Elastic solutions of displacements for a transversely isotropic half-space subjected to three-dimensional buried parabolic rectangular loads

Cheng-Der Wang [a,*], Jyh-Jong Liao [b]

[a] *Department of Civil Engineering, Nanya Institute of Technology, 414, Chung-Shan E. Rd. Sec. 3, Chung-Li 320, Taoyuan, Taiwan, ROC*
[b] *Department of Civil Engineering, National Chiao Tung University, 1001, Ta Hsueh Road, Hsin-Chu 300, Taiwan, ROC*

Abstract

This paper presents the closed-form solutions for estimating the displacements in a transversely isotropic half-space subjected to three-dimensional buried linearly varying, uniform, and parabolic rectangular loads. The loading types include an upward linearly varying load, a downward linearly varying load, a uniform load, a concave parabolic load, and a convex parabolic load on a rectangle. The planes of transverse isotropy are assumed to be parallel to the horizontal surface of the half-space. The presented solutions are obtained from integrating of the point load solutions in a Cartesian co-ordinate system for the transversely isotropic half-space. The solutions for a transversely isotropic medium subjected to parabolic rectangular loads are never mentioned in the literature. The parabolic loads might be more realistic than linear variable or uniform loads acting on foundations. The buried depth, the dimensions of loaded area, the type and degree of material anisotropy, and the loading type for transversely isotropic half-spaces influence these solutions. An illustrative example is presented to investigate the effect of the type and degree of rock anisotropy, and the loading types on the displacement. The proposed solutions can provide reasonably results to estimate the induced displacements in the isotropic/transversely isotropic geomaterials subjected to three-dimensional buried parabolic rectangular loads for practical purposes.
© 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Closed-form solutions; Displacements; Transversely isotropic half-space; Three-dimensional; Parabolic rectangular loads

1. Introduction

The deformation response of the materials is an important factor in the design of foundations. In general, the magnitude and distribution of the displacements in soils/rocks are predicted by using analytical solutions that model the constituted materials as a linearly elastic, homogeneous and isotropic continuum.

* Corresponding author. Tel.: +886-3-4361070x302; fax: +886-3-4372193.
E-mail address: cdwang@nanya.edu.tw (C.-D. Wang).

However, for soils are deposited through a process of sedimentation over a long period of time, or rock masses cut by discontinuities, such as cleavages, foliations, stratifications, schistosities, joints, these solutions should account for anisotropy. Especially, anisotropy rocks are often modeled as orthotropic or transversely isotropic materials from the standpoint of practical considerations in engineering. In this paper, elastic solutions of displacements for linearly varying, uniform, and parabolic loading problems in a transversely isotropic half-space are relevant.

Foundation displacements are usually estimated by the use of the classical Boussinesq solution (1885) for a point load acting on the surface of an isotropic half-space. It means that the soil/rock above the level of the foundation is assumed to be taking no part in the distribution of displacement due to the load on the foundation. In calculating the displacements within the soil/rock when the foundations are relatively deep, it might be advisable to use solutions derived for the case of loads applied within the elastic medium (Skopek, 1961). Moreover, Gazetas (1985) found that for most available solutions refer to loads acting directly on the ground surface. To estimate the displacement of an embedded foundation, engineers apply reduction factors (i.e., Fox, 1948) to the displacement of the corresponding surface foundation. However, most reduction factors are purely empirical, and may grossly exaggerate the effect of the embedment. Hence, to derive the closed-form solutions of displacements subjected to the buried loads applied at a regular shape should be useful for practical engineers.

Solution of a concentrated force acting at a point in the interior of an elastic space is known as the fundamental solution or the Green's function solution of three-dimensional elasticity (Tarn and Wang, 1987). The solutions of displacements induced by a concentrated force for a transversely isotropic half-space have been presented by numerous investigators (i.e., Lekhnitskii, 1963; Pan and Chou, 1979; Liao and Wang, 1998). Nevertheless, the types of external loads of a half-space should be more complex than a point load in most engineering cases. Hence, the closed-form solutions for the displacements in a half-space subjected to various loads applied on different regions are needed for engineering design. Regarding the transversely isotropic medium related to buried loads, Chowdhury (1987) utilized the methods of images and Hankel transforms to obtain the exact solutions of surface displacements for a vertical circular load applied in the interior of a transversely isotropic half-space. Hanson and Wang (1997) derived the elastic fields of displacements by potential function subjected to concentrated ring loadings embedded in a full or transversely isotropic half-space. Wang and Liao (1999) presented the elastic closed-form solutions for the displacements in a transversely isotropic half-space induced by various buried loading types. The loading types include finite line loads, uniform rectangular loads, and linearly varied rectangular loads in the x-direction, respectively. Recently, they also proposed the closed-form solutions for the displacements in a transversely isotropic half-space subjected to a buried right triangular loaded region (Wang and Liao, 2001). The loading types include uniform loads, linearly varied loads in the x-direction, and linearly varied loads in the y-direction, respectively.

It is well known that for computing the displacement in transversely isotropic half-spaces subjected to an arbitrary shape loaded area, the loaded area can be divided into many regularly-shaped sub-areas, such as rectangles. However, except the earlier work by the authors (Wang and Liao, 1999), Lin et al. (1991) presented the solutions of vertical surface displacement induced by uniform rectangular loads acting normally to the surface of a transversely isotropic half-space. In many engineering practices, the load actually is not uniformly distributed but more concentrated towards the center of the foundation (Hooper, 1976; Bauer et al., 1979). Therefore, the applied loads distributed in the forms of linearly varying or parabola of revolution might simulate more practical loading conditions. Nevertheless, corresponding to the existing closed-form solutions of this subject for anisotropic half-spaces was limited to solving axisymmetric problems. Gazetas (1982a,b) presented an analytical investigation on the effect of soil transversely isotropy on surface displacement distributions subjected to axisymmetric parabolic vertical surface loading. To the authors' knowledge, no analytical solutions of displacements for a transversely isotropic half-space subjected to three-dimensional buried parabolic rectangular loads have been presented. The closed-form

solutions proposed in this paper can be obtained from directly integrated the point load solutions in a Cartesian co-ordinate system (Wang and Liao, 1999), which were transformed by the same author's solutions in a cylindrical co-ordinate system (Liao and Wang, 1998). The presentation of these proposed solutions is clear and concise. Also, they indicate that the displacements in transversely isotropic half-spaces are affected by the buried depth, the dimensions of loaded region, the type and degree of material anisotropy, and the loading types. An illustrative example is given to investigate the effect of rock anisotropy, and loading types on the vertical surface displacement in the isotropic/transversely isotropic rocks subjected to a linearly varying, uniform, and parabolic rectangular load, respectively.

2. Point load solutions for a transversely isotropic half-space

In this paper, the solutions of displacements in transversely isotropic half-spaces subjected to three-dimensional buried linearly varying, uniform, and parabolic rectangular loads are directly integrated from the point load solutions in a Cartesian co-ordinate system (Wang and Liao, 1999). The planes of transverse isotropy are assumed to be parallel to its horizontal surface. The approaches for solving this boundary value problem of displacements subjected to a static point load with components (P_r, P_θ, P_z) in a cylindrical co-ordinate, which are expressed as the form of body forces, are shown in Fig. 1 (Liao and Wang, 1998). Fig. 1 depicts that a half-space is composed of two infinite spaces, one acting a point load in its interior and the other being free loading, and zero stress boundary conditions on the $z = 0$ plane ($\sigma_{zz} = \tau_{\theta z} = \tau_{rz} = 0$). The field equations can be referred to the paper of Liao and Wang (1998), and the Hankel, Fourier transforms with respect to r, θ are employed for solving this problem, respectively. Hence, the solutions can be derived from the governing equations for an infinite space (including the general solutions (I) and homogeneous solutions (II)) by satisfying the free traction on the surface of the half-space. The closed-form solutions of displacements subjected to a point load (P_x, P_y, P_z) acting at $z = h$ (from the surface) in the interior of a transversely isotropic half-space can be expressed as follows (Wang and Liao, 1999):

$$
\begin{aligned}
u_x^p = {} & \frac{P_x}{4\pi}\left[\frac{k}{m_1}p_{d11} - \frac{k}{m_2}p_{d12} - T_1 p_{d1a} + T_2 p_{d1b} + T_3 p_{d1c} - T_4 p_{d1d} + \frac{1}{u_3 A_{44}}(p_{d23} + p_{d2e})\right] \\
& + \frac{P_y}{4\pi}\left[-\frac{k}{m_1}p_{d31} + \frac{k}{m_2}p_{d32} + T_1 p_{d3a} - T_2 p_{d3b} - T_3 p_{d3c} + T_4 p_{d3d} + \frac{1}{u_3 A_{44}}(p_{d33} + p_{d3e})\right] \\
& - \frac{P_z}{4\pi}\left[k(p_{d41} - p_{d42}) + m_1(T_1 p_{d4a} - T_3 p_{d4c}) - m_2(T_2 p_{d4b} - T_4 p_{d4d})\right]
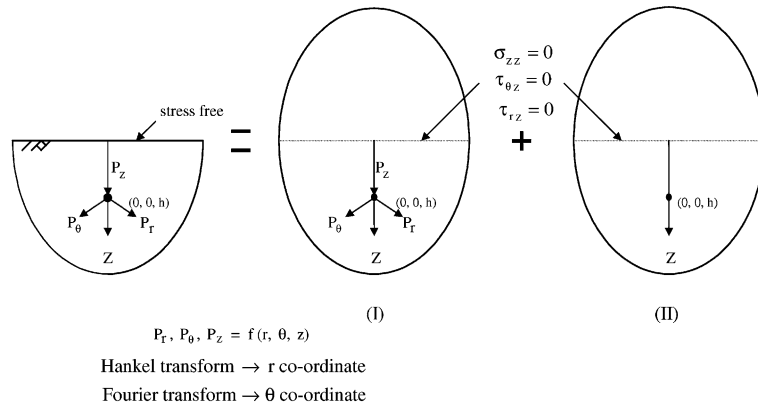\end{aligned}
\tag{1}
$$

Fig. 1. The approach for solving a point load problem (Liao and Wang, 1998).

