrium distribution is shown in the following section.

3.6. Derivation of the equilibrium distribution

Since the traffic field affects the movement of vehicles, density should be distributed by the field (or potential). The relation between density and potential is obtained from the assumption that density will tend to become its equilibrium state under a specific traffic situation. The equation is derived by solving the following mathematical programming:

$$\text{Max} \quad W = N! \Big/ \prod_i n_i! \quad \forall i \in \Omega\,, \tag{48}$$

$$\text{s.t.} \quad \sum_i n_i = n_1 + n_2 + \cdots + n_m = N \quad \forall i \in \Omega\,, \tag{49}$$

$$\sum_i n_i \Theta_i = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_m\Theta_m = \Theta_{\text{tol}} \quad \forall i \in \Omega\,, \tag{50}$$

where i is the number of intervals on the road, n_i is the vehicle number of interval i, Θ_{tol} is the total velocity variance, and Θ_i is the velocity variance in interval i. The velocity variance of individual car is defined as $\|\bar{\mathbf{u}}_i - \mathbf{u}_e\|^2$, where $\bar{\mathbf{u}}_i$ is the average velocity of interval i and $\mathbf{u}_e$ is the equilibrium velocity. Eq. (48) is the objective function and Eqs. (49) and (50) are the given macroscopic phenomena. Eq. (48) finds out the most possible combination of n_i if the total number of cars on the road is N. Eq. (48) is a simplified form, which neglects total number of all-possible combination at denominator, since the denominator is a constant. Eq. (49) is the conservation of vehicle numbers and Eq. (50) is the conservation of total variance. The mathematical programming can be solved by the KKT condition. The solution obtained is the most possible density distribution, which is denoted by $k = k(\Theta)$. However, variance is not considered in some models, such as, LWR model and PW model. Therefore, the function $k{=}k(\Theta)$ must be converted to $k{=}k(\phi)$. The transformation is made as follows. Since the traffic pressure $\mathbf{F}$ is proportional to the acceleration, i.e., $\mathbf{F} \propto \mathrm{d}\mathbf{u}/\mathrm{d}t$, we have $\mathbf{F} \propto -\nabla\Theta$ from the relation between acceleration and energy because variance Θ can be referred to energy. From the relation among the traffic pressure, traffic field and traffic potential, $\mathbf{F} \propto \mathbf{E} = -\nabla\phi$, we have $\Theta = e\phi$ while passenger car equivalent e is a relative scalar. Through the transformation, the equilibrium distribution of density not only can be coupled with the gas-kinetic model, but also can be coupled with the LWR-like model and PW-like model. Transforming velocity variance into potential, the equilibrium distribution is given as

$$k = K_0 \exp((e\psi - e\phi)/\Theta_e)\,, \tag{51}$$

where K_0 is the essential density, Θ_e is the equilibrium velocity variance, ψ is the potential equivalent of the velocity variance threshold. ψ is named as the potential barrier here. We can infer several points from Eq. (51). The first one is that density

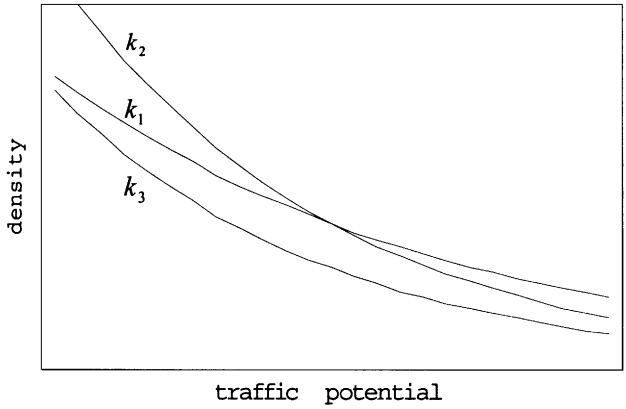

Fig. 1. The density curves of different potential barrier and equilibrium velocity variance, where $k_1 = K_0 \exp((e\psi_1 - e\phi)/\Theta_{e1})$, $k_2 = K_0 \exp((e\psi_1 - e\phi)/\Theta_{e2})$ and $k_3 = K_0 \exp((e\psi_2 - e\phi)/\Theta_{e1}) \cdot \Theta_{e1} > \Theta_{e2}$ and $\psi_1 > \psi_2$.

