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Stabilization of nonlinear systems in compound critical cases

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Abstract

In this paper, we study the stabilization of nonlinear systems in critical cases by using the center manifold reduction technique. Three degenerate cases are considered, wherein the linearized model of the system has two zero eigenvalues, one zero eigenvalue and a pair of nonzero pure imaginary eigenvalues, or two distinct pairs of nonzero pure imaginary eigenvalues; while the remaining eigenvalues are stable. Using a local nonlinear mapping (normal form reduction) and Liapunov stability criteria, one can obtain the stability conditions for the degenerate reduced models in terms of the original system dynamics. The stabilizing control laws, in linear and/or nonlinear feedback forms, are then designed for both linearly controllable and linearly uncontrollable cases. The normal form transformations obtained in this paper have been verified by using code MACSYMA.

Keywords: Nonlinear systems; Stabilization; Center manifold reduction

1. Introduction

Recently, the center manifold theorem has been applied to the stabilization of nonlinear systems (see, e.g., [8–16]). Aeyels [1] obtained a stabilizing control law for third-order systems which possess a pair of pure imaginary eigenvalues and one stable eigenvalue. This result has been extended in [2] to more general high-dimensional, nonlinear systems, in which the linearized model has either a

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pair of pure imaginary eigenvalues or one zero eigenvalue; while the remaining eigenvalues are stable or stabilizable.

More degenerate cases have been considered by Behtash and Sastry [3]. They obtained results for the nonlinear systems whose linear part has: two zero eigenvalues with geometric multiplicity 1; one zero eigenvalue and a pair of pure imaginary eigenvalues; or two distinct pairs of pure imaginary eigenvalues. Unfortunately, they consider only the case in which the state vector dimension is one more than the number of critical modes. In addition, most of their results are given in terms of the system dynamics after normal form reduction. In this paper, we extend their results to more general high-dimensional, nonlinear systems, where the noncritical modes are either stable or stabilizable and the number of these noncritical modes is not restricted. Moreover, the stabilizing control laws are given in terms of the original system dynamics before normal form reduction.

First, the normal form reduction technique is briefly recalled and applied to derive stability conditions for low-dimensional, critical nonlinear systems, specifically, where the linearized model of the system has exactly two zero eigenvalues with geometric multiplicity 1; one zero eigenvalue and a pair of pure imaginary eigenvalues; or two pairs of pure imaginary eigenvalues. This is followed by a study of stabilization of general high-dimensional, critical nonlinear systems. In Section 3, the stability condition derived in Section 2.1 for planar systems with two zero eigenvalues, along with the center manifold reduction technique, are employed to design the stabilizing feedback control laws for high-dimensional, nonlinear systems. A linear and/or nonlinear feedback stabilizing control law is proposed for linearly uncontrollable systems, while a purely nonlinear stabilizing control law is designed for linearly controllable systems. Similar results are obtained for the remaining two degenerate cases, in which the uncontrolled model has one zero eigenvalue and a pair of pure imaginary eigenvalues, or two distinct pairs of pure imaginary eigenvalues; while remaining eigenvalues are stable or stabilizable by linear feedback. These are given in Sections 4 and 5, respectively. Finally, concluding remarks are given in Section 6.

2. Stability conditions for critical reduced models

In this paper, we study the stabilization of critical nonlinear system

$$\dot{\eta} = A_{11}\eta + b_1 u + F(\eta, \xi), \tag{1a}$$

$$\dot{\xi} = A_{22}\xi + b_2 u + G(\eta, \xi), \tag{1b}$$

where functions F, G are sufficiently smooth with $F(0,0) = 0$, $DF(0,0) = 0$, $G(0,0) = 0$ and $DG(0,0) = 0$. Specifically, we consider three degenerate cases in which A_{11} has exactly two zero eigenvalues with geometric multiplicity 1; one zero eigenvalue and a pair of pure imaginary eigenvalues; or two distinct pairs of pure imaginary eigenvalues. Here, the control input u in (1a), (1b) is taken to be a scalar. So, b_1, b_2 are both vectors. It is not difficult to extend the results to the case of multiple inputs. Details are omitted.

First, the stability conditions for the low-dimensional critical system (1a) with $u = 0$, $\xi = 0$ are derived in this section by employing the technique of normal form reduction as recalled below and Liapunov stability criteria. These stability conditions and the center manifold reduction technique (e.g., [2]) are applied to study the stabilization of the system (1a), (1b) in Sections 2.1–2.3.

In the rest of this section, we focus on the derivation of stability conditions for the low-dimensional critical system (1a) with $u = 0$ and $\xi = 0$ as given by

$$\begin{aligned}\dot{\eta} &= A_{11}\eta + F(\eta)\\ &= A_{11}\eta + F_2(\eta,\eta) + F_3(\eta,\eta,\eta) + \mathrm{O}(\|\eta\|^4),\end{aligned} \tag{2}$$

where $F(\eta) := F(\eta, 0)$ and F_2, F_3 denote quadratic and cubic terms of the Taylor expansion of F, respectively. Here, we have presumed that F is at least four times continuously differentiable.

It is known (e.g., [7]) that a nonlinear transformation $\eta = \zeta + P(\zeta)$ can be applied to simplify the expressions of the critical nonlinear systems, where P is a purely nonlinear vector function

$$P(\zeta) = P_2(\zeta,\zeta) + P_3(\zeta,\zeta,\zeta) + \mathrm{O}(\|\zeta\|^4), \tag{3}$$

with P_2 and P_3 being the quadratic and cubic terms in P, respectively.

Applying this scheme to Eq. (2), we obtain

$$\begin{aligned}\dot{\zeta} &= (I + DP(\zeta))^{-1}F(\zeta + P(\zeta))\\ &= \mathscr{F}_1\zeta + \mathscr{F}_2(\zeta,\zeta) + \mathscr{F}_3(\zeta,\zeta,\zeta) + \mathrm{O}(\|\zeta\|^4),\end{aligned} \tag{4}$$

where $\mathscr{F}_1 = A_{11}$ and $\mathscr{F}_2, \mathscr{F}_3$ are as given by

$$\begin{aligned}\mathscr{F}_2(\zeta,\zeta) &= F_2(\zeta,\zeta) + F_1\cdot P_2(\zeta,\zeta) - DP_2(\zeta,\zeta)\cdot F_1\zeta,\\ \mathscr{F}_3(\zeta,\zeta,\zeta) &= F_3(\zeta,\zeta,\zeta) - DP_2(\zeta,\zeta)\cdot\mathscr{F}_2(\zeta,\zeta) + DF_2(\zeta,\zeta)\cdot P_2(\zeta,\zeta)\\ &\quad + F_1\cdot P_3(\zeta,\zeta,\zeta) - DP_3(\zeta,\zeta,\zeta)\cdot F_1\zeta.\end{aligned}$$

The main goal of this section is to obtain the homogeneous functions P_i for which the nonlinear terms $\mathscr{F}_i$ of the transformed model (4) allow a simple analysis of the local stability of the origin.

2.1. Stability of the second-order model

First, consider the case in which $\eta = (x, y)'$ is a two-dimensional vector, and Eq. (2) is a planar system

$$\dot{x} = y + f_{xx}x^2 + f_{xy}xy + f_{yy}y^2 + f_{xxx}x^3 + f_{xxy}x^2y + f_{xyy}xy^2 + f_{yyy}y^3 + O(\|(x,y)\|^4), \tag{5a}$$

$$\dot{y} = g_{xx}x^2 + g_{xy}xy + g_{yy}y^2 + g_{xxx}x^3 + g_{xxy}x^2y + g_{xyy}xy^2 + g_{yyy}y^3 + O(\|(x,y)\|^4). \tag{5b}$$

By using the technique recalled in [3,4], it is not difficult to obtain a normal form expression for (5a), (5b). For instance, a general form has been obtained by Takens [4]. A result of [4] for the normal form of (5a), (5b) up to sixth-order can be written as

$$\dot{x}_1 = x_2 + O(\|(x_1,x_2)\|^6), \tag{6a}$$

$$\dot{x}_2 = \delta_1 x_1^2 + \delta_2 x_1x_2 + \delta_3 x_1^3 + \delta_4 x_1^2x_2 + \delta_5 x_1^4 + \delta_6 x_1^3x_2 + \delta_7 x_1^5 + \delta_8 x_1^4x_2 + O(\|(x_1,x_2)\|^6), \tag{6b}$$

where x_1, x_2 are the transformed states after normal form reduction and δ_i are constants.

To study the local stability of (6a), (6b) by Liapunov's direct method, we invoke a special locally positive definite function. A class of such functions has been introduced by Fu and Abed [5] for constructing families of Liapunov functions for critical nonlinear systems the linear part of which has exactly one zero eigenvalue or a pair of nonzero pure imaginary eigenvalues with the remaining eigenvalues stable. This result is extended below to a more general case, which will provide a means to obtain the stability conditions for the model (6a), (6b).

Lemma 1. *The scalar function*

$$V(x_1,x_2) = v_1x_2^2 + v_2x_1x_2^2 + v_3x_2^3 + v_4x_1^4 + v_5x_1^3x_2 + v_6x_1^2x_2^2 + v_7x_1x_2^3 + v_8x_2^4 + v_9x_1^5 + v_{10}x_1^6 \tag{7}$$

is locally positive definite near the origin if $v_1, v_4 > 0$.

Lemma 1 follows directly from ([5], Lemma 1). Details are omitted. Next, we have the following obvious result.

Corollary 1. *The scalar function*

$$V(x_1,x_2) = x_2^4(\delta_1 + \rho_1(x_1,x_2)) + x_1^2x_2^2(\delta_2 + \rho_2(x_1,x_2)) + \delta_3x_1^6 + O(\|(x_1,x_2)\|^7) \tag{8}$$

is locally negative definite near the origin if $\delta_i < 0$ *for* $i = 1,2,3$ *and the smooth scalar functions* ρ_1, ρ_2 *satisfy* $\rho_i(0,0) = 0$ *for* $i = 1,2$.

Now, we employ Lemma 1 and Corollary 1 to study the local stability of Eqs. (6a), (6b). Choose as a Liapunov function candidate for (6a), (6b) a function V as in (7) with $v_2 = v_6 = 0$. The time derivative of V along trajectories of Eqs. (6a), (6b) is given by

$$\begin{aligned}\dot{V}1 &= 2v_1(\delta_1 x_1^2 x_2 + \delta_2 x_1 x_2^2) + v_7 x_2^4 + (2v_1\delta_3 + 4v_4)x_1^3 x_2 \\ &\quad + (2v_1\delta_4 + 3v_5 + 3v_3\delta_1 + \rho(x_1,x_2))x_1^2 x_2^2 + v_5\delta_1 x_1^5 \\ &\quad + (5v_9 + v_5\delta_2 + 2v_1\delta_5)x_1^4 x_2 + v_5\delta_3 x_1^6 \\ &\quad + (v_5\delta_4 + 2v_1\delta_7 + 6v_{10})x_1^5 x_2 + \mathrm{O}(\|(x_1,x_2)\|^7),\end{aligned} \tag{9}$$

where ρ is a smooth, scalar function with $\rho(0,0) = 0$.

According to Lemma 1, V is locally positive definite if $v_1, v_4 > 0$. By employing Corollary 1 to check the local negative definiteness of $\dot{V}$ (given in (9)) and applying Liapunov stability criteria, we have:

Proposition 1. *Let* $\delta_1 = \delta_2 = 0$. *Then the origin of* (6a) *and* (6b) *is asymptotically stable if the values of* v_i *in* (7) *can be chosen such that*

(i) $v_1, v_4 > 0$, $v_2 = v_6 = 0$,
(ii) $v_7, v_5\delta_3, 2v_1\delta_4 + 3v_5 < 0$,
(iii) $5v_9 + 2v_1\delta_5 = 0$, $v_5\delta_4 + 2v_1\delta_7 + 6v_{10} = 0$ *and* $2v_1\delta_3 + 4v_4 = 0$.

Assume $\delta_1 = \delta_2 = 0$ and $\delta_3, \delta_4 < 0$. With these assumptions we can choose v_i such that the stability conditions in Proposition 1 hold. As implied by ([7], Lemma 2.6), the local stability of the origin is preserved under normal form reduction. Thus, we have:

Lemma 2. *The origin is asymptotically stable for* (5a), (5b) *if* $\delta_1 = \delta_2 = 0$, $\delta_3, \delta_4 < 0$.