$$u_y^p = \frac{P_x}{4\pi}\left[-\frac{k}{m_1}p_{d31} + \frac{k}{m_2}p_{d32} + T_1p_{d3a} - T_2p_{d3b} - T_3p_{d3c} + T_4p_{d3d} + \frac{1}{u_3A_{44}}(p_{d33} + p_{d3e})\right]$$

$$+ \frac{P_y}{4\pi}\left[\frac{k}{m_1}p_{d21} - \frac{k}{m_2}p_{d22} - T_1p_{d2a} + T_2p_{d2b} + T_3p_{d2c} - T_4p_{d2d} + \frac{1}{u_3A_{44}}(p_{d13} + p_{d1e})\right]$$

$$- \frac{P_z}{4\pi}\left[k(p_{d51} - p_{d52}) + m_1(T_1p_{d5a} - T_3p_{d5c}) - m_2(T_2p_{d5b} - T_4p_{d5d})\right] \tag{2}$$

$$u_z^p = \frac{P_x}{4\pi}\left[-k(p_{d41} - p_{d42}) + m_1(T_1p_{d4a} - T_2p_{d4b}) - m_2(T_3p_{d4c} - T_4p_{d4d})\right]$$

$$+ \frac{P_y}{4\pi}\left[-k(p_{d51} - p_{d52}) + m_1(T_1p_{d5a} - T_2p_{d5b}) - m_2(T_3p_{d5c} - T_4p_{d5d})\right]$$

$$- \frac{P_z}{4\pi}\left[m_1(kp_{d61} + T_1m_1p_{d6a} - T_2m_2p_{d6b}) - m_2(kp_{d62} + T_3m_1p_{d6c} - T_4m_2p_{d6d})\right] \tag{3}$$

where:

- The generalized Hooke's law for the transversely isotropic medium in a cylindrical co-ordinate system can express the constitutive equations used in this paper as:

$$\sigma_{rr} = A_{11}\varepsilon_{rr} + (A_{11} - 2A_{66})\varepsilon_{\theta\theta} + A_{13}\varepsilon_{zz} \tag{4}$$

$$\sigma_{\theta\theta} = (A_{11} - 2A_{66})\varepsilon_{rr} + A_{11}\varepsilon_{\theta\theta} + A_{13}\varepsilon_{zz} \tag{5}$$

$$\sigma_{zz} = A_{13}(\varepsilon_{rr} + \varepsilon_{\theta\theta}) + A_{33}\varepsilon_{zz} \tag{6}$$

$$\tau_{r\theta} = A_{66}\gamma_{r\theta} \tag{7}$$

$$\tau_{\theta z} = A_{44}\gamma_{\theta z} \tag{8}$$

$$\tau_{rz} = A_{44}\gamma_{rz} \tag{9}$$

where A_{ij} $(i,j = 1 \sim 6)$ are the elastic moduli or elasticity constants of the medium. For a transversely isotropic material, only five independent elastic constants are needed to describe its deformational response. In this paper, the five engineering elastic constants, E, E', v, v' and G' are adopted and defined as:

1. E and E' are Young's moduli in the plane of transverse isotropy and in a direction normal to it, respectively.
2. v and v' are Poisson's ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel or normal to it, respectively.
3. G' is the shear modulus in planes normal to the plane of transverse isotropy. Hence, A_{ij} can be expressed in terms of these elastic constants as:

$$A_{11} = \frac{E\left(1 - \frac{E}{E'}v'^2\right)}{(1+v)\left(1 - v - \frac{2E}{E'}v'^2\right)}, \quad A_{13} = \frac{Ev'}{1 - v - \frac{2E}{E'}v'^2}, \quad A_{33} = \frac{E'(1-v)}{1 - v - \frac{2E}{E'}v'^2}, \quad A_{44} = G',$$

$$A_{66} = \frac{E}{2(1+v)} \tag{10}$$

- $m_j = \dfrac{(A_{13} + A_{44})u_j}{A_{33}u_j^2 - A_{44}} = \dfrac{A_{11} - A_{44}u_j^2}{(A_{13} + A_{44})u_j}$ $(j = 1, 2)$, $k = \dfrac{(A_{13} + A_{44})}{A_{33}A_{44}(u_1^2 - u_2^2)}$, $T_1 = \dfrac{k}{m_1}\dfrac{u_1 + u_2}{u_2 - u_1}$,

  $T_2 = \dfrac{k}{m_2}\dfrac{2u_1(u_2 + m_2)}{(u_2 - u_1)(u_1 + m_1)}$, $T_3 = \dfrac{k}{m_1}\dfrac{2u_2(u_1 + m_1)}{(u_2 - u_1)(u_2 + m_2)}$, $T_4 = \dfrac{k}{m_2}\dfrac{u_1 + u_2}{u_2 - u_1}$;

  $p_{d1i} = \dfrac{1}{R_i + z_i} - \dfrac{x^2}{R_i(R_i + z_i)^2}$, $p_{d2i} = \dfrac{1}{R_i + z_i} - \dfrac{y^2}{R_i(R_i + z_i)^2}$, $p_{d3i} = \dfrac{xy}{R_i(R_i + z_i)^2}$,

  $p_{d4i} = \dfrac{x}{R_i(R_i + z_i)}$, $p_{d5i} = \dfrac{y}{R_i(R_i + z_i)}$, $p_{d6i} = \dfrac{1}{R_i}$.

- $R_i = \sqrt{x^2 + y^2 + z_i^2}$ $(i = 1, 2, 3, a, b, c, d, e)$, $z_1 = u_1(z - h)$, $z_2 = u_2(z - h)$,

  $z_3 = u_3(z - h)$, $z_a = u_1(z + h)$, $z_b = u_1 z + u_2 h$, $z_c = u_1 h + u_2 z$, $z_d = u_2(z + h)$,

  $z_e = u_3(z + h)$.

- $u_3 = \sqrt{A_{66}/A_{44}}$, u_1 and u_2 are the roots of the following characteristic equation:

  $$u^4 - su^2 + q = 0 \tag{11}$$

whereas

$$s = \frac{A_{11}A_{33} - A_{13}(A_{13} + 2A_{44})}{A_{33}A_{44}}, \quad q = \frac{A_{11}}{A_{33}}.$$

The three categories of the characteristic roots, u_1 and u_2 are:

Case 1. $u_{1,2} = \pm\sqrt{\{1/2[s \pm \sqrt{(s^2 - 4q)}]\}}$ are two real distinct roots when $s^2 - 4q > 0$;

Case 2. $u_{1,2} = \pm\sqrt{s/2}, \pm\sqrt{s/2}$ are double equal real roots when $s^2 - 4q = 0$;

Case 3. $u_1 = \frac{1}{2}\sqrt{(s + 2\sqrt{q})} - i\frac{1}{2}\sqrt{(-s + 2\sqrt{q})} = \gamma - i\delta$, $u_2 = \gamma + i\delta$ are two complex conjugate roots (where γ cannot be equal to zero (Lekhnitskii, 1963; Tarn and Wang, 1987; Wang and Rajapakse, 1990)) when $s^2 - 4q < 0$. In order to discuss the displacement is not going to be complex, we choice u_z^P in Eq. (3) when subjected to a vertical point load (P_z) only, to support this point of view. Namely, Eq. (3) is reduced to the following equation:

$$u_z^p = -\frac{P_z}{4\pi}[m_1(kp_{d61} + T_1 m_1 p_{d6a} - T_2 m_2 p_{d6b}) - m_2(kp_{d62} + T_3 m_1 p_{d6c} - T_4 m_2 p_{d6d})] \tag{12}$$

Since u_1 and u_2 are complex conjugate, it is convenient to assume $p_{d61}, p_{d62}, p_{d6a}, p_{d6b}, p_{d6c}, p_{d6d}$ in Eq. (12) as:

$$p_{d61} = \frac{1}{\sqrt{x^2 + y^2 + [u_1(z - h)]^2}} = a - ib, \quad p_{d62} = \frac{1}{\sqrt{x^2 + y^2 + [u_2(z - h)]^2}} = a + ib,$$

$$p_{d6a} = \frac{1}{\sqrt{x^2 + y^2 + [u_1(z + h)]^2}} = c - id, \quad p_{d6d} = \frac{1}{\sqrt{x^2 + y^2 + [u_2(z + h)]^2}} = c + id,$$

$$p_{d6b} = \frac{1}{\sqrt{x^2 + y^2 + (u_1 z + u_2 h)^2}} = e - if, \quad p_{d6c} = \frac{1}{\sqrt{x^2 + y^2 + (u_2 z + u_1 h)^2}} = e + if.$$

Therefore, Eq. (12) can be expressed by replacing $u_1 = \gamma - i\delta$, $u_2 = \gamma + i\delta$, m_1, m_2, k, T_1, T_2, T_3, T_4, and $p_{d61}, p_{d62}, p_{d6a}, p_{d6b}, p_{d6c}, p_{d6d}$ as follows:

$$u_z^p = \frac{P_z}{8\pi A_{33}A_{44}\gamma(\gamma^2 + \delta^2)}\left\{A_{11}(a\delta - b\gamma) + A_{44}(\gamma^2 + \delta^2)(a\delta + b\gamma)\right.$$

$$-\frac{\gamma}{\delta}[A_{11}(c\gamma + d\delta) - A_{44}(\gamma^2 + \delta^2)(c\gamma - d\delta)]$$

$$\left.+ \frac{[A_{11}^2 + 2A_{11}A_{13}(\gamma^2 - \delta^2) + A_{13}^2(\gamma^2 + \delta^2)^2]}{\delta(A_{13} + A_{44})^2} \times \left\{e[A_{11} - A_{44}(\gamma^2 - \delta^2)] + A_{44}f\gamma\delta\right\}\right\} \tag{13}$$

From Eq. (13), it is proved that even though the roots of the characteristic equation are complex, the desired displacement is always real.