decreases as traffic potential increases. The second one is as the equilibrium velocity variance increases, the variation of density increases, which means the traffic is sensitive. The third one is as the potential barrier is low, the density is small; that is, drivers are aggressive. The three points are also illustrated in Fig. 1.

By coupling Eqs. (20) and (51), the nonlinear dispersion model is obtained. If a set of boundary conditions of the traffic potential is applied, vehicles are forced to drive through the road according to the path, which has the least resistance. Therefore, vehicles on the two-dimensional research domain will not move forward and backward or circle round. They will try to pass through the road as soon as possible.

3.7. Closure relations

The system presented herein also needs closure relations so as to determine the equilibrium velocity $\mathbf{u}_e(k, \mathbf{u}, \Theta)$, equilibrium variance $\Theta_e(k, \mathbf{u}, \Theta)$, and relaxation time τ_m and τ_e in Eqs. (37) and (46). There are a variety of possible closure relations, which could be adopted from previous studies. [3,8–11,13,14,20,40]. The $\mathbf{u}_e$ and Θ_e proposed in study are represented by

$$\mathbf{u}_e(k, \mathbf{u}, \Theta) = \mathbf{u}_0 - \tau_m \mathbf{p}_b(k) k\Theta \tag{52}$$

and

$$\Theta_e(k, \mathbf{u}, \Theta) = \tau_e \mathbf{p}_p(k) \cdot k\Theta\mathbf{u}\,, \tag{53}$$

respectively. $\mathbf{u}_0$ is the average desired velocity, $\mathbf{p}_b(k) \in [0, 1]$ is the braking probability vector, and $\mathbf{p}_p(k) \in [0, 1]$ is the passing probability vector. The explicit forms are obtained by specifying expressions for $\mathbf{p}_b(k)$ and $\mathbf{p}_p(k)$. Eq. (52) means that the equilibrium velocity decreases as $\mathbf{p}_b(k)$ increases. On the other hand, the equilibrium

variance increases as $\mathbf{p}_p(k)$ increases. Since $\mathbf{u}_e(k)$ and $\Theta_e(k,\mathbf{u})$ are equilibrium equations, the functions suggested in this study are

$$\mathbf{u}_e(k) = \frac{\int_{\mathbf{v}} \mathbf{v} f_e(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}}{\int_{\mathbf{v}} f_e(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}} \tag{54}$$

and

$$\Theta_e(k) = \frac{\int_{\mathbf{v}} \|\mathbf{v}-\mathbf{u}_e\|^2 f_e(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}}{\int_{\mathbf{v}} f_e(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}} \tag{55}$$

respectively. f_e denotes the steady state homogeneous solution of Boltzmann equation. The relaxation time τ_m and τ_e are shown as

$$\tau_m(k) = \frac{T_m}{g(k)} \quad \text{and} \quad \tau_e(k) = \frac{T_e}{g(k)}\,, \tag{56}$$

which are modified from the suggestion of Helbing [38]. $g(k)$ is the proportion of freely moving vehicles, T_m is the reaction time of velocity and T_e is the reaction time of variance. As f_e, $g(k)$ T_m and T_e are determined, the closure relations are expressed specifically. Then, the self-consistent system is complete.

4. Multiclass users model

Dynamic multilane traffic flow model proposed in the previous studies includes three conservation laws, which control the motion of vehicles, and a nonlinear Poisson equation, which distribute the density on the road. The set of equations can only describe one driving behavior. Fortunately, the system can be extended to multi-class users model by employing the concept of Hoogendoorn and Bovy [18–20]. They considered that:

(a) each class has different behavior and is described by different conservation laws;
(b) space of a road section is limited and all class of users share the space.