By suitable choice of nonlinear functions P_2 and P_3 (in (3)), we obtain the values of the δ_i as: $\delta_1 = g_{xx}$, $\delta_2 = g_{xy} + 2f_{xx}$ and

$$\delta_3 = g_{xxx} + g_{xx}f_{xy} - g_{xy}f_{xx}, \tag{10}$$

$$\delta_4 = g_{xxy} + 3f_{xxx} + \tfrac{1}{2}\{f_{yy}g_{xx} + (g_{xy} - 2f_{xx})g_{yy} + f_{xy}g_{xy}\}. \tag{11}$$

In the next corollary, the stability conditions of Lemma 2 are stated in terms of the functions f and g.

Corollary 2. *The origin of* (5a), (5b) *is asymptotically stable if* $g_{xx} = 0$, $g_{xy} + 2f_{xx} = 0$, $g_{xxx} + 2f_{xx}^2 < 0$ *and*

$$g_{xxy} + 3f_{xxx} - f_{xx}(f_{xy} + 2g_{yy}) < 0. \tag{12}$$

Note that the stability conditions for (5a), (5b) given in Corollary 2 agree with a result of Behtash and Sastry ([3], Lemma 1).

2.2. Stability of the third-order reduced model

Next, consider the case in which $\eta = (x, y, z)'$ and model (2) is the three-dimensional system

$$\dot{x} = \Omega_1 y + f(x, y, z), \tag{13a}$$

$$\dot{y} = -\Omega_2 x + g(x, y, z), \tag{13b}$$

$$\dot{z} = r(x, y, z), \tag{13c}$$

where $\Omega_1\Omega_2 > 0$ and functions f, g, r are sufficiently smooth and take the general form

$$\begin{aligned}\varphi(x, y, z) = {} & \varphi_{xx}x^2 + \varphi_{xy}xy + \varphi_{xz}xz + \varphi_{yy}y^2 + \varphi_{yz}yz + \varphi_{zz}z^2 \\ & + \varphi_{xxx}x^3 + \varphi_{xxy}x^2y + \varphi_{xxz}x^2z + \varphi_{xyy}xy^2 + \varphi_{xyz}xyz + \varphi_{xzz}xz^2 \\ & + \varphi_{yyy}y^3 + \varphi_{yyz}y^2z + \varphi_{yzz}yz^2 + \varphi_{zzz}z^3 + \mathrm{O}(\|(x, y, z)\|^4).\end{aligned} \tag{14}$$

As explained above, it is not difficult to derive the normal form for system (13a)–(13c). For instance, a normal form for the case of $\Omega_1 = \Omega_2 = -\omega$ up to the third-order approximation has been obtained in cylindrical polar coordinates by Guckenheimer and Holmes [6]. A similar result is also obtained by Behtash and Sastry [3] for designing a purely nonlinear feedback stabilizing control law for the case in which ξ in (1a), (1b) is a scalar. However, in both results mentioned above, the values of the coefficients in the normal form for (13a)–(13c) have not been expressed in terms of the original system dynamics (i.e., the functions f, g, r). In the following discussions, a normal form representation for a general system (13a)–(13c) up to third-order will be given explicitly in terms of the original system dynamics. The result will be easy to apply to the stability analysis and stabilization of higher-dimensional systems (1a), (1b). Note that, we do not assume $\Omega_1 = \Omega_2$ in the following discussions.

By employing the technique given in [3,4] with $P = P_2$ a quadratic function as given in Appendix A, we can remove parts of quadratic terms of the dynamics in (13a)–(13c), and Eqs. (13a)–(13c) become

$$\begin{aligned}\dot{z}_1 = \Omega_1\Bigg\{ & z_2 + \frac{1}{\Omega_1 + \Omega_2}(g_{yz} + f_{zz})z_1 z_3 \\ & + \frac{1}{2\Omega_1\Omega_2}(\Omega_2 f_{yz} - \Omega_1 g_{zz})z_2 z_3\Bigg\} + \tilde{f}(z_1, z_2, z_3),\end{aligned} \tag{15a}$$

$$\dot{z}_2 = \Omega_2\left\{ -z_1 + \frac{1}{\Omega_1+\Omega_2}(g_{yz}+f_{zz})z_2z_3 - \frac{1}{2\Omega_1\Omega_2}(\Omega_2 f_{yz} - \Omega_1 g_{zz})z_1z_3 \right\} + \tilde{g}(z_1,z_2,z_3) \tag{15b}$$

$$\dot{z}_3 = \frac{1}{\Omega_1+\Omega_2}(\Omega_1 r_{zz} + \Omega_2 r_{yy})\cdot(z_1^2+z_2^2) + r_{zz}z_3^2 + \tilde{r}(z_1,z_2,z_3). \tag{15c}$$

Assume that the nonlinear vector function P in the normal form transformation is chosen as $P(\eta) = P_2(\eta) + P_3(\eta)$ with P_2 and P_3 as given in Appendix A. The new transformed version of (13a)–(13c) then becomes

$$\dot{x}_1 = \Omega_1\left\{ x_2 + \frac{1}{\Omega_1+\Omega_2}(g_{yz}+f_{xz})x_1x_3 + \frac{1}{2\Omega_1\Omega_2}(\Omega_2 f_{yz} - \Omega_1 g_{xz})x_2x_3 + \delta_1 x_1(x_1^2+x_2^2) + \epsilon_1 x_2(x_1^2+x_2^2) + x_3^2(\delta_2 x_1 + \epsilon_2 x_2) \right\} + \mathrm{O}(\|(x,y,z)\|^4), \tag{16a}$$

$$\dot{x}_2 = \Omega_2\left\{ -x_1 + \frac{1}{\Omega_1+\Omega_2}(g_{yz}+f_{xz})x_2x_3 - \frac{1}{2\Omega_1\Omega_2}(\Omega_2 f_{yz} - \Omega_1 g_{xz})x_1x_3 + \delta_1 x_2(x_1^2+x_2^2) - \epsilon_1 x_1(x_1^2+x_2^2) + x_3^2(\delta_2 x_2 - \epsilon_2 x_1) \right\} + \mathrm{O}(\|(x,y,z)\|^4), \tag{16b}$$

$$\dot{x}_3 = \frac{1}{\Omega_1+\Omega_2}(\Omega_1 r_{xx} + \Omega_2 r_{yy})\cdot(x_1^2+x_2^2) + r_{zz}x_3^2 + \delta_3 x_3(x_1^2+x_2^2) + \tilde{r}_{zzz}x_3^3 + \mathrm{O}(\|(x,y,z)\|^4), \tag{16c}$$

where

$$\epsilon_1 = \frac{1}{4\Omega_1\Omega_2(\Omega_1+\Omega_2)}\cdot\{3\Omega_2^2\tilde{f}_{222} + \Omega_1\Omega_2(\tilde{f}_{112} - \tilde{g}_{122}) - 3\Omega_1^2\tilde{g}_{111}\}, \tag{17}$$

$$\epsilon_2 = \frac{1}{2\Omega_1\Omega_2}(\Omega_2\tilde{f}_{233} - \Omega_1\tilde{g}_{133}), \tag{18}$$

$$\delta_1 = \frac{1}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2}\cdot\{\Omega_1(3\tilde{f}_{111} + \tilde{g}_{112}) + \Omega_2(3\tilde{g}_{222} + \tilde{f}_{122})\}, \tag{19}$$

$$\delta_2 = \frac{1}{\Omega_1+\Omega_2}(\tilde{f}_{133} + \tilde{g}_{233}), \tag{20}$$

$$\delta_3 = \frac{1}{\Omega_1+\Omega_2}(\Omega_1\tilde{r}_{113} + \Omega_2\tilde{r}_{223}). \tag{21}$$

Here, φ_{ijk} denotes the coefficient of the cubic term $z_iz_jz_k$ of a function $\varphi \in \{\tilde{f},\tilde{g},\tilde{r}\}$ and $i,j,k = 1,2,3$.

Using Corollary 1 and Liapunov stability criteria, we obtain the following stability conditions for (13a)–(13c) based on the transformed model (16a)–(16c).

Lemma 3. *The origin of* (13a)–(13c) *is asymptotically stable if* $r_{zz} = 0$, $\Omega_1\delta_1, \tilde{r}_{333} < 0$, *and either of the following conditions hold*:

(i) $g_{yz} + f_{xz} = 0$, $\Omega_1 r_{xx} + \Omega_2 r_{yy} = 0$, *and* $\Omega_1\delta_2, \delta_3 \leqslant 0$ *or* $\Omega_1\delta_2$ *and* δ_3 *are nonzero and of opposite sign*,
(ii) $\Omega_1(g_{yz} + f_{xz})$ *and* $\Omega_1 r_{xx} + \Omega_2 r_{yy}$ *are nonzero and are of opposite sign*, *and* $\Omega_1\delta_2, \delta_3 \leqslant 0$,

where the values of δ_i, $i = 1, 2, 3$ *are given in* (19)–(21).

Proof. As discussed above, Eqs. (13a)–(13c) can be transformed into (16a)–(16c) by normal form reduction. Choose

$$V = p_1\left(x_1^2 + \frac{\Omega_1}{\Omega_2}x_2^2\right) + p_2 x_3^2 \tag{22}$$

with $p_1, p_2 > 0$ as a Liapunov function candidate for the transformed model (16a)–(16c).

The time derivative of V along trajectories of (16a)–(16c) is

$$\begin{aligned}\dot{V} &= 2\Omega_1 p_1\delta_1(x_1^2 + x_2^2)^2 + 2(p_1\Omega_1\delta_2 + p_2\delta_3)x_3^2(x_1^2 + x_2^2) \\ &\quad + 2p_2\tilde{r}_{333}x_3^4 + 2p_2 r_{zz}x_3^3 + \frac{2}{\Omega_1 + \Omega_2}\{\Omega_1 p_1(g_{yz} + f_{xz}) \\ &\quad + p_2(\Omega_1 r_{xx} + \Omega_2 r_{yy})\}x_3(x_1^2 + x_2^2) + \mathrm{O}(\|(x_1, x_2, x_3)\|^5).\end{aligned} \tag{23}$$

Since $p_1, p_2 > 0$, the scalar function V given in (22) is positive definite. Suppose $r_{zz} = 0$ and $\Omega_1\delta_1, \tilde{r}_{333} < 0$. From Corollary 1, it follows that $\dot{V}$ (given in (23)) is locally negative definite if either condition (i) or (ii) holds. The application of Liapunov stability criteria to (16a)–(16c) indicates that the origin is asymptotically stable. As implied by ([7], Lemma 2.6) the origin is also asymptotically stable for the model (13a)–(13c). □

Note that the stability condition (i) of Lemma 3 above agrees with that obtained by Behtash and Sastry ([3], Theorem 2).

By expressing the values δ_i in terms of the original system dynamics, we have the following result for the case (i) of Lemma 3.

Corollary 3. *The origin is asymptotically stable for* (13a)–(13c) *if* $r_{zz} = 0$, $\Omega_1 r_{xx} + \Omega_2 r_{yy} = 0$, $f_{xz} + g_{yz} = 0$, $S_1, S_2 < 0$ *and* $S_3, S_4 \leqslant 0$ *or* S_3 *and* S_4 *are nonzero and of opposite sign*, *where*

$$
\begin{aligned}
S_1 &:= \tilde{r}_{333} \\
&= \frac{1}{\Omega_1\Omega_2}\{\Omega_1\Omega_2 r_{zzz} - \Omega_2 f_{zz} r_{yz} + \Omega_1 g_{zz} r_{xz}\},
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
S_2 &:= \Omega_1\delta_1 \\
&= \frac{1}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2}\bigg\{(\Omega_1 g_{xz} + \Omega_2 f_{yz}) r_{yy} - \Omega_1 g_{yz} r_{xy} + 3\Omega_1\Omega_2 g_{yyy} \\
&\quad + (\Omega_1 g_{xy} - 2\Omega_2 f_{yy}) g_{yy} + \frac{\Omega_1^2}{\Omega_2} g_{xx} g_{xy} + \Omega_1^2 g_{xxy} + \frac{2\Omega_1^2}{\Omega_2} f_{xx} g_{xx} \\
&\quad - \Omega_2 f_{xy} f_{yy} + \Omega_1\Omega_2 f_{xyy} - \Omega_1 f_{xx} f_{xy} + 3\Omega_1^2 f_{xxx}\bigg\},
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
S_3 &:= \Omega_1\delta_2 \\
&= \frac{1}{\Omega_2(\Omega_1 + \Omega_2)}\{2\Omega_2 f_{zz} r_{yz} - 2\Omega_1 g_{zz} r_{xz} + \Omega_1 (g_{xy} + 2 f_{xx}) g_{zz} \\
&\quad + \Omega_1\Omega_2 g_{yzz} - 2\Omega_2 f_{zz} g_{yy} - \Omega_2 f_{xy} f_{zz} + \Omega_1\Omega_2 f_{xzz}\},
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
S_4 &:= \delta_3 \\
&= \frac{1}{\Omega_2(\Omega_1 + \Omega_2)}\bigg\{\Omega_2^2 r_{yyz} - \frac{\Omega_2}{\Omega_1}(\Omega_2 f_{yy} + \Omega_1 f_{xx}) r_{yz} - \frac{\Omega_2}{\Omega_1}(\Omega_1 g_{xz} + \Omega_2 f_{yz}) r_{yy} \\
&\quad + (\Omega_2 g_{yy} + \Omega_1 g_{xx}) r_{xz} + \Omega_2 g_{yz} r_{xy} + \Omega_1\Omega_2 r_{xxz}\bigg\}.
\end{aligned} \tag{27}
$$

Similarly, the case (ii) of Lemma 3 is addressed in terms of the original dynamics as follows.