3. Linearly varying and parabolic rectangular loads solutions for a transversely isotropic half-space

3.1. Linearly varying rectangular loads

A three-dimensional upward linearly varying load, P_j^{linear} ($j = x, y, z$) (forces per unit area) distributed on a rectangle with length L and width W at the buried depth of h as shown in Fig. 2 is considered. The loading type in Fig. 2 can be treated as the following form:

$$\widetilde{P}_j^{\text{linear}} = P_j^{\text{linear}}\left[1 + \alpha\left(\frac{|x|}{L} + \frac{|y|}{W} - \frac{|xy|}{LW}\right)\right] \tag{14}$$

where α is a constant. According to Eq. (14), α can be classified as three different loading cases:
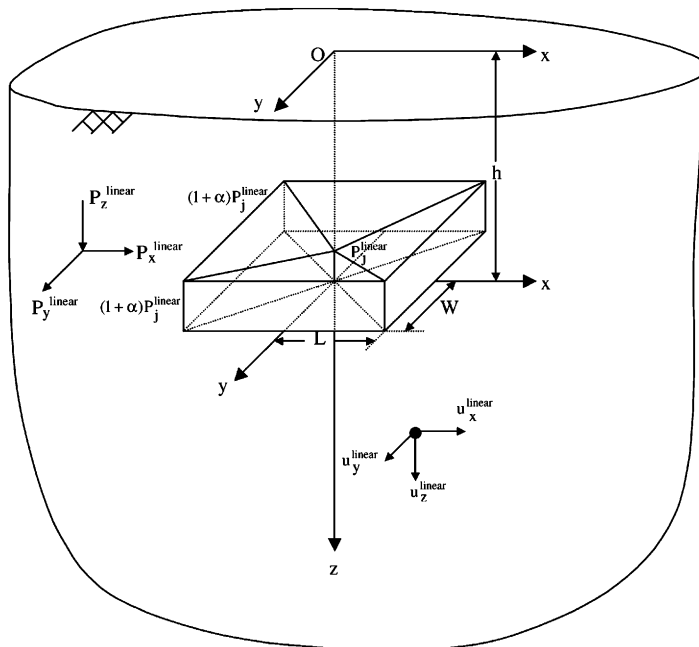

Fig. 2. The case of upward linearly varying rectangular loads with LW area at the buried depth of h ($\alpha > 0$).
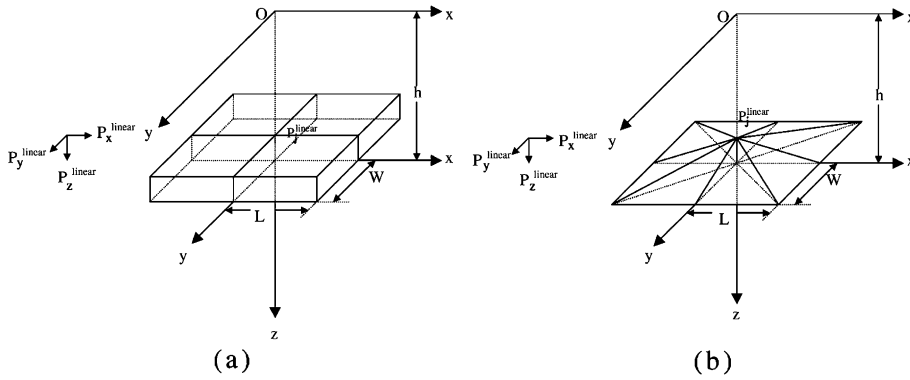
Fig. 3. (a) The case of uniform rectangular loads with LW area at the buried depth of h ($\alpha = 0$), (b) the case of a completely downward linearly varying rectangular loads with LW area at the buried depth of h ($\alpha = -1$).

Case 1. $\alpha > 0$, is the upward linearly varying load as depicted in Fig. 2;
Case 2. $\alpha = 0$, is the uniform load as seen in Fig. 3(a);
Case 3. $\alpha < 0$, is the downward linearly varying load. Fig. 3(b) shows the zero contact stress at the edges of Fig. 2 for the case of $\alpha = -1$.

To solve the displacements in the transversely isotropic half-spaces induced by this load, an elementary force $\widetilde{P}_j^{\text{linear}}\, \mathrm{d}\eta\, \mathrm{d}\varsigma$ acting on an elementary surface $\mathrm{d}\eta\, \mathrm{d}\zeta$ is extracted from the rectangle. Replacing the concentrated force P_j by $\widetilde{P}_j^{\text{linear}}\, \mathrm{d}\zeta\, \mathrm{d}\eta$, y by $(y - \eta)$, and x by $(x - \zeta)$ in Eqs. (1)–(3), the solutions of displacements for the elementary force acting in the half-space are obtained. Then, the complete solutions can be obtained by integrating the solutions induced by the elementary force with η from 0 to W, and ζ from 0 to L, respectively, as follows:

$$[U]^{\text{linear}} = \int_0^L \int_0^W [U]^{\text{p}}\, \mathrm{d}\eta\, \mathrm{d}\varsigma \tag{15}$$

where $[U] = [u_x, u_y, u_z]^{\text{T}}$ (superscript T denotes that the transpose matrix) and the superscripts linear and p express the displacement components that are induced by a linear varying rectangular load and a point load, respectively. By mathematics operations (Gradshteynn and Ryzhik, 1994), the explicit solutions of the displacements in a half-space can be regrouped as the forms of Eqs. (1)–(3). It means that the exact solutions of this case are the same as Eqs. (1)–(3) except that the displacement elementary functions $p_{d1i}, p_{d2i}, \ldots, p_{d6i}$ are replaced by the displacement integral functions $M_{d1i}^{\langle 1 \rangle} \sim M_{d1i}^{\langle 4 \rangle}$, $M_{d2i}^{\langle 1 \rangle} \sim M_{d2i}^{\langle 4 \rangle}, \ldots,$ $M_{d6i}^{\langle 1 \rangle} \sim M_{d6i}^{\langle 4 \rangle}$ ($i = 1, 2, 3, a, b, c, d, e$) for u_x^{linear}, u_y^{linear}, u_z^{linear} (Fig. 2), respectively. For example, p_{d1i} should be replaced by $M_{d1i}^{\langle 1 \rangle} \sim M_{d1i}^{\langle 4 \rangle}$ as:

$$[p_{d1i}] \Rightarrow M_{d1i}^{\langle 1 \rangle} + \alpha \left[\left(\frac{|x|}{L} + \frac{|y|}{W} - \frac{|xy|}{LW} \right) M_{d1i}^{\langle 1 \rangle} - \frac{1}{L}\left(1 - \frac{|y|}{W} \right) M_{d1i}^{\langle 2 \rangle} - \frac{1}{W}\left(1 - \frac{|x|}{L} \right) M_{d1i}^{\langle 3 \rangle} - \frac{1}{LW} M_{d1i}^{\langle 4 \rangle} \right] \tag{16}$$

Similarly, the solutions for different loading types given below also can be expressed as the forms of Eqs. (1)–(3), except for the integral functions. Hence, only the displacement integral functions will be presented.

$$M_{d1i}^{\langle 1 \rangle} = -z_i(D_1 - D_2 + D_3) - xD_4 + x^*D_5 \tag{17}$$

$$M_{d1i}^{\langle 2 \rangle} = -\frac{y}{2}(R_i - R_{x^*i}) + \frac{y^*}{2}(R_{y^*i} - R_{x^*y^*i}) - \frac{(x^2 + z_i^2)}{2}D_4 + \frac{(x^{*2} + z_i^2)}{2}D_5 - z_i(yD_6 - y^*D_7) \tag{18}$$

$$M_{d1i}^{\langle 3\rangle} = x(R_i - R_{y^*i}) - x^*(R_{x^*i} - R_{x^*y^*i}) + z_i(xD_8 - x^*D_9) \tag{19}$$

$$M_{d1i}^{\langle 4\rangle} = \tfrac{1}{6}(R_i^3 - R_{x^*i}^3 - R_{y^*i}^3 + R_{x^*y^*i}^3) + \frac{x^2}{2}(R_i - R_{y^*i}) - \frac{x^{*2}}{2}(R_{x^*i} - R_{x^*y^*i}) - \frac{y^2}{2}(R_i - R_{x^*i})$$
$$+ \frac{y^{*2}}{2}(R_{y^*i} - R_{x^*y^*i}) + \frac{z_i}{2}(x^2D_8 - x^{*2}D_9 - y^2D_6 + y^{*2}D_7) \tag{20}$$

$$M_{d2i}^{\langle 1\rangle} = z_i(D_1 - D_2) - yD_{10} + y^*D_{11} \tag{21}$$

$$M_{d2i}^{\langle 2\rangle} = y(R_i - R_{x^*i}) - y^*(R_{y^*i} - R_{x^*y^*i}) + z_i(yD_6 - y^*D_7) \tag{22}$$

$$M_{d2i}^{\langle 3\rangle} = -\frac{x}{2}(R_i - R_{y^*i}) + \frac{x^*}{2}(R_{x^*i} - R_{x^*y^*i}) - \frac{(y^2 + z_i^2)}{2}D_{10} + \frac{(y^{*2} + z_i^2)}{2}D_{11} - z_i(xD_8 - x^*D_9) \tag{23}$$

$$M_{d2i}^{\langle 4\rangle} = -\frac{(2x^2 - 4y^2 - z_i^2)}{6}R_i + \frac{(2x^{*2} - 4y^2 - z_i^2)}{6}R_{x^*i} + \frac{(2x^2 - 4y^{*2} - z_i^2)}{6}R_{y^*i}$$
$$- \frac{(2x^{*2} - 4y^{*2} - z_i^2)}{6}R_{x^*y^*i} - \frac{z_i}{2}(x^2D_8 - x^{*2}D_9 - y^2D_6 + y^{*2}D_7) \tag{24}$$

$$M_{d3i}^{\langle 1\rangle} = -R_i + R_{x^*i} + R_{y^*i} - R_{x^*y^*i} - z_i(D_8 - D_9) \tag{25}$$

$$M_{d3i}^{\langle 2\rangle} = -\frac{x}{2}(R_i - R_{y^*i}) + \frac{x^*}{2}(R_{x^*i} - R_{x^*y^*i}) - \frac{(y^2 - z_i^2)}{2}D_{10} + \frac{(y^{*2} - z_i^2)}{2}D_{11} + z_i(yD_1 - y^*D_2) \tag{26}$$

$$M_{d3i}^{\langle 3\rangle} = \text{exchanging with } x, y \text{ in } M_{d3i}^{\langle 2\rangle} \tag{27}$$

$$M_{d3i}^{\langle 4\rangle} = \frac{LWz_i}{2} - \frac{1}{3}\left[x(yR_i - y^*R_{y^*i}) - x^*(yR_{x^*i} - y^*R_{x^*y^*i})\right] - \frac{1}{3}(x^3D_4 - x^{*3}D_5 + y^3D_{10} - y^{*3}D_{11})$$
$$+ \frac{z_i}{2}\left(x^2D_{12} - x^{*2}D_{13} + y^2D_1 - y^{*2}D_2 - \frac{z_i^3}{3}D_3\right) \tag{28}$$

$$M_{d4i}^{\langle 1\rangle} = -xD_{12} + x^*D_{13} - yD_6 + y^*D_7 - z_i(D_4 - D_5) \tag{29}$$

$$M_{d4i}^{\langle 2\rangle} = \frac{1}{2}\left(LW - x^2D_{12} + x^{*2}D_{13} + y^2D_1 - y^{*2}D_2 + z_i^2D_3\right) + z_i(yD_{10} - y^*D_{11}) \tag{30}$$

$$M_{d4i}^{\langle 3\rangle} = \frac{z_i}{2}(R_i - R_{x^*i} - R_{y^*i} + R_{x^*y^*i}) - \frac{1}{2}(x^2D_8 - x^{*2}D_9 + y^2D_6 - y^{*2}D_7) \tag{31}$$

$$M_{d4i}^{\langle 4\rangle} = \frac{LW(y + y^*)}{3} - \frac{1}{3}(x^3D_8 - x^{*3}D_9 - y^3D_1 + y^{*3}D_2)$$
$$+ \frac{z_i}{6}[x(R_i - R_{y^*i}) - x^*(R_{x^*i} - R_{x^*y^*i}) + (3y^2 + z_i^2)D_{10} - (3y^{*2} + z_i^2)D_{11}] \tag{32}$$