The first assumption is easy to achieve. According to different driving behavior, a specific Boltzmann equation is derived. With the same procedure mentioned in Section 3, the macroscopic kinetic system of each class of user is derived from the specific Boltzmann equation. The concept of Hoogendoorn and Bovy's second assumption is the same as Poisson equation (Eq. (20)) of our system. If there are i classes of users or vehicles on the road, Eq. (19) is modified as

$$\mathbf{E} = \sum_i \frac{e_i}{\varepsilon_i} \int_\Omega \frac{k_i(x,y)}{\|\mathbf{X}\|^2}\,\mathrm{d}\Omega - \frac{e}{\varepsilon} k_s + K_a\,, \tag{57}$$

where the subscript i denotes the variables of user i. Thus,

$$div\,\mathbf{E} = -\Delta\phi = \sum_i \frac{e_i k_i}{\varepsilon_i} - \frac{e}{\varepsilon} k_s + K_a\,. \tag{58}$$

The other equations in the multi-class users' system are illustrated as follows:

$$\mathbf{q}_i(\mathbf{x},t) = k_i(\mathbf{x},t)\mathbf{u}_i(\mathbf{x},t)\,, \tag{59}$$

$$\frac{\partial k_i(\mathbf{x},t)}{\partial t} + \nabla_{\mathbf{x}} \cdot \mathbf{q}_i(\mathbf{x},t) = 0\,, \tag{60}$$

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \nabla_{\mathbf{x}} \cdot \mathbf{u}_i = -\frac{1}{k_c}\left[\nabla_{\mathbf{x}} \cdot (k_i \boldsymbol{\theta}_{ia})\right] - e_i \nabla\phi - \frac{\mathbf{u}_i - \mathbf{u}_{ie}}{\tau_{im}}\,, \tag{61}$$

$$\frac{\partial \Theta_i}{\partial t} + \mathbf{u}_i \cdot \nabla_{\mathbf{x}} \Theta_i = 2e_i \nabla\phi \cdot (\mathbf{u}_i - \mathbf{u}_{ie}) - \frac{\Theta_i - \Theta_{ie}}{\tau_{ie}} - \frac{2}{k_i}\nabla_{\mathbf{x}} \cdot [k_i(\mathbf{u}_i - \mathbf{u}_{ie})\boldsymbol{\theta}_{ia}]\,, \tag{62}$$

$$k_i = K_{i0} \exp\left(-\frac{e_i\phi - e_i\psi_i}{\Theta_{ie}}\right)\,. \tag{63}$$

The system equations above mean that each class of user is controlled by his own conservation laws. By coupling with the Poisson equation (Eq. (57)), a self-consistent multi-class users dynamic traffic flow model is obtained. Total flow density and total density are represented as

$$\mathbf{q}(\mathbf{x},t) = \sum_i e_i \mathbf{q}_i(\mathbf{x},t) = \sum_i e_i k_i(\mathbf{x},t)\mathbf{u}_i(\mathbf{x},t) \tag{64}$$

and

$$k(\mathbf{x},t) = \sum_i e_i k_i(\mathbf{x},t)\,, \tag{65}$$

respectively.

5. Conclusions and perspectives

In this paper we have derived a macroscopic multilane traffic model for multiple classes users. The system is a self-consistent system; it can be solved with proper initial conditions and boundary conditions. Our consideration is based on the following assumptions:

(a) A multilane road is considered as a two-dimensional domain;
(b) the whole system will tend toward equilibrium;
(c) each class of user has different behavior and is described by different conservation laws (the gas-kinetic model);
(d) the individual velocity variance is employed as the second moment function;
(e) the space of a road section is limited and all classes of users share the space.