Corollary 4. *The origin of* (13a)–(13c) *is asymptotically stable if*:

(i) $r_{zz} = 0$,
(ii) $\Omega_1 r_{xx} + \Omega_2 r_{yy}$ *and* $f_{xz} + g_{yz}$ *have nonzero values and of opposite sign*,
(iii) $S_1, \tilde{S}_2 < 0$ *and* $\tilde{S}_3, \tilde{S}_4 \leqslant 0$,

where S_1 *is given by* (24) *and*

$$
\begin{aligned}
\tilde{S}_2 = &\frac{1}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2}\bigg\{ -\frac{\Omega_1 g_{xz} + \Omega_2 f_{yz}}{(\Omega_1 + \Omega_2)\Omega_2}[(\Omega_2 + 2\Omega_1)\Omega_2 r_{yy} \\
&+ (2\Omega_2 + 3\Omega_1)\Omega_1 r_{xx}] + \frac{\Omega_1}{2}(g_{yz} + 3 f_{xz}) r_{xy} + 3\Omega_1\Omega_2 g_{yyy} \\
&+ (\Omega_1 g_{xy} - 2\Omega_2 f_{yy}) g_{yy} + \frac{\Omega_1^2}{\Omega_2} g_{xx} g_{xy} + \Omega_1^2 g_{xxy} + \frac{2\Omega_1^2}{\Omega_2} f_{xx} g_{xx} \\
&- \Omega_2 f_{xy} f_{yy} + \Omega_1\Omega_2 f_{xyy} - \Omega_1 f_{xx} f_{xy} + 3\Omega_1^2 f_{xxx}\bigg\},
\end{aligned} \tag{28}
$$

$$\tilde{S}_3 = \frac{1}{\Omega_2(\Omega_1+\Omega_2)}\Bigg\{2\Omega_2 f_{zz} r_{yz} - 2\Omega_1 g_{zz} r_{xz} + \Omega_1(g_{xy}+2f_{xx})g_{zz} + \Omega_1\Omega_2 g_{yzz} - 2\Omega_2 f_{zz} g_{yy} - \Omega_2 f_{xy} f_{zz} + \Omega_1\Omega_2 f_{xzz} - \frac{\Omega_1}{2(\Omega_1+\Omega_2)}(f_{xz}+g_{yz})\cdot(\Omega_1 g_{xz}+\Omega_2 f_{yz})\Bigg\}, \tag{29}$$

$$\tilde{S}_4 = \frac{1}{\Omega_2(\Omega_1+\Omega_2)}\Bigg\{\Omega_2^2 r_{yyz} - \frac{\Omega_2}{\Omega_1}(\Omega_2 f_{yy}+\Omega_1 f_{xx})r_{yz} + (\Omega_1 g_{xz}+\Omega_2 f_{yz})r_{xx} + (\Omega_2 g_{yy}+\Omega_1 g_{xx})r_{xz} - \frac{\Omega_2}{\Omega_1+\Omega_2}[\Omega_1 g_{yz}+(\Omega_2+2\Omega_1)f_{xz}]r_{xy} + \Omega_1\Omega_2 r_{xxz}\Bigg\}. \tag{30}$$

2.3. Stability of critical fourth-order nonlinear systems

In this section, we derive stability conditions for (2) in which $\eta := (x,y,z,w)'$, $F(\eta) = (f(\eta), g(\eta), r(\eta), s(\eta))'$ and

$$A_{11} = \begin{pmatrix} 0 & \Omega_1 & 0 & 0 \\ -\Omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_3 \\ 0 & 0 & -\Omega_4 & 0 \end{pmatrix}. \tag{31}$$

Here, $\Omega_1\Omega_2, \Omega_3\Omega_4 > 0$ and f, g, r, s are smooth, purely nonlinear scalar functions with the form as

$$\begin{aligned}\varphi &= \varphi_{xx}x^2 + \varphi_{xy}xy + \varphi_{xz}xz + \varphi_{xw}xw + \varphi_{yy}y^2 + \varphi_{yz}yz + \varphi_{yw}yw \\ &\quad + \varphi_{zz}z^2 + \varphi_{zw}zw + \varphi_{ww}w^2 + \varphi_{xxx}x^3 + (\varphi_{xxy}y + \varphi_{xxz}z + \varphi_{xxw}w)x^2 \\ &\quad + (\varphi_{xyy}x + \varphi_{yyy}y + \varphi_{yyz}z + \varphi_{yyw}w)y^2 + \varphi_{xyz}xyz + \varphi_{xyw}xyw + \varphi_{xzw}zxw \\ &\quad + \varphi_{yzw}yzw + (\varphi_{xzz}x + \varphi_{yzz}y + \varphi_{zzz}z + \varphi_{zzw}w)z^2 \\ &\quad + (\varphi_{xww}x + \varphi_{yww}y + \varphi_{zww}z + \varphi_{www}w)w^2 + \mathrm{O}(\|(x,y,z,w)\|^4).\end{aligned} \tag{32}$$

For the case in which $\Omega_1 = \Omega_2 = -1$ and $\Omega_3 = \Omega_4 = -\omega \notin \{\pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 3\}$, a normal form for the model (2) has been obtained by using the technique given in (3) and (4); see for instance, [3,6]. In the following analysis, we do not assume that $\Omega_1 = \Omega_2$ nor that $\Omega_3 = \Omega_4$ for facilitating possible applications.

Assume that $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. By using the technique of normal form reduction to let $\eta = \zeta + P(\zeta)$ with P defined in Eq. (3), we can write model (2) as Eq. (4). First, consider the case in which the nonlinear function P is a purely quadratic function only (i.e., $P = P_2$) as given in Appendix B, we can make $\mathscr{F}_2$ (given in Section 2) 0 and Eq. (4) then becomes

$$\dot{\zeta} = A\zeta + \tilde{F}(\zeta), \tag{33}$$

where $\tilde{F}(\zeta) = (\tilde{f}(\zeta), \tilde{g}(\zeta), \tilde{r}(\zeta), \tilde{s}(\zeta))'$. Now, let P be a nonlinear function as given in (3) with P_2 having being as discussed above such that $\mathscr{F}_2 = 0$. By a suitable choice of cubic function P_3, as detailed in Appendix B, the transformed model (4) takes the form

$$\dot{x}_1 = \Omega_1\{x_2 + (\delta_1 x_1 + \epsilon_1 x_2)(x_1^2 + x_2^2) + (\delta_2 x_1 + \epsilon_2 x_2)(x_3^2 + x_4^2)\} + \mathrm{O}(\|(x_1, x_2, x_3, x_4)\|^4), \tag{34a}$$

$$\dot{x}_2 = \Omega_2\{-x_1 + (\delta_1 x_2 - \epsilon_1 x_1)(x_1^2 + x_2^2) + (\delta_2 x_2 - \epsilon_2 x_1)(x_3^2 + x_4^2)\} + \mathrm{O}(\|(x_1, x_2, x_3, x_4)\|^4), \tag{34b}$$

$$\dot{x}_3 = \Omega_3\{x_4 + (\delta_3 x_3 + \epsilon_3 x_4)(x_1^2 + x_2^2) + (\delta_4 x_3 + \epsilon_4 x_4)(x_3^2 + x_4^2)\} + \mathrm{O}(\|(x_1, x_2, x_3, x_4)\|^4), \tag{34c}$$

$$\dot{x}_4 = \Omega_4\{-x_3 + (\delta_3 x_4 - \epsilon_3 x_3)(x_1^2 + x_2^2) + (\delta_4 x_4 - \epsilon_4 x_3)(x_3^2 + x_4^2)\} + \mathrm{O}(\|(x_1, x_2, x_3, x_4)\|^4), \tag{34d}$$

where

$$\delta_1 = \frac{\Omega_2(3\tilde{g}_{222} + \tilde{f}_{122}) + \Omega_1(\tilde{g}_{112} + 3\tilde{f}_{111})}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2} \tag{35}$$

$$\epsilon_1 = \frac{\Omega_1\Omega_2(\tilde{f}_{112} - \tilde{g}_{122}) + 3\Omega_2^2\tilde{f}_{222} - 3\Omega_1^2\tilde{g}_{111}}{4\Omega_1\Omega_2(\Omega_1 + \Omega_2)} \tag{36}$$

$$\delta_2 = \frac{\Omega_3(\tilde{f}_{133} + \tilde{g}_{233}) + \Omega_4(\tilde{f}_{144} + \tilde{g}_{244})}{(\Omega_1 + \Omega_2) \cdot (\Omega_3 + \Omega_4)} \tag{37}$$

$$\epsilon_2 = \frac{\Omega_2(\Omega_3\tilde{f}_{233} + \Omega_4\tilde{f}_{244}) - \Omega_1(\Omega_3\tilde{g}_{133} + \Omega_4\tilde{g}_{144})}{2\Omega_1\Omega_2(\Omega_3 + \Omega_4)} \tag{38}$$

$$\delta_3 = \frac{\Omega_1(\tilde{r}_{113} + \tilde{s}_{114}) + \Omega_2(\tilde{r}_{223} + \tilde{s}_{224})}{(\Omega_1 + \Omega_2) \cdot (\Omega_3 + \Omega_4)} \tag{39}$$

$$\epsilon_3 = \frac{\Omega_4(\Omega_1\tilde{r}_{114} + \Omega_2\tilde{r}_{224}) - \Omega_3(\Omega_1\tilde{s}_{113} + \Omega_2\tilde{s}_{223})}{2\Omega_3\Omega_4(\Omega_1 + \Omega_2)} \tag{40}$$

$$\delta_4 = \frac{\Omega_4(3\tilde{s}_{444} + \tilde{r}_{344}) + \Omega_3(\tilde{s}_{334} + 3\tilde{r}_{333})}{3\Omega_3^2 + 2\Omega_3\Omega_4 + 3\Omega_4^2} \tag{41}$$

$$\epsilon_4 = \frac{\Omega_3\Omega_4(\tilde{r}_{334} - \tilde{s}_{344}) + 3\Omega_4^2\tilde{r}_{444} - 3\Omega_3^2\tilde{s}_{333}}{4\Omega_3\Omega_4(\Omega_3 + \Omega_4)}. \tag{42}$$

Here, let $\zeta := (z_1, z_2, z_3, z_4)'$ in (33). Then φ_{ijk} denotes the coefficient of the cubic term $z_i z_j z_k$ of a function φ, for $\varphi = \tilde{f}, \tilde{g}, \tilde{r}, \tilde{s}$ and $i, j, k = 1, \cdots, 4$.

Referring to the transformed model (34a)–(34d), we readily obtain the following stability conditions for the original model (2).