$$M_{d5i}^{\langle 1\rangle} = \text{exchanging with } x, y \text{ in } M_{d4i}^{\langle 1\rangle} \tag{33}$$

$$M_{d5i}^{\langle 2\rangle} = \text{exchanging with } x, y \text{ in } M_{d4i}^{\langle 3\rangle} \tag{34}$$

$$M_{d5i}^{\langle 3 \rangle} = \text{exchanging with } x, y \text{ in } M_{d4i}^{\langle 2 \rangle} \tag{35}$$

$$M_{d5i}^{\langle 4 \rangle} = \frac{lw(x + x^*)}{3} + \frac{1}{3}\left(x^3 D_{12} - x^{*3} D_{13} - y^3 D_6 + y^{*3} D_7\right)$$
$$+ \frac{z_i}{6}\left[y(R_i - R_{x^*i}) - y^*(R_{y^*i} - R_{x^*y^*i}) + (3x^2 + z_i^2)D_4 - (3x^{*2} + z_i^2)D_5\right] \tag{36}$$

$$M_{d6i}^{\langle 1 \rangle} = M_{d1i}^{\langle 1 \rangle} + M_{d2i}^{\langle 1 \rangle} \tag{37}$$

$$M_{d6i}^{\langle 2 \rangle} = M_{d1i}^{\langle 2 \rangle} + M_{d2i}^{\langle 2 \rangle} \tag{38}$$

$$M_{d6i}^{\langle 3 \rangle} = M_{d1i}^{\langle 3 \rangle} + M_{d2i}^{\langle 3 \rangle} \tag{39}$$

$$M_{d6i}^{\langle 4 \rangle} = M_{d1i}^{\langle 4 \rangle} + M_{d2i}^{\langle 4 \rangle} \tag{40}$$

where $x^* = x - L$, $y^* = y - W$, $R_{x^*i} = \sqrt{x^{*2} + y^2 + z_i^2}$, $R_{y^*i} = \sqrt{x^2 + y^{*2} + z_i^2}$, $R_{x^*y^*i} = \sqrt{x^{*2} + y^{*2} + z_i^2}$ ($i = 1, 2, 3, a, b, c, d, e$);

$$D_1 = \tan^{-1}\frac{y^2 + z_i(R_i + z_i)}{xy} - \tan^{-1}\frac{y^2 + z_i(R_{x^*i} + z_i)}{x^*y},$$

$$D_2 = \tan^{-1}\frac{y^{*2} + z_i(R_{y^*i} + z_i)}{xy^*} - \tan^{-1}\frac{y^{*2} + z_i(R_{x^*y^*i} + z_i)}{x^*y^*},$$

$$D_3 = \tan^{-1}\frac{xy}{z_i R_i} - \tan^{-1}\frac{x^*y}{z_i R_{x^*i}} - \tan^{-1}\frac{xy^*}{z_i R_{y^*i}} + \tan^{-1}\frac{x^*y^*}{z_i R_{x^*y^*i}},$$

$$D_4 = \ln\left|\frac{R_{y^*i} + y^*}{R_i + y}\right|, \quad D_5 = \ln\left|\frac{R_{x^*y^*i} + y^*}{R_{x^*i} + y}\right|, \quad D_6 = \ln\left|\frac{R_{x^*i} + z_i}{R_i + z_i}\right|, \quad D_7 = \ln\left|\frac{R_{x^*y^*i} + z_i}{R_{y^*i} + z_i}\right|,$$

$$D_8 = \ln\left|\frac{R_{y^*i} + z_i}{R_i + z_i}\right|, \quad D_9 = \ln\left|\frac{R_{x^*y^*i} + z_i}{R_{x^*i} + z_i}\right|, \quad D_{10} = \ln\left|\frac{R_{x^*i} + x^*}{R_i + x}\right|, \quad D_{11} = \ln\left|\frac{R_{x^*y^*i} + x^*}{R_{y^*i} + x}\right|,$$

$$D_{12} = \tan^{-1}\frac{x^2 + z_i(R_i + z_i)}{xy} - \tan^{-1}\frac{x^2 + z_i(R_{y^*i} + z_i)}{xy^*},$$

$$D_{13} = \tan^{-1}\frac{x^{*2} + z_i(R_{x^*i} + z_i)}{x^*y} - \tan^{-1}\frac{x^{*2} + z_i(R_{x^*y^*i} + z_i)}{x^*y^*}.$$

The Eqs. (1)–(3), (16), and (17)–(40) can be easily automated to calculate the displacements in transversely isotropic half-spaces subjected to three-dimensional buried linearly varying rectangular loads.

3.2. Parabolic rectangular loads

For the case of subjected loads with non-linear distributions, we use a non-linear three-dimensional buried load with concave parabolas on a rectangle (Fig. 4) to present the results. Fig. 4 depicts that the concave parabolic load is applied over a rectangular region with sides L and W. The loading type in Fig. 4 can be expressed similarly as the form of Davis and Selvadurai (1996):

$$\widetilde{P}_j^{\mathrm{par}} = P_j^{\mathrm{par}}\left[1 + \beta\left(\frac{x^2}{L^2} + \frac{y^2}{W^2} - \frac{x^2 y^2}{L^2 W^2}\right)\right] \tag{41}$$
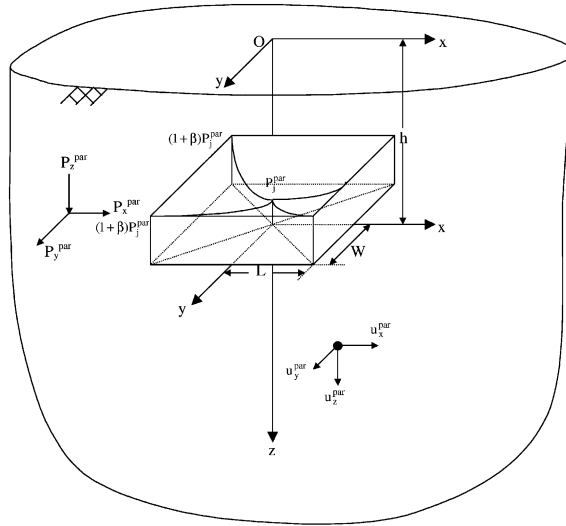
Fig. 4. The case of concave parabolic rectangular loads with LW area at the buried depth of h ($\beta > 0$).

where β is a constant, and also can be presented as the following three cases:

Case 1. $\beta > 0$, is the concave parabolic load as shown in Fig. 4;
Case 2. $\beta = 0$, is the uniform load; and this case is identical with the condition of $\alpha = 0$ (Fig. 3(a));
Case 3. $\beta < 0$, is the convex parabolic load. Fig. 5 depicts the zero contact stress at the edges of Fig. 4 for the case of $\beta = -1$.

The elementary force P_j acting on a small rectangle also can be expressed as $\tilde{P}_j^{\mathrm{par}}\,\mathrm{d}\zeta\,\mathrm{d}\eta$ ($j = x, y, z$) (forces per unit area). Similarly, by the same approach as the case of a linear varying rectangular load, the solutions of displacements for this case can be obtained by directly integrating as follows:

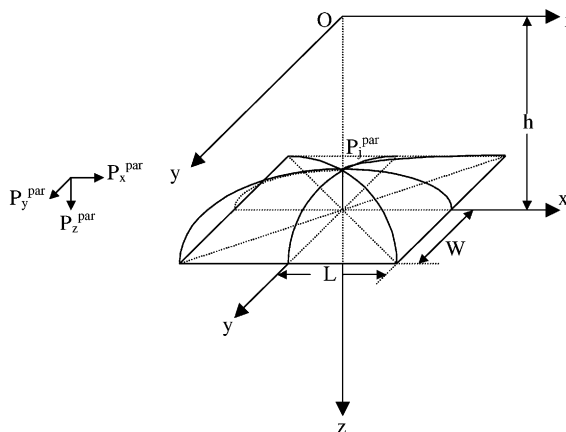$$[U]^{\mathrm{par}} = \int_0^L \int_0^W [U]^{\mathrm{p}}\,\mathrm{d}\eta\,\mathrm{d}\varsigma \tag{42}$$

Fig. 5. The case of completely convex parabolic rectangular loads with LW area at the buried depth of h ($\beta = -1$).

where the superscripts par and p express the displacement components that are induced by a parabolic load and a point load, respectively. The explicit solutions of displacements in the half-space can also be regrouped as the forms of Eqs. (1)–(3). Hence, the displacement elementary functions $p_{d1i}, p_{d2i}, \ldots, p_{d6i}$ in Eqs. (1)–(3) are replaced by the displacement integral functions $M_{d1i}^{\langle 1 \rangle} \sim M_{d1i}^{\langle 9 \rangle}$, $M_{d2i}^{\langle 1 \rangle} \sim M_{d2i}^{\langle 9 \rangle}, \ldots, M_{d6i}^{\langle 1 \rangle} \sim M_{d6i}^{\langle 9 \rangle}$ ($i = 1, 2, 3, a, b, c, d, e$) for u_x^{par}, u_y^{par}, u_z^{par} (Fig. 4), respectively. In this loading case, for instance, p_{d1i} should be replaced by $M_{d1i}^{\langle 1 \rangle} \sim M_{d1i}^{\langle 9 \rangle}$ as:

$$[p_{d1i}] \Rightarrow M_{d1i}^{\langle 1 \rangle} + \beta \left[\left(\frac{x^2}{L^2} + \frac{y^2}{W^2} * D_{14} \right) * M_{d1i}^{\langle 1 \rangle} - \frac{2x}{L^2} * D_{15} * M_{d1i}^{\langle 2 \rangle} - \frac{2y}{W^2} * D_{14} * M_{d1i}^{\langle 3 \rangle} - \frac{4xy}{L^2 W^2} * M_{d1i}^{\langle 4 \rangle} \right.$$

$$\left. + \frac{D_{15}}{L^2} * M_{d1i}^{\langle 5 \rangle} + \frac{D_{14}}{W^2} * M_{d1i}^{\langle 6 \rangle} + \frac{1}{L^2 W^2} \left(2x * M_{d1i}^{\langle 7 \rangle} + 2y * M_{d1i}^{\langle 8 \rangle} - M_{d1i}^{\langle 9 \rangle} \right) \right] \tag{43}$$

where $D_{14} = 1 - (x^2/L^2)$, $D_{15} = 1 - (y^2/W^2)$; The displacement integral functions for $M_{d1i}^{\langle 1 \rangle} \sim M_{d1i}^{\langle 4 \rangle}$, $M_{d2i}^{\langle 1 \rangle} \sim M_{d2i}^{\langle 4 \rangle}, \ldots, M_{d6i}^{\langle 1 \rangle} \sim M_{d6i}^{\langle 4 \rangle}$ were presented in Eqs. (17)–(40). Hence, only the integral functions for $M_{d1i}^{\langle 5 \rangle} \sim M_{d1i}^{\langle 9 \rangle}$, $M_{d2i}^{\langle 5 \rangle} \sim M_{d2i}^{\langle 9 \rangle}, \ldots, M_{d6i}^{\langle 5 \rangle} \sim M_{d6i}^{\langle 9 \rangle}$ will be given as follows:

$$M_{d1i}^{\langle 5 \rangle} = LWz_i - \frac{1}{3}[x(yR_i - y^*R_{y^*i}) - x^*(yR_{x^*i} - y^*R_{x^*y^*i})] + z_i \left(y^2 D_1 - y^{*2}D_2 + \frac{z_i^2}{3}D_3 \right) - \frac{x^3}{3}D_4$$

$$+ \frac{x^{*3}}{3}D_5 - \frac{y(y^2 - 3z_i^2)}{3}D_{10} + \frac{y^*(y^{*2} - 3z_i^2)}{3}D_{11} \tag{44}$$

$$M_{d1i}^{\langle 6 \rangle} = -LWz_i + \frac{1}{2}[x(yR_i - y^*R_{y^*i}) - x^*(yR_{x^*i} - y^*R_{x^*y^*i})] + \frac{x(x^2 - z_i^2)}{2}D_4 - \frac{x^*(x^{*2} - z_i^2)}{2}D_5$$

$$- z_i(x^2 D_{12} - x^{*2}D_{13}) \tag{45}$$

$$M_{d1i}^{\langle 7 \rangle} = -\frac{2LWz_i(x + x^*)}{3} + \frac{y(9x^2 - 6y^2 + z_i^2)}{24}R_i - \frac{y(9x^{*2} - 6y^2 + z_i^2)}{24}R_{x^*i} - \frac{y^*(9x^2 - 6y^2 + z_i^2)}{24}R_{y^*i}$$

$$+ \frac{y^*(9x^{*2} - 6y^2 + z_i^2)}{24}R_{x^*y^*i} + \frac{(3x^2 - z_i^2)^2}{24}D_4 - \frac{(3x^{*2} - z_i^2)^2}{24}D_5 - \frac{2z_i}{3}\left(x^3 D_{12} - x^{*3}D_{13} + \frac{y^3}{2}D_6 - \frac{y^{*3}}{2}D_7 \right) \tag{46}$$

$$M_{d1i}^{\langle 8 \rangle} = \frac{2LWz_i(y + y^*)}{3} + \frac{x(6x^2 - 3y^2 + z_i^2)}{12}R_i - \frac{x^*(6x^{*2} - 3y^2 + z_i^2)}{12}R_{x^*i} - \frac{x(6x^2 - 3y^{*2} + z_i^2)}{12}R_{y^*i}$$

$$+ \frac{x^*(6x^{*2} - 3y^{*2} + z_i^2)}{12}R_{x^*y^*i} - \frac{(3y^4 - 6y^2 z_i^2 - z_i^4)}{12}D_{10} + \frac{(3y^{*4} - 6y^{*2}z_i^2 - z_i^4)}{12}D_{11}$$

$$+ \frac{2z_i}{3}\left(y^3 D_1 - y^{*3}D_2 + \frac{x^3}{2}D_8 - \frac{x^{*3}}{2}D_9 \right) \tag{47}$$

$$M_{d1i}^{\langle 9 \rangle} = -\frac{LWz_i(x^2 + xx^* + x^{*2} - y^2 - yy^* - y^{*2})}{2} + \frac{xy(9x^2 - 6y^2 + z_i^2)}{30}R_i - \frac{x^*y(9x^{*2} - 6y^2 + z_i^2)}{30}R_{x^*i}$$

$$- \frac{xy^*(9x^2 - 6y^{*2} + z_i^2)}{30}R_{y^*i} + \frac{x^*y^*(9x^{*2} - 6y^{*2} + z_i^2)}{30}R_{x^*y^*i} + \frac{x^3(9x^2 - 5z_i^2)}{30}D_4 - \frac{x^{*3}(9x^{*2} - 5z_i^2)}{30}D_5$$

$$- \frac{y^3(3y^2 - 5z_i^2)}{15}D_{10} + \frac{y^{*3}(3y^{*2} - 5z_i^2)}{15}D_{11} - \frac{z_i}{2}\left(x^4 D_{12} - x^{*4}D_{13} - y^4 D_1 + y^{*4}D_2 + \frac{z_i^4}{15}D_3 \right) \tag{48}$$

$$M_{d2i}^{\langle 5\rangle} = -LWz_i + \frac{1}{2}\left[x(yR_i - y^*R_{y^*i}) - x^*(yR_{x^*i} - y^*R_{x^*y^*i})\right] - z_i(y^2D_1 - y^{*2}D_2) + \frac{y(y^2 - z_i^2)}{2}D_{10}$$
$$- \frac{y^*(y^{*2} - z_i^2)}{2}D_{11} \tag{49}$$

$$M_{d2i}^{\langle 6\rangle} = LWz_i - \frac{1}{3}\left[x(yR_i - y^*R_{y^*i}) - x^*(yR_{x^*i} - y^*R_{x^*y^*i})\right] + z_i\left(x^2D_{12} - x^{*2}D_{13} + \frac{z_i^2}{3}D_3\right)$$
$$- \frac{x(x^2 - 3z_i^2)}{3}D_4 + \frac{x^*(x^{*2} - 3z_i^2)}{3}D_5 - \frac{y^3}{3}D_{10} + \frac{y^{*3}}{3}D_{11} \tag{50}$$

$$M_{d2i}^{\langle 7\rangle} = \frac{2LWz_i(x + x^*)}{3} - \frac{y(3x^2 - 6y^2 - z_i^2)}{12}R_i + \frac{y(3x^{*2} - 6y^2 - z_i^2)}{12}R_{x^*i} + \frac{y^*(3x^2 - 6y^{*2} - z_i^2)}{12}R_{y^*i}$$
$$- \frac{y^*(3x^{*2} - 6y^{*2} - z_i^2)}{12}R_{x^*y^*i} - \frac{(3x^4 - 6x^2z_i^2 - z_i^4)}{12}D_4 + \frac{(3x^{*4} - 6x^{*2}z_i^2 - z_i^4)}{12}D_5$$
$$+ \frac{2z_i}{3}\left(x^3D_{12} - x^{*3}D_{13} + \frac{y^3}{2}D_6 - \frac{y^{*3}}{2}D_7\right) \tag{51}$$

$$M_{d2i}^{\langle 8\rangle} = -\frac{2LWz_i(y + y^*)}{3} - \frac{x(6x^2 - 9y^2 - z_i^2)}{24}R_i + \frac{x^*(6x^{*2} - 9y^2 - z_i^2)}{24}R_{x^*i} + \frac{x(6x^2 - 9y^{*2} - z_i^2)}{24}R_{y^*i}$$
$$- \frac{x^*(6x^{*2} - 9y^{*2} - z_i^2)}{24}R_{x^*y^*i} + \frac{(3y^2 - z_i^2)^2}{24}D_{10} - \frac{(3y^{*2} - z_i^2)^2}{24}D_{11}$$
$$- \frac{2z_i}{3}\left(\frac{x^3}{2}D_8 - \frac{x^{*3}}{2}D_9 + y^3D_1 - y^{*3}D_2\right) \tag{52}$$

$$M_{d2i}^{\langle 9\rangle} = \frac{LWz_i(x^2 + xx^* + x^{*2} - y^2 - yy^* - y^{*2})}{2} - \frac{xy(6x^2 - 9y^2 - z_i^2)}{30}R_i + \frac{x^*y(6x^{*2} - 9y^2 - z_i^2)}{30}R_{x^*i}$$
$$+ \frac{xy^*(6x^2 - 9y^{*2} - z_i^2)}{30}R_{y^*i} - \frac{x^*y^*(6x^{*2} - 9y^{*2} - z_i^2)}{30}R_{x^*y^*i} - \frac{x^3(3x^2 - 5z_i^2)}{15}D_4 + \frac{x^{*3}(3x^{*2} - 5z_i^2)}{15}D_5$$
$$+ \frac{y^3(9y^2 - 5z_i^2)}{30}D_{10} - \frac{y^{*3}(9y^{*2} - 5z_i^2)}{30}D_{11} + \frac{z_i}{2}\left(x^4D_{12} - x^{*4}D_{13} - y^4D_1 + y^{*4}D_2 - \frac{z_i^4}{15}D_3\right) \tag{53}$$

$$M_{d3i}^{\langle 5\rangle} = -\frac{(x^2 - 2y^2 + z_i^2)}{3}R_i + \frac{(x^{*2} - 2y^2 + z_i^2)}{3}R_{x^*i} + \frac{(x^2 - 2y^{*2} + z_i^2)}{3}R_{y^*i} - \frac{(x^{*2} - 2y^{*2} + z_i^2)}{3}R_{x^*y^*i}$$
$$+ z_i(y^2D_6 - y^{*2}D_7) \tag{54}$$

$$M_{d3i}^{\langle 6\rangle} = \text{exchanging with } x, y \text{ in } M_{d3i}^{\langle 5\rangle} \tag{55}$$

$$M_{d3i}^{\langle 7\rangle} = \frac{LWz_i(y + y^*)}{3} + \frac{x(6x^2 - 3y^2 - z_i^2)}{12}R_i - \frac{x^*(6x^{*2} - 3y^2 - z_i^2)}{12}R_{x^*i} - \frac{x(6x^2 - 3y^{*2} - z_i^2)}{12}R_{y^*i}$$
$$+ \frac{x^*(6x^{*2} - 3y^{*2} - z_i^2)}{12}R_{x^*y^*i} - \frac{(3y^4 + z_i^4)}{12}D_{10} + \frac{(3y^{*4} + z_i^4)}{12}D_{11} + \frac{2z_i}{3}\left(x^3D_8 - x^{*3}D_9 + \frac{y^3}{2}D_1 - \frac{y^{*3}}{2}D_2\right) \tag{56}$$

$$M_{d3i}^{\langle 8\rangle} = \text{exchanging with } x, y \text{ in } M_{d3i}^{\langle 7\rangle} \tag{57}$$

$$M_{d3i}^{\langle 9 \rangle} = \frac{1}{5}[(2x^4 - x^2y^2 + 2y^4)R_i - (2x^{*4} - x^{*2}y^2 + 2y^4)R_{x^*i} - (2x^4 - x^2y^{*2} + 2y^4)R_{y^*i}$$
$$+ (2x^{*4} - x^{*2}y^{*2} + 2y^4)R_{x^*y^*i}] + \frac{LWz_i(x+x^*)(y+y^*)}{4} + \frac{z_i}{2}(x^4D_8 - x^{*4}D_9 + y^4D_6 - y^{*4}D_7)$$
$$- \frac{z_i^2}{30}(R_i^3 - R_{x^*i}^3 - R_{y^*i}^3 + R_{x^*y^*i}^3) + \frac{z_i^4}{10}(R_i - R_{x^*i} - R_{y^*i} + R_{x^*y^*i}) \tag{58}$$