Considering a multilane road as a two-dimensional domain allows us to handle the driving behaviors, which are not restricted to drive one by one in a single lane. Another advantage of this consideration is to avoid modeling complicated lane-changing behavior. Lane-changing behavior is controlled by the nonlinear Poisson equation. If the

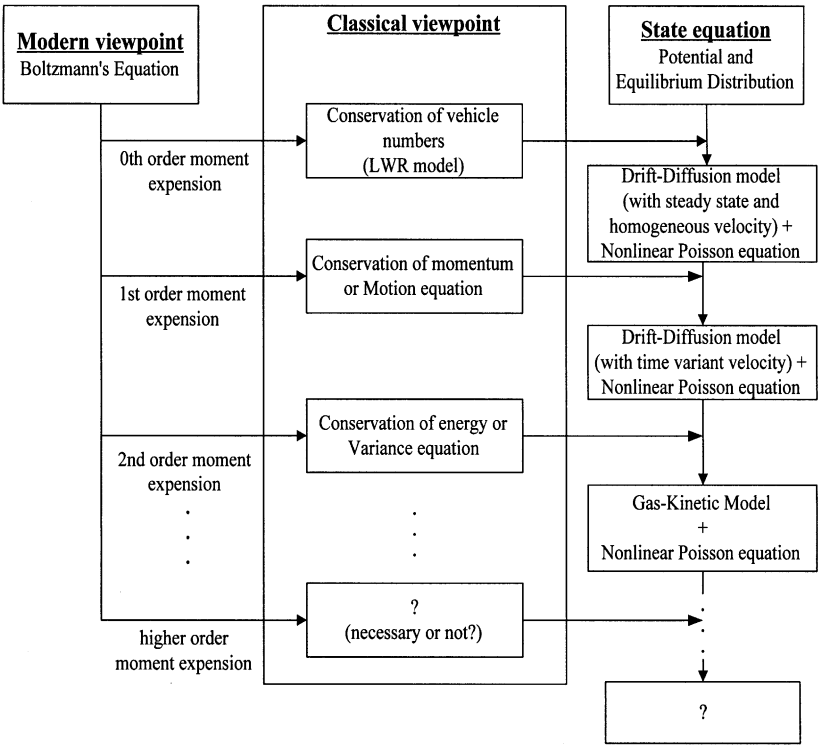

Fig. 2. The relationship between models.

research area is only a single lane road, the system can be reduced to a one-dimensional model.

This study derives a dynamic macroscopic traffic flow system from Boltzmann equation. Boltzmann equation employed herein includes accelerated effect, which is controlled by Poisson equation. Helbing [30–40] and Hoogendoorn and Bovy [18,19] followed Paveri-Fontana [22] to multiply Boltzmann equation by 1, $\mathbf{v}$ and $\mathbf{v}^2$ so as to derive macroscopic systems. However, the second-order moment $\mathbf{v}^2$ does not make sense in traffic flow. Although multiplying Boltzmann equation by $\mathbf{v}^2$ can obtain velocity variance equation, it may also generate some meaningless terms. This study modifies the second moment function as $\|\mathbf{v} - \mathbf{u}_e\|^2$, which is individual velocity variance, and reformulates velocity variance equation to be more reasonable. This study exposes three moment functions. If there still exists the other meaningful moment, it should be considered as its influence is significant.

In addition, the Poisson equation plays an important role in the system. Since Poisson equation distributes vehicles on a road, it becomes the key point to extend the system to multiclass user traffic. Besides, the equilibrium distribution is employed to determine the state of traffic flow. Thus, the system is self-consistent.

The system equations and the relationship among them are illustrated in Fig. 2. From Fig. 2, the system equations can be simplified to adapt different traffic condition because

not all variables are significant in each traffic condition. For example, in uniform and equilibrium traffic flow, the influence of velocity variance equation may be ignored. The simplification is needed because computation of the whole system takes a lot of time. According to different traffic conditions, simplification should be discussed and validated further. Also, numerical methods should be developed to solve the system.

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