Lemma 4. *Let* $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, *for each* $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. *The origin is asymptotically stable for system* (2) *if* $\Omega_1\delta_1 < 0$, $\Omega_3\delta_4 < 0$ *and either* $\Omega_1\delta_2 \leqslant 0$ *and* $\Omega_3\delta_3 \leqslant 0$, *or* $\Omega_1\delta_2$ *and* $\Omega_3\delta_3$ *are nonzero and of opposite sign.*

Proof. As discussed above, system (2) can be transformed into Eqs. (34a)–(34d) if $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. Let

$$V = \frac{1}{2}p_1\left(x_1^2 + \frac{\Omega_1}{\Omega_2}x_2^2\right) + \frac{1}{2}p_2\left(x_3^2 + \frac{\Omega_3}{\Omega_4}x_4^2\right) \tag{43}$$

be a Liapunov function candidate for model (34a)–(34d) with $p_1, p_2 > 0$. Taking the time derivative of V along trajectories of the model (34a)–(34d), we then have

$$\begin{aligned}\dot{V} &= p_1\Omega_1\delta_1(x_1^2 + x_2^2)^2 + (p_1\Omega_1\delta_2 + p_2\Omega_3\delta_3)\cdot(x_1^2 + x_2^2)\cdot(x_3^2 + x_4^2) \\ &\quad + p_2\Omega_3\delta_4(x_3^2 + x_4^2)^2 + \mathrm{O}(\|(x_1, x_2, x_3, x_4)\|^5).\end{aligned} \tag{44}$$

Since $p_1, p_2 > 0$ and $\Omega_1\Omega_2, \Omega_3\Omega_4 > 0$, the scalar function V given in (43) is positive definite. First, consider the case in which $\Omega_1\delta_1 < 0$, $\Omega_3\delta_4 < 0$, $\Omega_1\delta_2 \leqslant 0$ and $\Omega_3\delta_3 \leqslant 0$. Since $p_1, p_2 > 0$, $\dot{V}$ given in (44) is locally negative definite. So the origin is asymptotically stable for the transformed model (34a)–(34d). By ([7], Lemma 2.6), the origin is also asymptotically stable for the original model (2).

Next, consider the case in which $\Omega_1\delta_1 < 0$, $\Omega_3\delta_4 < 0$, $\Omega_1\delta_2$ and $\Omega_3\delta_3$ are nonzero and of opposite sign. Similarly, we can show that $\dot{V}$ given in (44) is locally negative definite by choosing $p_1, p_2 > 0$ such that $p_1\Omega_1\delta_2 + p_2\Omega_3\delta_3 = 0$. The stability of the origin for model (2) is hence implied by the Liapunov stability criteria and ([7], Lemma 2.6). □

Note that, for the case in which $\Omega_1 = \Omega_2 = -1$ and $\Omega_3 = \Omega_4 = -\omega \notin \{\pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 3\}$, Lemma 4 agrees with a result of Behtash and Sastry ([3], Theorem 3). The stability conditions of Lemma 4 expressed in terms of the original nonlinear dynamics before normal form reduction are given in the next result.

Corollary 5. *Suppose* $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, *for each* $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. *The origin of* (2) *is asymptotically stable if* $S_1, S_2 < 0$ *and* $S_3, S_4 \leqslant 0$ *or* S_3 *and* S_4 *are nonzero and of opposite sign*, *where*

$$S_1 = \frac{1}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2}\Big\{\Omega_1[3(\Omega_2 g_{yyy} + \Omega_1 f_{xxx}) + (\Omega_1 g_{xxy} + \Omega_2 f_{xyy})]$$
$$+ g_{yy}(\Omega_1 g_{xy} - 2\Omega_2 f_{yy}) - f_{xy}(\Omega_2 f_{yy} + \Omega_1 f_{xx}) + \frac{\Omega_1^2}{\Omega_2} g_{xx}(g_{xy} + 2f_{xx})$$
$$+ \frac{\Omega_1}{\Omega_4}[(3\Omega_2 s_{yy} + \Omega_1 s_{xx})g_{yz} + (3\Omega_1 s_{xx} + \Omega_2 s_{yy})f_{xz}]$$
$$- \frac{\Omega_1}{\Omega_3}[(\Omega_1 r_{xx} + 3\Omega_2 r_{yy})g_{yw} + (\Omega_2 r_{yy} + 3\Omega_1 r_{xx})f_{xw}]$$
$$+ \frac{\Omega_1}{(4\Omega_1\Omega_2 - \Omega_3\Omega_4)\Omega_4}[\Omega_1(\Omega_4 g_{xw} - 2\Omega_2 g_{yz}) + \Omega_2(\Omega_4 f_{yw} + 2\Omega_1 f_{xz})]$$
$$\cdot (\Omega_4 r_{xy} - 2\Omega_1 s_{xx} + 2\Omega_2 s_{yy}) - \frac{\Omega_1}{(4\Omega_1\Omega_2 - \Omega_3\Omega_4)\Omega_3}[\Omega_1(2\Omega_2 g_{yw} + \Omega_3 g_{xz})$$
$$- \Omega_2(2\Omega_1 f_{xw} - \Omega_3 f_{yz})] \cdot (\Omega_3 s_{xy} - 2\Omega_2 r_{yy} + 2\Omega_1 r_{xx})\Big\}, \tag{45}$$

$$S_2 = \frac{1}{3\Omega_3^2 + 2\Omega_3\Omega_4 + 3\Omega_4^2}\Big\{\Omega_3[3(\Omega_4 s_{www} + \Omega_3 r_{zzz}) + (\Omega_3 s_{zzw} + \Omega_4 r_{zww})]$$
$$+ s_{ww}(\Omega_3 s_{zw} - 2\Omega_4 r_{ww}) - r_{zw}(\Omega_4 r_{ww} + \Omega_3 r_{zz}) + \frac{\Omega_3^2}{\Omega_4} s_{zz}(s_{zw} + 2r_{zz})$$
$$+ \frac{\Omega_3}{\Omega_2}[(3\Omega_4 g_{ww} + \Omega_3 g_{zz})s_{xw} + (3\Omega_3 g_{zz} + \Omega_4 g_{ww})r_{xz}]$$
$$- \frac{\Omega_3}{\Omega_1}[(\Omega_3 f_{zz} + 3\Omega_4 f_{ww})s_{yw} + (\Omega_4 f_{ww} + 3\Omega_3 f_{zz})r_{yz}]$$
$$+ \frac{\Omega_3}{(4\Omega_3\Omega_4 - \Omega_1\Omega_2)\Omega_2}[\Omega_3(\Omega_2 s_{yz} - 2\Omega_4 s_{xw})$$
$$+ \Omega_4(\Omega_2 r_{yw} + 2\Omega_3 r_{xz})] \cdot (\Omega_2 f_{zw} - 2\Omega_3 g_{zz} + 2\Omega_4 g_{ww})$$
$$- \frac{\Omega_3}{(4\Omega_3\Omega_4 - \Omega_1\Omega_2)\Omega_1}[\Omega_3(2\Omega_4 s_{yw} + \Omega_1 s_{xz})$$
$$- \Omega_4(2\Omega_3 r_{yz} - \Omega_1 r_{xw})] \cdot (\Omega_1 g_{zw} - 2\Omega_4 f_{ww} + 2\Omega_3 f_{zz})\Big\}, \tag{46}$$

$$S_3 = \frac{\Omega_3}{(\Omega_1 + \Omega_2)\cdot(\Omega_3 + \Omega_4)}\Big\{2f_{zz}r_{yz} + \frac{1}{\Omega_3}[2\Omega_4 f_{ww}s_{yw}$$
$$+ \Omega_1\Omega_4(f_{xww} + g_{yww})] + \Omega_1(f_{xzz} + g_{yzz})$$
$$- \frac{2\Omega_1}{\Omega_2} g_{zz}r_{xz} + \frac{\Omega_1}{\Omega_3^2\Omega_4}[\Omega_3(\Omega_4 s_{ww} + \Omega_3 s_{zz})(g_{yz} + f_{xz})$$

$$
\begin{aligned}
&- \Omega_4(\Omega_4 r_{ww} + \Omega_3 r_{zz})(g_{yw} + f_{xw})] - \frac{2\Omega_1\Omega_4}{\Omega_3\Omega_2} g_{ww}s_{xw} \\
&+ \frac{1}{\Omega_3\Omega_2}[\Omega_1(\Omega_3 g_{zz} + \Omega_4 g_{ww})(g_{xy} + 2f_{xx}) \\
&- \Omega_2(\Omega_3 f_{zz} + \Omega_4 f_{ww})(f_{xy} + 2g_{yy})] \\
&+ \frac{1}{(4\Omega_3\Omega_4 - \Omega_1\Omega_2)\Omega_3}[\Omega_4(\Omega_1 r_{xw} - 2\Omega_3 r_{yz}) \\
&+ \Omega_3(\Omega_1 s_{xz} + 2\Omega_4 s_{yw})] \cdot (\Omega_1 g_{zw} - 2\Omega_4 f_{ww} + 2\Omega_3 f_{zz}) \\
&- \frac{\Omega_1}{(4\Omega_3\Omega_4 - \Omega_1\Omega_2)\Omega_3\Omega_2}[\Omega_4(\Omega_2 r_{yw} + 2\Omega_3 r_{xz}) \\
&+ \Omega_3(\Omega_2 s_{yz} - 2\Omega_4 s_{xw})] \cdot (\Omega_2 f_{zw} - 2\Omega_3 g_{zz} + 2\Omega_4 g_{ww})\Big\},
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
S_4 = {}& \frac{\Omega_1}{(\Omega_1 + \Omega_2)\cdot(\Omega_3 + \Omega_4)}\Big\{2r_{xx}f_{xw} + \frac{1}{\Omega_1}[2\Omega_2 r_{yy}g_{yw} + \Omega_3\Omega_2(r_{yyz} + s_{yyw})] \\
&+ \frac{\Omega_3}{\Omega_1^2\Omega_2}[\Omega_1(\Omega_2 g_{yy} + \Omega_1 g_{xx})(s_{xw} + r_{xz}) - \Omega_2(\Omega_2 f_{yy} + \Omega_1 f_{xx})(s_{yw} + r_{yz})] \\
&- \frac{2\Omega_3}{\Omega_4}s_{xx}f_{xz} + \frac{1}{\Omega_1\Omega_4}[\Omega_3(\Omega_1 s_{xx} + \Omega_2 s_{yy})(s_{zw} + 2r_{zz}) \\
&- \Omega_4(\Omega_1 r_{xx} + \Omega_2 r_{yy})(r_{zw} + 2s_{ww})] - \frac{2\Omega_3\Omega_2}{\Omega_1\Omega_4}s_{yy}g_{yz} \\
&+ \frac{1}{(4\Omega_1\Omega_2 - \Omega_3\Omega_4)\Omega_1}[\Omega_2(\Omega_3 f_{yz} - 2\Omega_1 f_{xw}) \\
&+ \Omega_1(\Omega_3 g_{xz} + 2\Omega_2 g_{yw})] \cdot (\Omega_3 s_{xy} - 2\Omega_2 r_{yy} + 2\Omega_1 r_{xx}) \\
&- \frac{\Omega_3}{(4\Omega_1\Omega_2 - \Omega_3\Omega_4)\Omega_1\Omega_4}[\Omega_2(\Omega_4 f_{yw} + 2\Omega_1 f_{xz}) + \Omega_1(\Omega_4 g_{xw} \\
&- 2\Omega_2 g_{yz})] \cdot (\Omega_4 r_{xy} - 2\Omega_1 s_{xx} + 2\Omega_2 s_{yy}) + \Omega_3(r_{xxz} + s_{xxw})\Big\}.
\end{aligned} \tag{48}
$$

3. Double zero eigenvalue

In the following three sections, we apply the stability results obtained in Section 2 to the stabilization problem of system (1a), (1b). First, we consider the case in which A_{11} is in the form of (47) below. Thus, both η and b_1 are both two-dimensional vectors. Thus, $\eta := (x, y)'$ and $b_1 = (b_{11}, b_{12})'$, $F := (f, g)'$ and

$$
A_{11} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{49}
$$

$$\begin{aligned} G(x,y,\xi) = {} & x^2G_{xx} + xyG_{xy} + y^2G_{yy} + (xG_{x\xi} + yG_{y\xi})\xi \\ & + G_{\xi\xi}(\xi,\xi) + x^3G_{xxx} + x^2yG_{xxy} + xy^2G_{xyy} \\ & + y^3G_{yyy} + (x^2G_{xx\xi} + xyG_{xy\xi} + y^2G_{yy\xi})\xi + xG_{x\xi\xi}(\xi,\xi) \\ & + yG_{y\xi\xi}(\xi,\xi) + G_{\xi\xi\xi}(\xi,\xi,\xi) \ + \mathrm{O}(\|(x,y,\xi)\|^4). \end{aligned} \tag{50}$$

The scalar functions f, g are taken to be the form

$$\begin{aligned} \varphi(x,y,\xi) = {} & \varphi_{xx}x^2 + \varphi_{xy}xy + \varphi_{yy}y^2 + (x\varphi_{x\xi} + y\varphi_{y\xi})\xi \\ & + \xi'\varphi_{\xi\xi}\xi + \varphi_{xxx}x^3 + \varphi_{xxy}x^2y + \varphi_{xyy}xy^2 \\ & + \varphi_{yyy}y^3 + (x^2\varphi_{xx\xi} + xy\varphi_{xy\xi} + y^2\varphi_{yy\xi})\xi \\ & + \xi'(x\varphi_{x\xi\xi} + y\varphi_{y\xi\xi})\xi + \varphi_{\xi\xi\xi}(\xi,\xi,\xi) + \mathrm{O}(\|(x,y,\xi)\|^4). \end{aligned} \tag{51}$$

The coefficients in the expansion in (50) and (51) are either constants or symmetric multilinear functions of their arguments. For instance, $\varphi_{\xi\xi\xi}$ and $G_{\xi\xi}$ denote a symmetric trilinear function and a symmetric bilinear function, respectively.