$$M_{d4i}^{\langle 5 \rangle} = \frac{LW(x+x^*)}{6} - \frac{1}{3}[x^3D_{12} - x^{*3}D_{13} - y^3D_6 + y^{*3}D_7 - z_i^3(D_4 - D_5)]$$
$$- \frac{2z_i}{3}[y(R_i - R_{x^*i}) - y^*(R_{y^*i} - R_{x^*y^*i})] \tag{59}$$

$$M_{d4i}^{\langle 6 \rangle} = M_{d5i}^{\langle 4 \rangle} \tag{60}$$

$$M_{d4i}^{\langle 7 \rangle} = \frac{LW(x^2 + xx^* + x^{*2} + y^2 + yy^* + y^{*2})}{4} + \frac{1}{4}(x^4D_{12} - x^{*4}D_{13} + y^4D_1 - y^{*4}D_2)$$
$$+ \frac{z_i}{12}[x(yR_i - y^*R_{y^*i}) - x^*(yR_{x^*i} - y^*R_{x^*y^*i})] + \frac{z_i}{3}\left(x^3D_4 - x^{*3}D_5 + y^3D_{10} - y^{*3}D_{11} - \frac{z_i^3}{4}D_3\right) \tag{61}$$

$$M_{d4i}^{\langle 8 \rangle} = \frac{LW(x+x^*)(y+y^*)}{8} - \frac{1}{4}(x^4D_8 - x^{*4}D_9 - y^4D_6 + y^{*4}D_7) + \frac{z_i}{12}[(x^2 - 5y^2 - 2z_i^2)R_i$$
$$- (x^{*2} - 5y^2 - 2z_i^2)R_{x^*i} - (x^2 - 5y^{*2} - 2z_i^2)R_{y^*i} + (x^{*2} - 5y^{*2} - 2z_i^2)R_{x^*y^*i}] \tag{62}$$

$$M_{d4i}^{\langle 9 \rangle} = \frac{LW(x+x^*)(2x^2 + 2x^{*2} + y^2 + yy^* + y^{*2})}{10} + \frac{1}{5}(x^5D_{12} - x^{*5}D_{13} + y^5D_6 + y^{*5}D_7)$$
$$+ \frac{z_i}{20}\{[(x^2 - 6y^2 - z_i^2)R_i - (x^{*2} - 6y^2 - z_i^2)R_{x^*i}]y - [(x^2 - 6y^{*2} - z_i^2)R_{y^*i}$$
$$- (x^{*2} - 6y^{*2} - z_i^2)R_{x^*y^*i}]y^* + (5x^4 - z_i^4)D_4 - (5x^{*4} - z_i^4)D_5\} \tag{63}$$

$$M_{d5i}^{\langle 5 \rangle} = M_{d4i}^{\langle 4 \rangle} \tag{64}$$

$$M_{d5i}^{\langle 6 \rangle} = \text{exchanging with } x, y \text{ in } M_{d4i}^{\langle 5 \rangle} \tag{65}$$

$$M_{d5i}^{\langle 7 \rangle} = \text{exchanging with } x, y \text{ in } M_{d4i}^{\langle 8 \rangle} \tag{66}$$

$$M_{d5i}^{\langle 8 \rangle} = M_{d4i}^{\langle 7 \rangle} \tag{67}$$

$$M_{d5i}^{\langle 9 \rangle} = \text{exchanging with } x, y \text{ in } M_{d4i}^{\langle 9 \rangle} \tag{68}$$

$$M_{d6i}^{\langle 5 \rangle} = M_{d1i}^{\langle 5 \rangle} + M_{d2i}^{\langle 5 \rangle} \tag{69}$$

$$M_{d6i}^{\langle 6 \rangle} = M_{d1i}^{\langle 6 \rangle} + M_{d2i}^{\langle 6 \rangle} \tag{70}$$

$$M_{d6i}^{\langle 7 \rangle} = M_{d1i}^{\langle 7 \rangle} + M_{d2i}^{\langle 7 \rangle} \tag{71}$$

$$M_{d6i}^{\langle 8 \rangle} = M_{d1i}^{\langle 8 \rangle} + M_{d2i}^{\langle 8 \rangle} \tag{72}$$
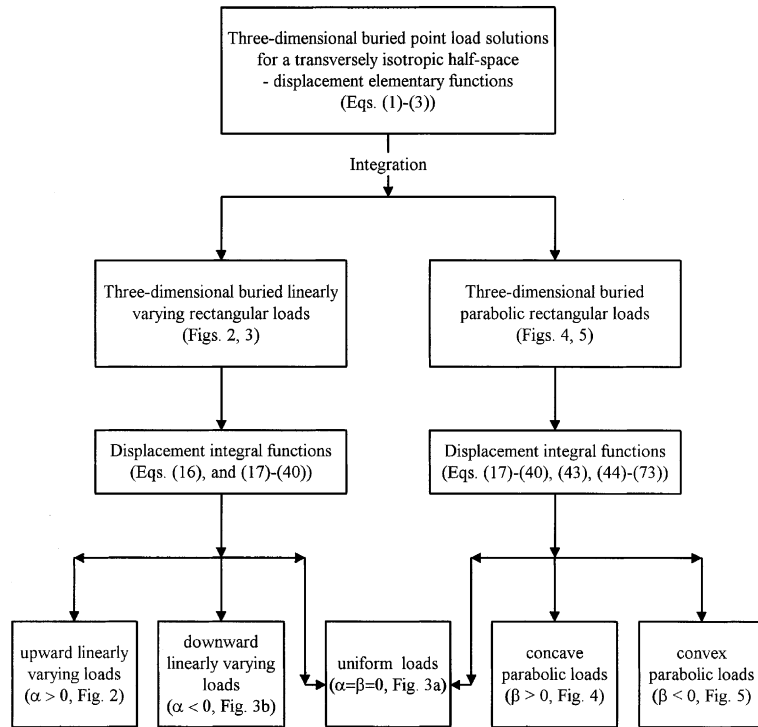
Fig. 6. Flow chart for computing the displacements for a transversely isotropic half-space subjected to presented loading types.

$$M_{d6i}^{\langle 9 \rangle} = M_{d1i}^{\langle 9 \rangle} + M_{d2i}^{\langle 9 \rangle} \tag{73}$$

The Eqs. (1)–(3), (17)–(40), (43), and (44)–(73) can be utilized to compute the displacements in a transversely isotropic half-space subjected to three-dimensional buried parabolic rectangular loads. Also, the displacements in the media due to non-uniform irregularly shaped loads can be estimated by superposition of values corresponding to the rectangular sub-areas. A flow chart that illustrates the presented solutions for computing the displacements induced by linearly varying, uniform, and parabolic rectangular loads in a transversely isotropic half-space is presented in Fig. 6.

4. Illustrative example

A series of parametric study is conducted to verify the solutions derived and investigate the effect of the type and degree of rock anisotropy, and the loading types on the displacements. An illustrative example as depicted in Figs. 7 and 8 includes a vertical linear-varying, uniform, and parabolic load acting on a rectangle. The hypothetical constituted foundation materials include several types of isotropic and transversely isotropic rocks. Their elastic properties are listed in Table 1 with E/E' and G/G' ranging between 1 and 3, and v/v' varying between 0.75 and 1.5. The values adopted in Table 1 of E and v are 50 GPa and 0.25, respectively. The chosen domains of variation are based on the suggestions of Gerrard (1975) and Amadei et al. (1987). The loads act on the horizontal surface ($h = 0$) of isotropic/transversely isotropic rocks. The degree of anisotropy including the ratios E/E', v/v', and G/G' is accounted for investigating its effect on the displacements.
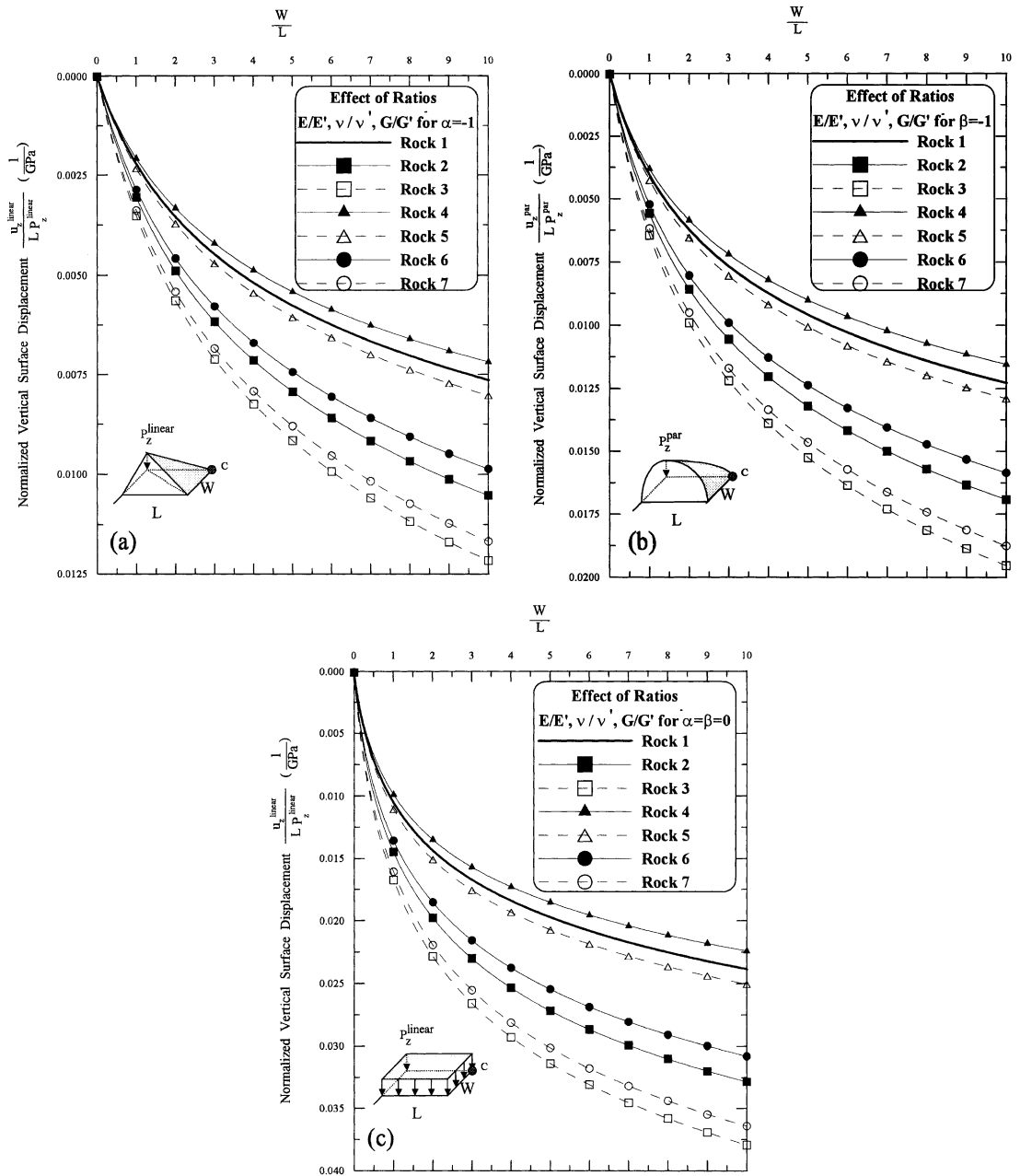
Fig. 7. Effect of the type and degree of rock anisotropy on vertical surface displacement induced by the case of (a) $\alpha = -1$, (b) $\beta = -1$, (c) $\alpha = \beta = 0$, (d) $\alpha = 1$, (e) $\beta = 1$.

Based on Eqs. (1)–(3), (16), (17)–(40), and Eqs. (1)–(3), (17)–(40), (43), (44)–(73) for linearly varying, and parabolic loading types, a FORTRAN program was written to calculate the displacements, respectively. In this program, all the components of displacement at any point in the half-space can be computed. However, the vertical settlement is usually the most interesting quantity in foundation analysis. Hence, in this study,
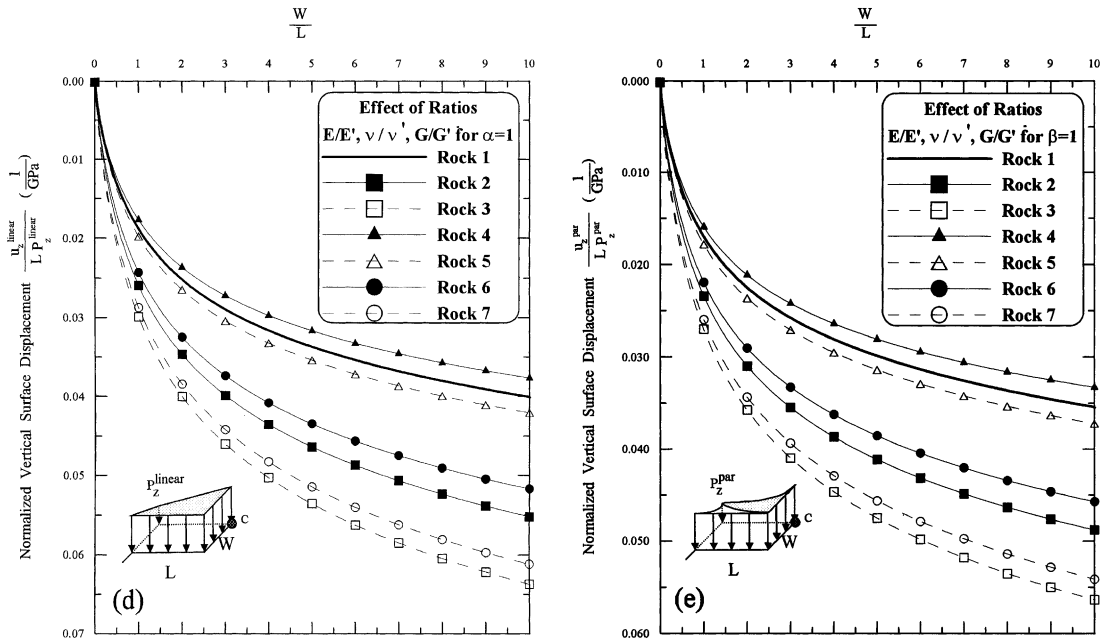
Fig. 7 (*continued*)

only the vertical surface displacement at the right corner (point C) of the loaded area was presented. Figs. 7 and 8 show the results for this example. The normalized vertical surface displacement ($u_z^{\text{linear}}/LP_z^{\text{linear}}$ or $u_z^{\text{par}}/LP_z^{\text{par}}$) at the corner induced by a completely downward linearly varying load ($\alpha = -1$), a completely convex parabolic load ($\beta = -1$), a uniform load ($\alpha = \beta = 0$), an upward linearly varying load ($\alpha = 1$), and a concave parabolic load ($\beta = 1$) over a rectangular area vs. the non-dimensional ratio of the loaded side (W/L) is given in Fig. 7(a)–(e), respectively. Knowing the loading types and magnitudes, the dimensions of loaded area, and rock types, the vertical surface displacement at point C can be estimated from these figures. The same displacement that induced by as already mentioned loads ($\alpha = -1$, $\beta = -1$, $\alpha = \beta = 0$, $\alpha = 1$, $\beta = 1$) for Rocks 1–7 (Table 1) is also given in Fig. 8 (a)–(g), respectively. According to the results reported in Figs. 7 and 8, the effect of the type and degree of rock anisotropy, the loading types, and the dimensions of loaded region on the displacement induced by surface loads is investigated below.

Fig. 7(a)–(e) indicate that the vertical surface displacement increases with the increase of E/E' with $v/v' = G/G' = 1$ (Rocks 1, 2, and 3), v/v' with $E/E' = G/G' = 1$ (Rocks 1, 4, and 5), and G/G' with $E/E' = v/v' = 1$ (Rocks 1, 6 and 7) for a given shape. Especially, the increases of the ratio of E/E' and G/G' do have a great influence on the vertical displacement. It reflects the fact that the displacement increases with the increase of deformability in the direction parallel to the applied load. However, the variation of v/v' in this displacement is little for all the cases. The results of this figure indicate that the displacement induced by the presented loading types strongly depend on the type and degree of rock anisotropy. Fig. 8 (a)–(g) show the vertical surface displacement for Rocks 1–7 that induced by loading cases of $\alpha = -1$, $\beta = -1$, $\alpha = \beta = 0$, $\alpha = 1$, $\beta = 1$. The trend of these figures (Fig. 8(a)–(g)) for each rock is similar. However, the calculated results for each loading case are quite different. The vertical surface displacement induced by a uniform load ($\alpha = \beta = 0$) is approximately 2.4 times of that induced by the average of a completely downward linearly varying load ($\alpha = -1$) and a completely convex parabolic load ($\beta = -1$) for all constituted rocks. Nevertheless, the induced displacement by the average of an upward linearly varying load ($\alpha = 1$) and a concave parabolic load ($\beta = 1$) is probably 1.6 times of that induced by a uniform
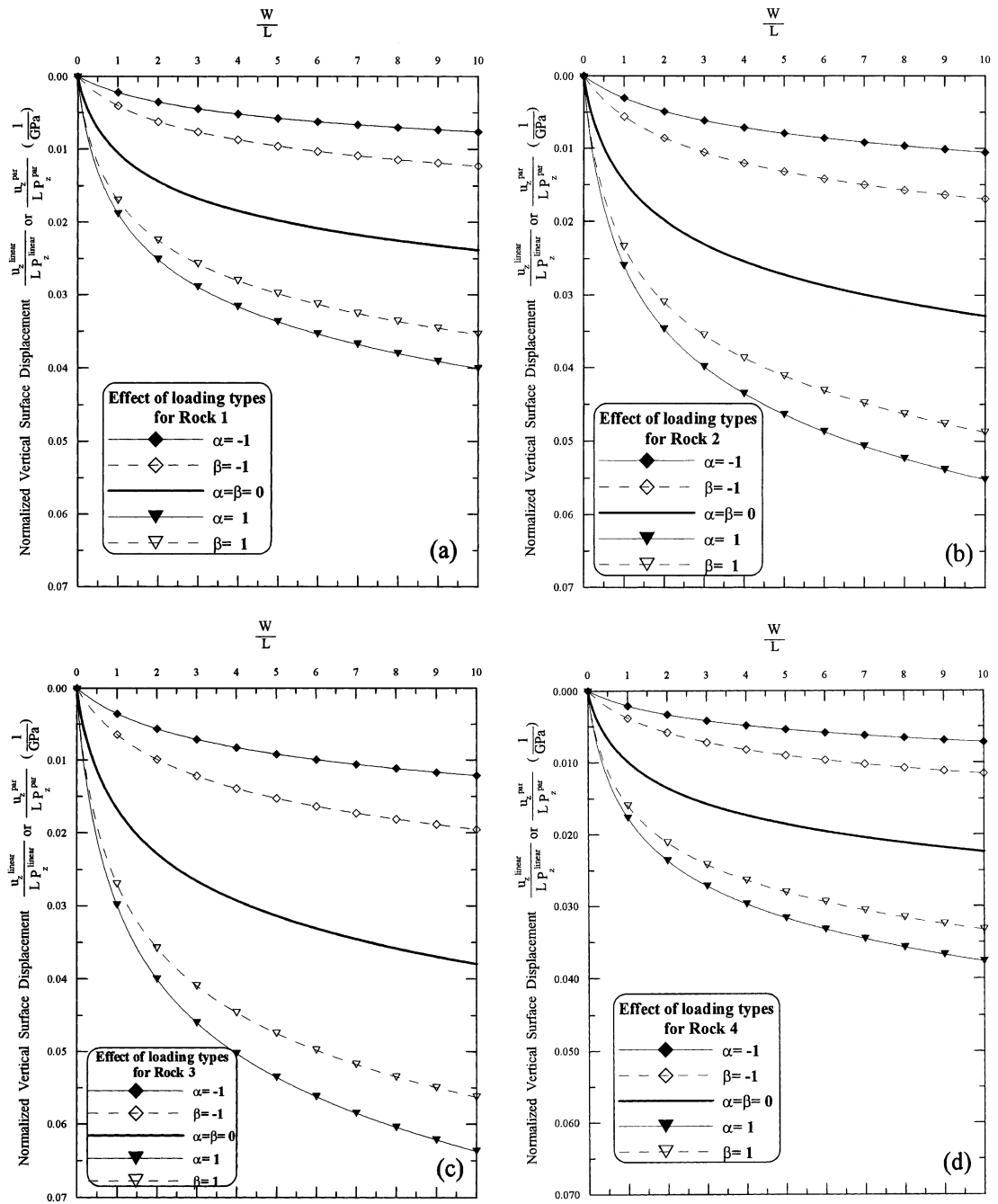
Fig. 8. Effect of the loading types on vertical surface displacement for (a) Rock 1, (b) Rock 2, (c) Rock 3, (d) Rock 4, (e) Rock 5, (f) Rock 6, (g) Rock 7.