3.1. The case $b_1 = 0$

In this section, we consider the case in which $b_1 = 0$ and let the feedback control u be given by

$$u(x,y,\xi) = k_{11}x + k_{12}y + K_2\xi + U(x,y,\xi), \tag{52}$$

where k_{11}, k_{12} are scalars and U is a smooth function with $U(0,0,0) = 0$ and $DU(0,0,0) = 0$.

Suppose $A_{22} + b_2K_2$ is stable. According to the results of [2], the stability of system (1a), (1b) agrees with the stability of the reduced model

$$\dot{x} = y + f(x,y,E_1x + E_2y + h(x,y)), \tag{53a}$$

$$\dot{y} = g(x,y,E_1x + E_2y + h(x,y)), \tag{53b}$$

where $E = (E_1, E_2)$ and $h(x,y)$ solve Eqs. (54) and (55), respectively:

$$b_2K_1 + (A_{22} + b_2K_2)E - EA_{11} = 0, \tag{54}$$

$$\begin{aligned} & Dh(\eta) \cdot \{A_{11}\eta + F(\eta, h(\eta) + E\eta)\} \\ & \quad = (A_{22} + b_2K_2)h(\eta) + b_2U(\eta, h(\eta) + E\eta) + G(\eta, h(\eta) + E\eta), \end{aligned} \tag{55}$$

with boundary conditions $h(0,0) = 0$ and $Dh(0,0) = 0$.

The boundary conditions above dictate that h be of the form

$$h(x,y) = x^2h_{xx} + xyh_{xy} + y^2h_{yy} + \mathrm{O}(\|(x,y)\|^3), \tag{56}$$

where h_{xx}, h_{xy}, h_{yy} are constant vectors.

Let the nonlinear control function U have the form (51) and

$$\begin{aligned} H(x,y) &:= b_2U(x,y,E_1x+E_2y) + G(x,y,E_1x+E_2y) \\ &\quad - f(x,y,E_1x+E_2y)E_1 - g(x,y,E_1x+E_2y)E_2 \\ &= x^2H_{xx} + xyH_{xy} + y^2H_{yy} + \mathrm{O}(\|(x,y)\|^3). \end{aligned} \tag{57}$$

By solving Eqs. (54) and (55), we then have

$$E_1 = -k_{11}(A_{22}+b_2K_2)^{-1}b_2, \tag{58}$$

$$E_2 = -\{(A_{22}+b_2K_2)^2\}^{-1}\cdot\{k_{12}(A_{22}+b_2K_2)+k_{11}I\}b_2 \tag{59}$$

and

$$h_{xx} = -(A_{22}+b_2K_2)^{-1}H_{xx}, \tag{60}$$

$$h_{xy} = -2\{(A_{22}+b_2K_2)^2\}^{-1}H_{xx} - (A_{22}+b_2K_2)^{-1}H_{xy}, \tag{61}$$

$$h_{yy} = -(A_{22}+b_2K_2)^{-1}(H_{yy}-h_{xy}). \tag{62}$$

The reduced model (53a), (53b) is hence obtained as

$$\begin{aligned} \dot{x} &= y + \hat{f}_{xx}x^2 + \hat{f}_{xy}xy + \hat{f}_{yy}y^2 + \hat{f}_{xxx}x^3 \\ &\quad + \hat{f}_{xxy}x^2y + \hat{f}_{xyy}xy^2 + \hat{f}_{yyy}y^3 + \mathrm{O}(\|(x,y)\|^4), \end{aligned} \tag{63a}$$

$$\begin{aligned} \dot{y} &= \hat{g}_{xx}x^2 + \hat{g}_{xy}xy + \hat{g}_{yy}y^2 + \hat{g}_{xxx}x^3 \\ &\quad + \hat{g}_{xxy}x^2y + \hat{g}_{xyy}xy^2 + \hat{g}_{yyy}y^3 + \mathrm{O}(\|(x,y)\|^4), \end{aligned} \tag{63b}$$

where φ_{ij} and φ_{ijk} denote the coefficients of quadratic terms ij and cubic terms ijk of function φ, for $\varphi = \hat{f}, \hat{g}$ and $i,j,k \in \{x,y\}$, respectively, and are given in Appendix C.

Now, referring to the stability criterion given in Corollary 2 and the foregoing discussions, we have:

Proposition 2. *Assume that* $b_{11} = b_{12} = 0$, *the control input is given by* (52) *and the nonlinear function U has the form as the one given in* (51). *Then the origin of* (1a), (1b) *is asymptotically stable if*:

(i) $A_{22} + b_2K_2$ *is stable*,
(ii) $\hat{g}_{xx} = 0$, $\hat{g}_{xy} + 2\hat{f}_{xx} = 0$,
(iii) $\hat{g}_{xxx} + 2\hat{f}_{xx}^2 < 0$ *and* (*iv*) $\hat{g}_{xxy} + 3\hat{f}_{xxx} - \hat{f}_{xx}(\hat{f}_{xy} + 2\hat{g}_{yy}) < 0$.

It can be seen from Proposition 2 and Appendix C that only the quadratic terms of the function G, and the linear and quadratic terms of control input u contribute to the stability conditions. Thus, a linear and/or quadratic feedback stabilizing control law is implied by Proposition 2. Although a purely linear feedback stabilizing control law might conceivably be obtained by using Proposition 2, in general, construction of such a control law is not feasible.

Consider a special case of system (1a), (1b) in which ξ is a scalar. So, b_2 is a scalar. Referring to Eqs. (57)–(62), we can determine the values of E_1, E_2, h_{xx}, h_{xy} and h_{yy} from the linear and quadratic gains of the control input. A linear-plus-quadratic stabilizing control law can hence be obtained as follows.

Lemma 5. *Assume that ξ is a scalar and $b_{11} = b_{12} = 0$. If $A_{22} + b_2K_2$ is stable and $g_{x\xi} \neq 0$, then a* linear-plus-quadratic *feedback can be designed to guarantee the stability of the origin of* (1a), (1b). *The proposed feedback control has the form*

$$u = k_{11}x + k_{12}y + K_2\xi + u_{xx}x^2 + u_{xy}xy + u_{yy}y^2.$$

Note that a purely quadratic feedback stabilizing control law, under the assumptions: $g_{xx} = 0$, $g_{xy} + 2f_{xx} = 0$ and $g_{x\xi} \neq 0$, given by Behtash and Sastry ([3], Corollary 1) for a three-dimensional version of (1a), (1b) is a special case of Lemma 5.

Suppose the control input u is a purely nonlinear function. Then a purely quadratic stabilizing control law follows readily from Proposition 2.

Lemma 6. *Assume that $b_{11} = b_{12} = 0$ and A_{22} is stable. Then there exists a purely quadratic stabilizing feedback $u = u_{xx}x^2 + u_{xy}xy + u_{yy}y^2$ for the origin of* (1a) *and* (1b) *if the following conditions hold*:

(i) $g_{xx} = 0$, $g_{xy} + 2f_{xx} = 0$,
(ii) $g_{xxx} + g_{x\xi}h_{xx} + 2f_{xx}^2 < 0$,
(iii) $g_{xxy} + g_{x\xi}h_{xy} + g_{y\xi}h_{xx} + 3(f_{xxx} + f_{x\xi}h_{xx}) - f_{xx}(f_{xy} + 2g_{yy}) < 0$,

where

$$h_{xx} = -A_{22}^{-1}(u_{xx}b_2 + G_{xx}), \tag{64}$$

$$h_{xy} = -2(A_{22}^2)^{-1}(u_{xx}b_2 + G_{xx}) - A_{22}^{-1}(u_{xy}b_2 + G_{xy}). \tag{65}$$

A stability criterion for the uncontrolled version of (1a), (1b) is obtained as follows.

Corollary 6. *Assume that $u = 0$. The origin of* (1a) *and* (1b) *is asymptotically stable if*:

(i) *A_{22} is stable*,
(ii) $g_{xx} = 0$, $g_{xy} + 2f_{xx} = 0$,
(iii) $g_{xxx} + g_{x\xi}h_{xx} + 2f_{xx}^2 < 0$,
(iv) $g_{xxy} + g_{x\xi}h_{xy} + g_{y\xi}h_{xx} + 3(f_{xxx} + f_{x\xi}h_{xx}) - f_{xx}(f_{xy} + 2g_{yy}) < 0$,

where h_{xx} and h_{xy} are given in (64) *and* (65) *by letting $u_{xx} = u_{xy} = 0$.*

3.2. The case $b_1 \neq 0$

Next, we consider the case in which either b_{11} or b_{12} is nonzero. It is known that $b_{12} \neq 0$ implies the controllability of system (1a). For simplicity, the control law is restricted to be purely nonlinear such that the control input u has the form as given in (51).

Let A_{22} be stable. Similarly, from [2], the stability of system (1a), (1b) agrees with that of the reduced model

$$\dot{x} = y + b_{11}u(x, y, h(x, y)) + f(x, y, h(x, y)), \tag{66a}$$

$$\dot{y} = b_{12}u(x, y, h(x, y)) + g(x, y, h(x, y)), \tag{66b}$$

where h is the solution of

$$\begin{aligned} &Dh(\eta) \cdot \{A_{11}\eta + b_1 u(\eta, h(\eta)) + F(\eta, h(\eta))\} \\ &\quad = A_{22}h(\eta) + b_2 u(\eta, h(\eta)) + G(\eta, h(\eta)) \end{aligned} \tag{67}$$

with boundary conditions $h(0) = 0$ and $Dh(0) = 0$.

Here, the function h is assumed to be given by Eq. (56). Choose the control input to be a function of only x and y as follows:

$$\begin{aligned} u(x, y, \xi) = {} & u_{xx}x^2 + u_{xy}xy + u_{yy}y^2 + u_{xxx}x^3 \\ & + u_{xxy}x^2y + u_{xyy}xy^2 + u_{yyy}y^3. \end{aligned} \tag{68}$$

A stability criterion for control system (1a) and (1b) is obtained as follows.

Proposition 3. *Assume that* $b_1 \neq 0$ *and* A_{22} *is stable. Then the origin is asymptotically stable for* (1a), (1b) *if*:

(i) $g_{xx} + b_{12}u_{xx} = 0$, $g_{xy} + b_{12}u_{xy} + 2(f_{xx} + b_{11}u_{xx}) = 0$,
(ii) $g_{xxx} + b_{12}u_{xxx} + g_{x\xi}h_{xx} + 2(f_{xx} + b_{11}u_{xx})^2 < 0$,
(iii) $g_{xxy} + b_{12}u_{xxy} + g_{x\xi}h_{xy} + g_{y\xi}h_{xx} + 3(f_{xxx} + b_{11}u_{xxx} + f_{x\xi}h_{xx}) - (f_{xx} + b_{11}u_{xx}) \cdot \{f_{xy} + b_{11}u_{xy} + 2(g_{yy} + b_{12}u_{yy})\} < 0$,

where h_{xx} *and* h_{xy} *are given in Eqs.* (64) *and* (65).

According to Proposition 3, b_{12} plays a key role in all stability conditions (i)–(iii). Thus we have the following result.

Lemma 7. *Let* A_{22} *be stable, but the full system need not be stable. If* $b_{12} \neq 0$, *then the stability of the origin of* (1a), (1b) *can be guaranteed by a* purely quadratic-plus-cubic *state feedback of the form* (68).

4. One zero and a pair of pure imaginary eigenvalues

In this section, we apply Corollaries 3 and 4 to design stabilizing control laws for control system (1a), (1b), where $\eta := (x, y, z)'$ and $b_1 = (b_{11}, b_{12}, b_{13})'$ are both three-dimensional vectors, $F := (f, g, r)'$ and

$$A_{11} = \begin{pmatrix} 0 & \Omega_1 & 0 \\ -\Omega_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{69}$$

Also, in the following analysis, φ_{ij} and φ_{ijk} denote the coefficients of the quadratic terms ij and the cubic terms ijk of function φ, respectively, for all $i, j, k \in \{x, y, z, \xi\}$ and $\varphi \in \{f, g, r, G\}$. As usual, these coefficients are either constants or symmetric multilinear functions of their arguments.

4.1. The case $b_1 = 0$

Let the control input u be of the form

$$u(x, y, z, \xi) = k_{11}x + k_{12}y + k_{13}z + K_2\xi + U(x, y, z, \xi), \tag{70}$$

where k_{1i}, $i = 1, 2, 3$, are scalars and function U is smooth enough with $U(0, 0, 0, 0) = 0$ and $DU(0, 0, 0, 0) = 0$.

Let $A_{22} + b_2K_2$ be stable. From [2], the stability of (1a), (1b) agrees with the stability of the reduced model

$$\dot{x} = \Omega_1 y + f(x, y, z, E\eta + h(x, y, z)), \tag{71a}$$

$$\dot{y} = -\Omega_2 x + g(x, y, z, E\eta + h(x, y, z)), \tag{71b}$$

$$\dot{z} = r(x, y, z, E\eta + h(x, y, z)), \tag{71c}$$

where $E = (E_1, E_2, E_3)$ and $h(x, y, z)$ solve Eqs. (54) and (55), respectively, with $\eta := (x, y, z)'$ and boundary conditions $h(0, 0, 0) = 0$ and $Dh(0, 0, 0) = 0$.

Referring to the boundary conditions above, we can write h as

$$\begin{aligned} h(x, y, z) = {} & x^2h_{xx} + xyh_{xy} + xzh_{xz} + y^2h_{yy} + yzh_{yz} \\ & + z^2h_{zz} + \mathrm{O}(\|(x, y, z)\|^3), \end{aligned} \tag{72}$$

where h_{ij}, $i, j \in \{x, y, z\}$ are constant vectors.

Let

$$\begin{aligned} H(x, y, z) := {} & b_2U(x, y, z, E\eta) + G(x, y, z, E\eta) - f(x, y, z, E\eta)E_1 \\ & - g(x, y, z, E\eta)E_2 - r(x, y, z, E\eta)E_3 \\ = {} & x^2H_{xx} + xyH_{xy} + xzH_{xz} + y^2H_{yy} + yzH_{yz} \\ & + z^2H_{zz} + \mathrm{O}(\|(x, y, z)\|^3). \end{aligned} \tag{73}$$

Solving Eqs. (54) and (55), we have

$$E_1 = -\{(A_{22} + b_2K_2)^2 + \Omega_1\Omega_2 I\}^{-1} \cdot \{k_{11}(A_{22} + b_2K_2) - \Omega_2 k_{12} I\} b_2, \tag{74}$$

$$E_2 = -\{(A_{22} + b_2K_2)^2 + \Omega_1\Omega_2 I\}^{-1} \cdot \{k_{12}(A_{22} + b_2K_2) + \Omega_1 k_{11}\} b_2, \tag{75}$$

$$E_3 = -k_{13}(A_{22} + b_2K_2)^{-1} b_2, \tag{76}$$

and

$$\begin{aligned} h_{xy} = & -\{(A_{22} + b_2K_2)^2 + 4\Omega_1\Omega_2 I\}^{-1} \\ & \cdot \{-2\Omega_2 H_{yy} + 2\Omega_1 H_{xx} + (A_{22} + b_2K_2) H_{xy}\}, \end{aligned} \tag{77}$$

$$h_{xx} = -(A_{22} + b_2K_2)^{-1}(H_{xx} + \Omega_2 h_{xy}), \tag{78}$$

$$h_{yy} = -(A_{22} + b_2K_2)^{-1}(H_{yy} - \Omega_1 h_{xy}), \tag{79}$$

$$h_{xz} = -\{(A_{22} + b_2K_2)^2 + \Omega_1\Omega_2 I\}^{-1} \cdot \{(A_{22} + b_2K_2) H_{xz} - \Omega_2 H_{yz}\} \tag{80}$$

$$h_{yz} = -\{(A_{22} + b_2K_2)^2 + \Omega_1\Omega_2 I\}^{-1} \cdot \{(A_{22} + b_2K_2) H_{yz} + \Omega_1 H_{xz}\} \tag{81}$$

$$h_{zz} = -(A_{22} + b_2K_2)^{-1} H_{zz}. \tag{82}$$

Let $\hat{\varphi}(x,y,z) := \varphi(x,y,z,E\eta + h(x,y,z))$, for $\varphi = f,g,r$, where the elements of E are given in (74)–(76) and function h is defined in (72) with h_{ij} given in (77)–(82). The coefficients of the quadratic terms and the cubic terms of functions $\hat{f},\hat{g},\hat{r}$ expressed in terms of E_i and h_{jk} are also given in Appendix C.

The reduced model (71a)–(71c) can hence be rewritten as

$$\dot{x} = \Omega_1 y + \hat{f}(x,y,z), \tag{83a}$$

$$\dot{y} = -\Omega_2 x + \hat{g}(x,y,z), \tag{83b}$$

$$\dot{z} = \hat{r}(x,y,z). \tag{83c}$$

As discussed above, the stability of the overall system (1a), (1b) agrees with that of the reduced model (83a)–(83c) if $A_{22} + b_2K_2$ is stable. In the following design, we will focus on the stabilization of (83a)–(83c) by assuming $A_{22} + b_2K_2$ is stable.

The next result follows readily from Corollaries 3 and 4 and the foregoing discussions.

Proposition 4. *Let* $b_{11} = b_{12} = 0$ *and the control input be given by* (70). *Then the origin of* (1a), (1b) *is asymptotically stable if* $A_{22} + b_2K_2$ *is stable,* $\hat{r}_{zz} = 0$, *and either of the following two conditions holds*:

(i) $\Omega_1\hat{r}_{xx} + \Omega_2\hat{r}_{yy} = 0$, $\hat{f}_{xz} + \hat{g}_{yz} = 0$, $S_1, S_2 < 0$ *and* $S_3, S_4 \leqslant 0$ *or* S_3 *and* S_4 *are nonzero and of opposite sign, where* S_i, $i = 1,\ldots,4$, *are given in* (24)–(27) *with coefficients* $\varphi_{ij},\varphi_{ijk}$ *replaced by* $\hat{\varphi}_{ij}$ *and* $\hat{\varphi}_{ijk}$, *respectively, for all* $\varphi = f,g,r$.

(ii) $\Omega_1\hat{r}_{xx} + \Omega_2\hat{r}_{yy}$ *and* $\hat{f}_{xz} + \hat{g}_{yz}$ *have nonzero values and of opposite sign,* $S_1, \tilde{S}_2 < 0$ *and* $\tilde{S}_3, \tilde{S}_4 \leqslant 0$, *where* S_1 *is given in* (24) *and* $\tilde{S}_i$, $i = 2,3,4$, *are given in* (28)–(30) *with coefficients* $\varphi_{ij},\varphi_{ijk}$ *replaced by* $\hat{\varphi}_{ij}$ *and* $\hat{\varphi}_{ijk}$, *respectively, for all* $\varphi = f,g,r$.

Here, $\hat{\varphi}(x,y,z) := \varphi(x,y,z,E\eta + h(x,y,z))$ *for* $\varphi = f,g,r$, *as defined above.*

It is obvious from Proposition 4 and Appendix C that only up to the quadratic terms of function G and the control input u contribute to the stability conditions of Proposition 4 in the case $b_1 = 0$. A linear and/or quadratic feedback stabilizing control law can hence obtained from Proposition 4. Similar to the results given in Proposition 2, a purely linear feedback stabilizing control law might conceivably be obtained by using Proposition 4, however, in general construction of such a control law is not feasible. A stability criterion for the uncontrolled version of (1a) and (1b) can also be obtained from Proposition 4 by letting $u = 0$.

Consider a special case of system (1a), (1b) in which ξ is a scalar. So, b_2 is a scalar. Suppose the nonlinear control function U in (70) is a function of x, y and z only and has the form given in (14). According to Eqs. (74)–(82), the values of E_i, and h_{ij} can be determined by the linear and quadratic gains of control input. A linear-plus-quadratic stabilizing control law can hence be obtained from Proposition 4 as follows.

Lemma 8. *Let* ξ *be a scalar and* $b_{1i} = 0$ *for* $i = 1,2,3$. *Then a* linear-plus-quadratic *feedback can be designed to guarantee the stability of the origin for* (1a) *and* (1b), *if*:

(i) $A_{22} + b_2K_2$ *is stable*,
(ii) $r_{\xi\xi} = 0$,
(iii) $r_{z\xi} \neq 0$
(iv) $\Omega_1 r_{x\xi}g_{z\xi} - \Omega_2 r_{y\xi}f_{z\xi} \neq 0$,
(v) $\Omega_1 g_{x\xi} + \Omega_2 f_{y\xi} \neq 0$, *or* $g_{y\xi} + \alpha f_{x\xi} \neq 0$ *for* $\alpha = 1$ *and* $\alpha = \frac{1}{3}$.

This feedback control has the form

$$u(x,y,z,\xi) = k_{11}x + k_{12}y + k_{13}z + K_2\xi + u_{xx}x^2 + u_{xy}xy + u_{xz}xz + u_{yy}y^2 + u_{yz}yz + u_{zz}z^2. \quad (84)$$

Proof. In the following, we check the stability conditions of Proposition 4 under the assumptions of Lemma 8. Suppose ξ is a scalar, $b_{1i} = 0$ for $i = 1,2,3$, and conditions (i)–(iii) hold. Then the values of $\hat{r}_{zz}$ and S_1 (given in (24)) can be made to be real numbers through $r_{z\xi}$ by the choice of E_3 and h_{zz}. Moreover, since condition (iv) holds, the values of $\Omega_1\hat{r}_{xx} + \Omega_2\hat{r}_{yy}$ and $\hat{f}_{xz} + \hat{g}_{yz}$ can be assigned arbitrarily by a proper choice of E_1 and E_2, while the values of S_3 and S_4 (given in (26) and (27)) or $\tilde{S}_3$ and $\tilde{S}_4$ in (given in (29) and (30)) can be assigned by proper choice of h_{xz} and h_{yz}.

Finally, condition (v) provides the opportunity for assigning the values of S_2 (given in (25)) and $\tilde{S}_2$ (given in (28)) by proper choice of h_{xx} or h_{yy}. According to Appendix C and Eqs. (73)–(82), $\hat{\varphi}_{ij}$ and $\hat{\varphi}_{ijk}$ can be determined by the linear and quadratic control gains through the linear matrix E and the vector function h. The conclusions of the lemma follow. □

A purely quadratic feedback stabilizing control law can also be obtained as given below. The proof is similar to that of Lemma 8. Details are omitted.

Lemma 9. *Let A_{22} be stable, ξ be a scalar and $b_{1i} = 0$ for $i = 1, 2, 3$. Then a purely quadratic feedback*

$$u(x, y, z) = u_{xx}x^2 + u_{xy}xy + u_{xz}xz + u_{yy}y^2 + u_{yz}yz + u_{zz}z^2 \tag{85}$$

can be designed to guarantee the stability of the origin of (1a) *and* (1b), *if the following conditions hold*:

(i) $\Omega_1 r_{xx} + \Omega_2 r_{yy} = 0$ *and* $f_{xz} + g_{yz} = 0$, *or* $\Omega_1 r_{xx} + \Omega_2 r_{yy}$ *and* $f_{xz} + g_{yz}$ *have nonzero values and of opposite sign*,
(ii) $r_{zz} = 0$ *and* $r_{z\xi} \neq 0$, *and*
(iii) $\Omega_1 g_{x\xi} + \Omega_2 f_{y\xi} \neq 0$, *and* $g_{z\xi} \neq 0$ *or* $f_{z\xi} \neq 0$.

4.2. The case $b_1 \neq 0$

Next, we consider the case in which one of b_{1i}, $i = 1, 2, 3$, is nonzero. It is known that $b_{13} \neq 0$, and $b_{11} \neq 0$ or $b_{12} \neq 0$ implies the controllability of system (1a). For simplicity, the control law is restricted here to be a purely nonlinear function of x, y and z only and to have the form (14).