load ($\alpha = \beta = 0$). It means that the induced displacement also strongly depends on the loading types for isotropic/transversely isotropic rocks. From Figs. 7 and 8, it also can be found that the vertical surface
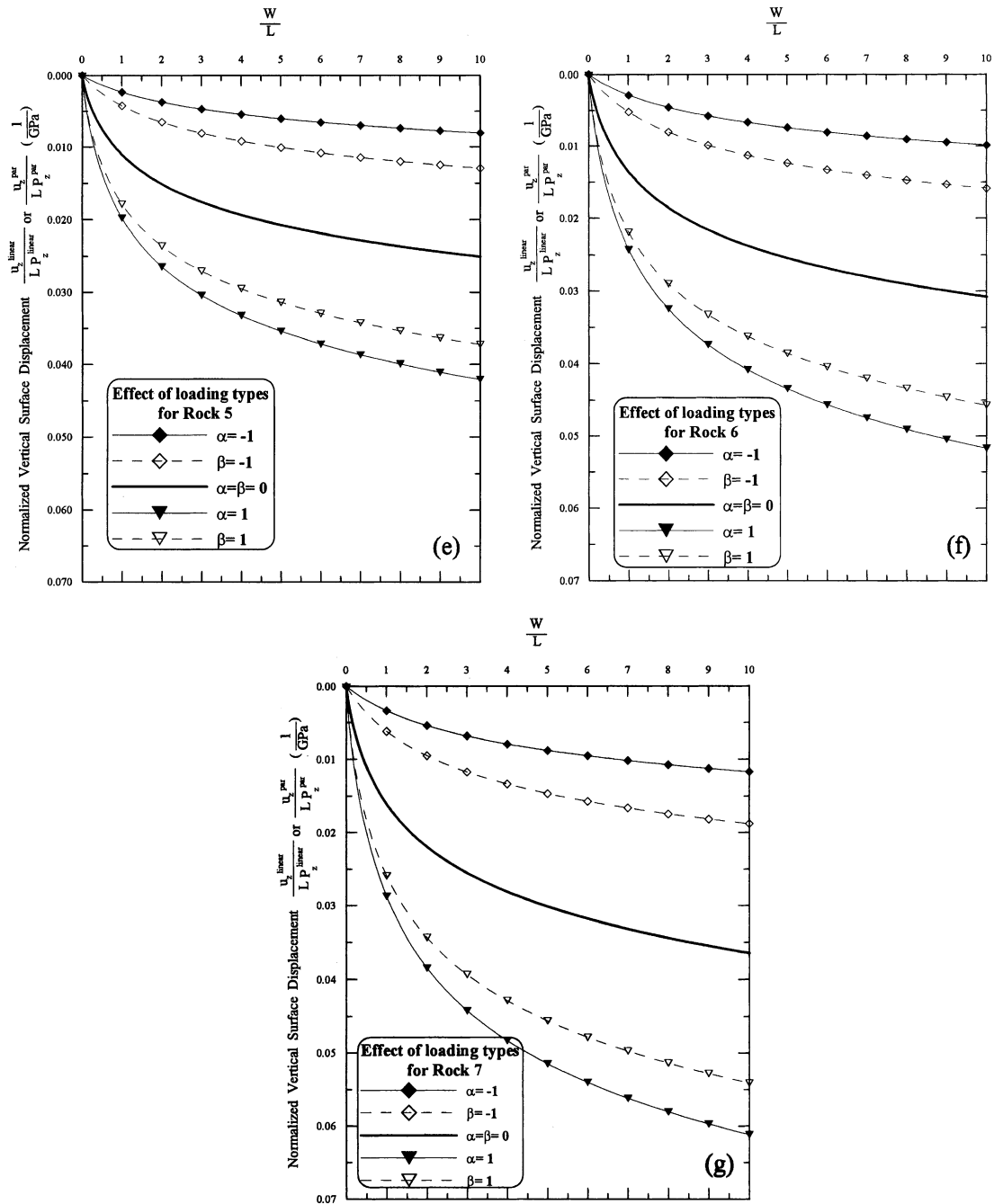
Fig. 8 (*continued*)

displacement increases with the increase of the ratio W/L for all rocks. It implicates that the displacement calculated from plane strain solution is larger than that obtained from three-dimensional solution.

Table 1
Elastic properties and root types for different rocks

Rock type	E/E'	v/v'	G/G'	Root type
Rock 1: isotropic	1.0	1.0	1.0	Equal
Rock 2: transversely isotropic	2.0	1.0	1.0	Complex
Rock 3: transversely isotropic	3.0	1.0	1.0	Complex
Rock 4: transversely isotropic	1.0	0.75	1.0	Complex
Rock 5: transversely isotropic	1.0	1.5	1.0	Distinct
Rock 6: transversely isotropic	1.0	1.0	2.0	Distinct
Rock 7: transversely isotropic	1.0	1.0	3.0	Distinct

Employing the example, the results show that the displacement in the isotropic/transversely isotropic rocks subjected to various loading types on a rectangle are easy and correct to be calculated by the presented solutions. Also, the results indicate that the displacement are affected by the effect of rock anisotropy, and different loading types. Hence, in engineering practice, it is not suitable to compute the displacements by the traditional isotropic solutions, or assuming the load is always uniformly distributed over a rectangular area in a transversely isotropic half-space.

5. Conclusions

The point load solutions of displacements in a Cartesian co-ordinate system for a transversely isotropic half-space can be expressed in terms of several elementary functions. Integrating of these elementary functions, the solutions of displacements subjected to three-dimensional buried linearly varying, uniform, and parabolic rectangular loads can be derived. The planes of transverse isotropy in this paper are assumed to be parallel to the horizontal surface of the half-space. The loading types include an upward linearly varying load, a downward linearly varying load, a uniform load, a concave parabolic load, and a convex parabolic load distributed over a rectangular region. The presented elastic solutions of displacements are influenced by the buried depth (h), the dimensions of loaded area (L or W), the type and degree of material anisotropy, and the loading types ($\alpha > 0$, $\alpha = \beta = 0$, $\alpha < 0$, $\beta > 0$, $\beta < 0$) for a transversely isotropic half-space.

The following conclusions can be drawn from the results of parametric studies: (1) The vertical surface displacement increases with the increase of deformability in the direction parallel to the applied load; especially, the increases of the ratio of E/E' and G/G' do have a great influence on it; (2) the usage of plain strain solution will overestimate the induced displacement than that calculated from three-dimensional solution; (3) the induced displacement strongly depends on the type and degree of rock anisotropy, and different loading cases.

Since the presentation of the derived solutions for displacements is clear and concise, the computation of induced displacements by various loading types distributed over a rectangular region in transversely isotropic half-spaces is fast and correct. These solutions can simulate more actually loading conditions as well as provide reasonably results for practical purposes. Regarding the elastic solutions of stresses for the identical loading cases acting on the transversely isotropic half-space can be derived by the same approach. The results will be presented in the forthcoming papers.

Acknowledgements

References

Amadei, B., Savage, W.Z., Swolfs, H.S., 1987. Gravitational stresses in anisotropic rock masses. International Journal of Rock Mechanics and Mining Sciences & Geomechanics Abstracts 24 (1), 5–14.

Bauer, G.E., Shields, D.H., Scott, J.D., Nwabuokei, S.O., 1979. Normal and shear measurements on a strip footing. Canadian Geotechnical Journal 16 (1), 177–189.

Boussinesq, J., 1885. Application des Potentiels à L'Étude de L'Équilibre et due Mouvement des Solides Élastiques. Gauthier-Villars, Paris.

Chowdhury, K.L., 1987. On the axisymmetric Mindlin's problem for a semi-space of granular material. Acta Mechanics 66, 145–160.

Davis, R.O., Selvadurai, A.P.S., 1996. Elasticity and Geomechanics. Cambridge University Press, New York, USA.

Fox, E.N., 1948. The mean elastic settlement of a uniformly loaded area at a depth below the ground surface. In: Proceedings of the 2nd International Conference on Soil Mechanics and Foundation Engineering, Rotterdam, pp. 129–132.

Gazetas, G., 1982a. Stresses and displacements in cross-anisotropic soils. Journal of Geotechnical Engineering Division, ASCE 108 (4), 532–553.

Gazetas, G., 1982b. Axisymmetric parabolic loading of anisotropic halfspace. Journal of Geotechnical Engineering Division, ASCE 108 (4), 654–660.

Gazetas, G., 1985. Elastic settlement of arbitrarily shaped foundations embedded in half-space. Geotechnique 35 (3), 339–346.

Gerrard, C.M., 1975. Background to mathematical modeling in geomechanics: the roles of fabric and stress history. In: Proceedings of the International Symposium on Numerical Methods, Karlsruhe, pp. 33–120.

Gradshteynn, I.S., Ryzhik, I.M., 1994. Tables of Integrals, Series, and Products. Academic Press, San Diego, CA.

Hanson, M.T., Wang, Y., 1997. Concentrated ring loadings in a full space or half space: solutions for transverse isotropy and isotropy. International Journal of Solids and Structures 34 (11), 1379–1418.

Hooper, J.A., 1976. Parabolic adhesive loading of a flexible raft foundation. Geotechnique 26 (3), 511–525.

Lekhnitskii, S.G., 1963. Theory of Elasticity of an Anisotropic Elastic Body. Holden-Day, San Francisco.

Liao, J.J., Wang, C.D., 1998. Elastic solutions for a transversely isotropic half-space subjected to a point load. International Journal for Numerical and Analytical Methods in Geomechanics 22 (6), 425–447.

Lin, W., Kuo, C.H., Keer, L.M., 1991. Analysis of a transversely isotropic half space under normal and tangential loadings. Journal of Tribology, ASME 113 (4), 335–338.

Pan, Y.C., Chou, T.W., 1979. Green's function solutions for semi-infinite transversely isotropic materials. International Journal of Engineering Science 17, 545–551.

Skopek, J., 1961. The influence of foundation depth on stress distribution. In: Proceedings of the 5th International Conference on Soil Mechanics and Foundation Engineering, pp. 815–818.

Tarn, J.Q., Wang, Y.M., 1987. A fundamental solution for a transversely isotropic elastic space. Journal of the Chinese Institute of Engineers 10 (1), 13–21.

Wang, C.D., Liao, J.J., 1999. Elastic solutions for a transversely isotropic half-space subjected to buried asymmetric-loads. International Journal for Numerical and Analytical Methods in Geomechanics 23 (2), 115–139.

Wang, C.D., Liao, J.J., 2001. Elastic solution for a transversely isotropic half-space subjected to arbitrarily shaped loads using triangulating technique. The International Journal of Geomechanics 1 (2), 193–224.

Wang, Y., Rajapakse, R.K.N.D., 1990. Asymmetric boundary-value problems for a transverse isotropic elastic medium. International Journal of Solids and Structures 26 (8), 833–849.