Let A_{22} be stable. According to the results of [2], the stability of system (1a), (1b) agrees with that of the reduced model (83a)–(83c). Here,

$$\hat{f}(x, y, z) = b_{11}u(x, y, z) + f(x, y, z, h(x, y, z)), \tag{86a}$$

$$\hat{g}(x, y, z) = b_{12}u(x, y, z) + g(x, y, z, h(x, y, z)), \tag{86b}$$

$$\hat{r}(x, y, z) = b_{13}u(x, y, z) + r(x, y, z, h(x, y, z)), \tag{86c}$$

and h is the solution for (67) with boundary conditions $h(0) = 0$ and $Dh(0) = 0$. Similarly, function h is assumed to be given by Eq. (72).

By letting

$$\begin{aligned} H(x, y, z) &:= b_2 u(x, y, z) + G(x, y, z, 0) \\ &= x^2 H_{xx} + xyH_{xy} + xzH_{xz} + y^2 H_{yy} + yzH_{yz} \\ &\quad + z^2 H_{zz} + \mathrm{O}(\|(x, y, z)\|^3), \end{aligned} \tag{87}$$

we can obtain h_{ij} as given in (77)–(82) with $K_2 = 0$ and H_{ij} given in (87).

A stability criterion for control system (1a), (1b) in the case of $b_1 \neq 0$ is obtained as follows.

Proposition 5. *Let $b_1 \neq 0$ and A_{22} be stable. Then the origin of* (1a), (1b) *is asymptotically stable if $\hat{r}_{zz} = 0$, and either of conditions* (i) *and* (ii) *given in Proposition* 4 *hold. Here, $\hat{\varphi}_{ij}$ and $\hat{\varphi}_{ijk}$ denote the coefficients of quadratic terms and cubic terms of function $\hat{\varphi}$ $(= \hat{f}, \hat{g}, \hat{r}$ given in* (86a)–(86c)), *respectively.*

It is obvious from Proposition 5 that the vector b_1 plays a key role in all stability conditions (i)–(iii). The next two results follow readily from Proposition 5.

Lemma 10. *Let A_{22} be stable, but the whole system may not be stable. If $b_{13} \neq 0$ and one of b_{11} and b_{12} is not zero, then the stability of the origin of* (1a), (1b) *can be guaranteed by a* purely quadratic-plus-cubic *state feedback as follows*:

$$\begin{aligned} u(x,y,z) = {} & u_{xx}x^2 + u_{xy}xy + u_{xz}xz + u_{yy}y^2 + u_{yz}yz + u_{zz}z^2 \\ & + u_{xxx}x^3 + u_{xxy}x^2y + u_{xxz}x^2z + u_{xyy}xy^2 + u_{xyz}xyz + u_{xzz}xz^2 \\ & + u_{yyy}y^3 + u_{yyz}y^2z + u_{yzz}yz^2 + u_{zzz}z^3. \end{aligned}$$

Lemma 11. *Let A_{22} be stable, but the full system need not be stable. Then the stability of the origin for* (1a), (1b) *can be guaranteed by a* purely cubic *state feedback*

$$\begin{aligned} u(x,y,z) = {} & u_{xxx}x^3 + u_{xxy}x^2y + u_{xxz}x^2z + u_{xyy}xy^2 + u_{xzz}xz^2 \\ & + u_{yyy}y^3 + u_{yyz}y^2z + u_{yzz}yz^2 + u_{zzz}z^3, \end{aligned} \tag{88}$$

if $r_{zz} = 0$ and following conditions hold:

(i) *$b_{13} \neq 0$ and one of b_{11} and b_{12} is not zero, and*
(ii) *$\Omega_1 r_{xx} + \Omega_2 r_{yy} = 0$ and $f_{xz} + g_{yz} = 0$, or the expressions $\Omega_1 r_{xx} + \Omega_2 r_{yy}$ and $f_{xz} + g_{yz}$ have nonzero values and of opposite sign.*

5. Two distinct pairs of pure imaginary eigenvalues

In this section, we continue the stabilization study of the system (1a), (1b) in which $\eta := (x, y, z, w)'$ and $b_1 = (b_{11}, b_{12}, b_{13}, b_{14})'$ are both four-dimensional vectors, $F := (f, g, r, s)'$ and

$$A_{11} = \begin{pmatrix} 0 & \Omega_1 & 0 & 0 \\ -\Omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_3 \\ 0 & 0 & -\Omega_4 & 0 \end{pmatrix}. \tag{89}$$

As in the previous two sections, in the following analysis, φ_{ij} and φ_{ijk} denote the coefficients of the quadratic terms ij and the cubic terms ijk of function φ, respectively, for all $i, j, k \in \{x, y, z, w, \xi\}$ and $\varphi \in \{f, g, r, s, G\}$. As usual, these coefficients are either constants or symmetric multilinear functions of their arguments.

5.1. The case $b_1 = 0$

First, we consider the case in which $b_1 = 0$, and

$$u(x, y, z, w, \xi) = k_{11}x + k_{12}y + k_{13}z + k_{14}w + K_2\xi + U(x, y, z, w, \xi), \tag{90}$$

where k_{1i}, $i = 1, \dots, 4$, are scalars and U is sufficiently smooth with $U(0, 0, 0, 0, 0) = 0$ and $DU(0, 0, 0, 0, 0) = 0$.

Let $A_{22} + b_2K_2$ be stable. Similarly, the stability of (1a), (1b) is known to agree with the stability of the reduced model

$$\dot{x} = \Omega_1 y + f(x, y, z, w, E\eta + h(x, y, z, w)), \tag{91a}$$

$$\dot{y} = -\Omega_2 x + g(x, y, z, w, E\eta + h(x, y, z, w)), \tag{91b}$$

$$\dot{z} = \Omega_3 w + r(x, y, z, w, E\eta + h(x, y, z, w)), \tag{91c}$$

$$\dot{w} = -\Omega_4 z + s(x, y, z, w, E\eta + h(x, y, z, w)), \tag{91d}$$

where $E = (E_1, E_2, E_3, E_4)$ and $h(x, y, z, w)$ solve Eqs. (54) and (55), respectively, with $\eta := (x, y, z, w)'$ and boundary conditions $h(0, 0, 0, 0) = 0$ and $Dh(0, 0, 0, 0) = 0$.

The boundary conditions above require h to have the form

$$\begin{aligned} h(x, y, z, w) = {} & x^2h_{xx} + xyh_{xy} + xzh_{xz} + xwh_{xw} + y^2h_{yy} + yzh_{yz} \\ & + ywh_{yw} + z^2h_{zz} + zwh_{zw} + w^2h_{ww} + \mathrm{O}(\|(x, y, z, w)\|^3), \end{aligned} \tag{92}$$

where h_{ij}, $i, j \in \{x, y, z, w\}$ are constant vectors.

Similarly, let

$$\begin{aligned} H(x, y, z, w) := {} & b_2U(x, y, z, w, E\eta) + G(x, y, z, w, E\eta) - f(x, y, z, w, E\eta)E_1 \\ & - g(x, y, z, w, E\eta)E_2 - r(x, y, z, w, E\eta)E_3 - s(x, y, z, w, E\eta)E_4 \\ = {} & x^2H_{xx} + xyH_{xy} + xzH_{xz} + xwH_{xw} + y^2H_{yy} + yzH_{yz} + ywH_{yw} \\ & + z^2H_{zz} + zwH_{zw} + w^2H_{ww} + \mathrm{O}(\|(x, y, z, w)\|^3). \end{aligned} \tag{93}$$

By solving Eqs. (54) and (55), we have

$$E_1 = -\{M_1^2 + \Omega_1\Omega_2 I\}^{-1} \cdot \{k_{11}M_1 - \Omega_2 k_{12} I\}b_2, \tag{94}$$

$$E_2 = -\{M_1^2 + \Omega_1\Omega_2 I\}^{-1} \cdot \{k_{12}M_1 + \Omega_1 k_{11}\}b_2, \tag{95}$$

$$E_3 = -\{M_1^2 + \Omega_3\Omega_4 I\}^{-1} \cdot \{k_{13}M_1 - \Omega_4 k_{14} I\}b_2, \tag{96}$$

$$E_4 = -\{M_1^2 + \Omega_3\Omega_4 I\}^{-1} \cdot \{k_{14}M_1 + \Omega_3 k_{13}\}b_2, \tag{97}$$

$$h_{zw} = -(M_1^2 + 4\Omega_3\Omega_4 I)^{-1}(-2\Omega_4 H_{ww} + 2\Omega_3 H_{zz} + M_1 H_{zw}), \tag{98}$$

$$h_{zz} = -M_1^{-1}(H_{zz} + \Omega_4 h_{zw}), \tag{99}$$

$$h_{ww} = -M_1^{-1}(H_{ww} - \Omega_3 h_{zw}), \tag{100}$$

$$\begin{pmatrix} h_{xz} \\ h_{xw} \end{pmatrix} = (M_2^2 + \Omega_1\Omega_2 I)^{-1}\left\{ M_2 \begin{pmatrix} H_{xz} \\ H_{xw} \end{pmatrix} - \Omega_2 \begin{pmatrix} H_{yz} \\ H_{yw} \end{pmatrix} \right\}, \tag{101}$$

$$\begin{pmatrix} h_{yz} \\ h_{yw} \end{pmatrix} = (M_2^2 + \Omega_1\Omega_2 I)^{-1}\left\{ \Omega_1 \begin{pmatrix} H_{xz} \\ H_{xw} \end{pmatrix} + M_2 \begin{pmatrix} H_{yz} \\ H_{yw} \end{pmatrix} \right\}, \tag{102}$$

where the expressions of h_{xx}, h_{xy}, h_{yy} are given in Eqs. (77)–(79) with H_{ij} defined in (93), $M_1 := A_{22} + b_2 K_2$ and

$$M_2 := \begin{pmatrix} M_1 & \Omega_4 I \\ -\Omega_3 I & M_1 \end{pmatrix}. \tag{103}$$

The reduced model (91a)–(91d) can hence be obtained as

$$\dot{x} = \Omega_1 y + \hat{f}(x, y, z, w), \tag{104a}$$

$$\dot{y} = -\Omega_2 x + \hat{g}(x, y, z, w), \tag{104b}$$

$$\dot{z} = \Omega_3 w + \hat{r}(x, y, z, w), \tag{104c}$$

$$\dot{w} = -\Omega_4 z + \hat{s}(x, y, z, w). \tag{104d}$$

Here, $\hat{\varphi}(x, y, z, w) := \varphi(x, y, z, w, E\eta + h(x, y, z, w))$ for $\varphi = f, g, r, s$ with E_i given in (94)–(97) and h defined in (92). The values of h_{ij} are given in (77)–(79) and (98)–(102), and the coefficients of the quadratic terms and cubic terms of the functions $\hat{f}, \hat{g}, \hat{r}, \hat{s}$ expressed in terms of E_i and h_{jk} are given in Appendix C.

A linear and/or quadratic feedback stabilizing control law readily follows from Corollary 5 and the foregoing discussions.

Proposition 6. *Let* $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, *for each* $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$ *and* $b_{1i} = 0$ *for* $i = 1, \ldots, 4$. *The origin is asymptotically stable for control system* (1a), (1b) *if* $S_1, S_2 < 0$ *and* $S_3, S_4 \leqslant 0$ *or* S_3 *and* S_4 *are nonzero and of opposite sign, where* S_i *are given in* (45)–(48) *with coefficients* $\varphi_{ij}, \varphi_{ijk}$ *replaced by* $\hat{\varphi}_{ij}, \hat{\varphi}_{ijk}$, *respectively, for all* $\varphi = f, g, r, s$. *Here,* $\hat{f}, \hat{g}, \hat{r}, \hat{s}$ *are defined above and the control input is given by* (90).

Note that a stability criterion for the uncontrolled model of (1a), (1b) can also be obtained from Proposition 6 by letting $u = 0$. Next, consider a special case in which ξ is a scalar. Referring to Eqs. (93), (77)–(79) and (98)–(102), we can determine h_{ij} from the quadratic gains of the control input. A purely quadratic stabilizing control law is hence obtained as follows.

Lemma 12. *Let ξ be a scalar, $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$ and $b_{1i} = 0$ for $i = 1, \ldots, 4$. A* purely quadratic *feedback*

$$u(x,y,z,w) = u_{xx}x^2 + x(u_{xy}y + u_{xz}z + u_{xw}w) + u_{yy}y^2 + y(u_{yz}z + u_{yw}w) + u_{zz}z^2 + u_{zw}zw + u_{ww}w^2 \quad (105)$$

exists guaranteeing the asymptotic stability of the origin for (1a), (1b), *if $f_{x\xi} + g_{y\xi} \neq 0$, $r_{z\xi} + s_{w\xi} \neq 0$ and either of the following two conditions hold*:

(i) $f_{x\xi} \neq g_{y\xi}$ *and* $r_{z\xi} \neq s_{w\xi}$,
(ii) $\Omega_1 g_{x\xi} + \Omega_2 f_{y\xi} \neq 0$ *and* $\Omega_3 s_{z\xi} + \Omega_4 r_{w\xi} \neq 0$.

Proof. In the following, we check the stability conditions of Proposition 6 under the hypotheses of Lemma 12. Suppose ξ is a scalar, $b_{1i} = 0$, $i = 1, \ldots, 4$, $f_{x\xi} + g_{y\xi} \neq 0$ and $r_{z\xi} + s_{w\xi} \neq 0$. Then the values of S_3 and S_4 (given in (47) and (48)) can be made equal to any real numbers by a proper choice of $\Omega_1 h_{xx} + \Omega_2 h_{yy}$ and $\Omega_3 h_{zz} + \Omega_4 h_{ww}$.

If condition (i) holds, then the value of S_1 (given in (45)) will be determined by h_{xx} and h_{yy}, independent of the value of S_4. Similarly, the value of S_2 is determined by h_{zz} and h_{ww}, irrespective of the value of S_3. The values of S_1 and S_2 can also be adjusted by the choice of h_{xy} and h_{zw} when condition (ii) holds.

According to Eqs. (77)–(79), (93) and (98)–(102), the values of h_{ij} can be directly determined by the quadratic feedback gains when ξ is scalar. The conclusion is hence implied. □

A similar stabilizing control law can also be designed as follows.

Lemma 13. *Suppose ξ is a scalar, $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$ and $b_{1i} = 0$ for $i = 1, \ldots, 4$. A* purely quadratic *feedback as given in* (105) *can be designed to guarantee the stability of the origin for* (1a), (1b) *if $f_{x\xi} \neq \alpha g_{y\xi}$ and $r_{z\xi} \neq \alpha s_{w\xi}$ for $\alpha = -3$ and $\alpha = -\frac{1}{3}$ and either of the following conditions holds*:

(i) $\Omega_2 f_{w\xi} s_{y\xi} - \Omega_1 g_{w\xi} s_{x\xi} \neq 0$ *or* $\Omega_2 f_{z\xi} r_{y\xi} - \Omega_1 g_{z\xi} r_{x\xi} \neq 0$,
(ii) $\Omega_4 f_{w\xi} r_{x\xi} - \Omega_3 f_{z\xi} s_{x\xi} \neq 0$ *or* $\Omega_2\Omega_4 f_{w\xi} r_{y\xi} - \Omega_1\Omega_3 g_{z\xi} s_{x\xi} \neq 0$, *or*
(iii) $\Omega_1\Omega_4 g_{w\xi} r_{x\xi} - \Omega_2\Omega_3 f_{z\xi} s_{y\xi} \neq 0$ *or* $\Omega_4 g_{w\xi} r_{y\xi} - \Omega_3 g_{z\xi} s_{y\xi} \neq 0$.

Proof. The proof is very similar to that of Lemma 12. Suppose $f_{x\xi} \neq \alpha g_{y\xi}$ and $r_{z\xi} \neq \alpha s_{w\xi}$ for $\alpha = -3$ and $\alpha = -\frac{1}{3}$. The values of S_1 and S_2 (given in (45) and (46)) can then be adjusted by h_{xx} (or h_{yy}) and h_{zz} (or h_{ww}). Moreover, the values of S_3 and S_4 (given in (47) and (48)) can be any real numbers by a proper choice of h_{xw}, h_{yw}, h_{xz} or h_{yz}, when either of conditions (i)–(iii) holds. Since the values of h_{ij} can be directly determined from the quadratic control gains when ξ is a scalar, the conclusion is hence implied. □

5.2. The case $b_1 \neq 0$

In this section, we consider the case in which one of b_{1i}, $i = 1, \dots, 4$ is nonzero. It is known that $b_{11} \neq 0$ or $b_{12} \neq 0$, and $b_{13} \neq 0$ or $b_{14} \neq 0$ imply the controllability of system (1a). Similar to Section 2, the control law, here, is also restricted to be a purely nonlinear function of x, y, z, w and has the form as given in (32).

Let A_{22} be stable. Then according to the discussions in [2], the stability of (1a), (1b) is determined from the reduced model (104a)–(104d), where

$$\hat{f}(x,y,z) = b_{11}u(x,y,z) + f(x,y,z,h(x,y,z)), \tag{106a}$$

$$\hat{g}(x,y,z) = b_{12}u(x,y,z) + g(x,y,z,h(x,y,z)), \tag{106b}$$

$$\hat{r}(x,y,z) = b_{13}u(x,y,z) + r(x,y,z,h(x,y,z)), \tag{106c}$$

$$\hat{s}(x,y,z) = b_{14}u(x,y,z) + s(x,y,z,h(x,y,z)), \tag{106d}$$

and h is the solution for (67) with boundary conditions $h(0) = 0$ and $Dh(0) = 0$.

Suppose h is given by Eq. (92) and let

$$\begin{aligned} H(x,y,z,w) &:= b_2u(x,y,z,w) + G(x,y,z,w,0) \\ &= x^2H_{xx} + xyH_{xy} + xzH_{xz} + xwH_{xw} + y^2H_{yy} + yzH_{yz} + ywH_{yw} \\ &\quad + z^2H_{zz} + zwH_{zw} + w^2H_{ww} + O(\|(x,y,z,w)\|^3). \end{aligned} \tag{107}$$

h_{ij} are hence obtained as given in (77)–(79) and (98)–(102) with $K_2 = 0$ and H_{ij} given in (107). A stability criterion for control system (1a), (1b) in the case $b_1 \neq 0$ readily follows from Corollary 5.

Proposition 7. *Suppose $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$ and $b_{1i} = 0$ for $i = 1, \dots, 4$. The origin is asymptotically stable for control system* (1a), (1b) *if $S_1, S_2 < 0$ and $S_3, S_4 \leqslant 0$ or S_3 and S_4 are nonzero and of opposite sign, where S_i are given in* (45)–(48) *with coefficients φ_{ij}, φ_{ijk} replaced by $\hat{\varphi}_{ij}$, $\hat{\varphi}_{ijk}$, respectively, for all $\varphi = f, g, r, s$. Here, $\hat{f}, \hat{g}, \hat{r}, \hat{s}$ are defined in* (106a)–(106d) *and the control input u is a purely nonlinear function and has the form as given in* (32).

A purely cubic stabilizing control law is obtained as follows.

Lemma 14. *Let A_{22} be stable, but the full system need not be stable. If $b_{11} \neq 0$ or $b_{12} \neq 0$, and $b_{13} \neq 0$ or $b_{14} \neq 0$, then the stability of the origin of* (1a), (1b) *can be guaranteed by a* purely cubic *state feedback*

$$\begin{aligned} u(x,y,z,w) &= u_{xxx}x^3 + (u_{xxy}y + u_{xxz}z + u_{xxw}w)x^2 \\ &\quad + (u_{xyy}x + u_{yyy}y + u_{yyz}z + u_{yyw}w)y^2 \\ &\quad + (u_{xzz}x + u_{yzz}y + u_{zzz}z + u_{zzw}w)z^2 \\ &\quad + (u_{xww}x + u_{yww}y + u_{zww}z + u_{www}w)w^2. \end{aligned} \tag{108}$$

6. Conclusions

The center manifold reduction technique proposed in [2], along with the normal form reduction recalled in Section 2, are applied in this paper to study the stability and stabilization of smooth, nonlinear autonomus systems in doubly critical cases. Specifically, the linearized model of the system has two zero eigenvalues with geometric multiplicity 1; one zero eigenvalue and a pair of nonzero pure imaginary eigenvalues; or two distinct pairs of nonzero pure imaginary eigenvalues. The feedback stabilizing control laws are proposed for both linearly controllable and linearly uncontrollable cases, while a purely nonlinear feedback design is considered in the former case and linear and/or nonlinear control designs are obtained for the latter case.

Some of the results given in this paper agree with those obtained by Behtash and Sastry [3]. However, the results obtained in this paper cover more detailed design for general high-dimensional systems. For instance, the stability criteria and stabilizing control laws are given in terms of the original system dynamics before normal form reduction. Moreover, there is no restriction on the number of the noncritical modes and the stabilizing control algorithms proposed in this paper can be coded easily.

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Appendix A

The polynomial functions P_2 and P_3 for deriving the normal form for the case in which A_{11} has exactly one zero eigenvalue and a pair of nonzero, pure imaginary eigenvalues are given below.

Let $P_2(x,y,z) = (P_2^1, P_2^2, P_2^3)'$, where $P_2^i(x,y,z)$ has the form as

$$\varphi = \varphi_{xx}x^2 + \varphi_{xy}xy + \varphi_{xz}xz + \varphi_{yy}y^2 + \varphi_{yz}yz + \varphi_{zz}z^2,$$

for all $\varphi = P_2^i$, $i = 1,\ldots,3$.

The coefficients of polynomial functions P_2^i are

$$P^1_{2,xx} = \frac{(2g_{yy} + f_{xy})\Omega_2 + g_{xx}\Omega_1}{3\Omega_1\Omega_2},$$

$$P^1_{2,xy} = -\frac{(g_{xy} + 2f_{xx})\Omega_1 - 2f_{yy}\Omega_2}{3\Omega_1\Omega_2},$$

$$P^1_{2,xz} = \frac{f_{yz}\Omega_2 + g_{xz}\Omega_1}{4\Omega_1\Omega_2},$$
$$P^1_{2,yy} = -\frac{(f_{xy} - g_{yy})\Omega_2 - 2g_{xx}\Omega_1}{3\Omega_2^2},$$
$$P^1_{2,yz} = -\frac{f_{xz}\Omega_2 - g_{yz}\Omega_1}{2(\Omega_2^2 + \Omega_1\Omega_2)},$$
$$P^1_{2,zz} = \frac{g_{zz}}{\Omega_2},$$

$$P^2_{2,xx} = -\frac{2f_{yy}\Omega_2 + (f_{xx} - g_{xy})\Omega_1}{3\Omega_1^2},$$
$$P^2_{2,xy} = \frac{(2g_{yy} + f_{xy})\Omega_2 - 2g_{xx}\Omega_1}{3\Omega_1\Omega_2},$$
$$P^2_{2,xz} = -\frac{f_{xz}\Omega_2 - g_{yz}\Omega_1}{2(\Omega_1^2 + \Omega_1\Omega_2)},$$
$$P^2_{2,yy} = -\frac{f_{yy}\Omega_2 + (g_{xy} + 2f_{xx})\Omega_1}{3\Omega_1\Omega_2},$$
$$P^2_{2,yz} = -\frac{f_{yz}\Omega_2 + g_{xz}\Omega_1}{4\Omega_1\Omega_2},$$
$$P^2_{2,zz} = -\frac{f_{zz}}{\Omega_1},$$

$$P^3_{2,xx} = \frac{r_{xy}}{4\Omega_1},$$
$$P^3_{2,xy} = \frac{r_{yy} - r_{xx}}{\Omega_2 + \Omega_1},$$
$$P^3_{2,xz} = \frac{r_{yz}}{\Omega_1},$$
$$P^3_{2,yy} = -\frac{r_{xy}}{4\Omega_2},$$
$$P^3_{2,yz} = -\frac{r_{xz}}{\Omega_2},$$
$$P^3_{2,zz} = 0.$$

Next, let $P_3(z_1, z_2, z_3) = (P_3^1, P_3^2, P_3^3)'$, where $P_3^i(z_1, z_2, z_3)$ has the form as

$$\varphi = \varphi_{111}z_1^3 + (\varphi_{112}z_2 + \varphi_{113}z_3)z_1^2 + (\varphi_{122}z_1 + \varphi_{222}z_2)z_2^2 + \varphi_{223}z_2^2z_3 + \varphi_{123}z_1z_2z_3 + (\varphi_{133}z_1 + \varphi_{233}z_2 + \varphi_{333}z_3)z_3^2$$

for all $\varphi = P_3^i$, $i = 1, \ldots, 3$.

The coefficients are given as follows.

$$P^1_{3,111} = \frac{(-3\tilde{f}_{222} + 2\tilde{f}_{112})\Omega_2^2 + (\tilde{g}_{122} - 2\tilde{g}_{111} + \tilde{f}_{112})\Omega_1\Omega_2 + \tilde{g}_{111}\Omega_1^2}{4\Omega_1\Omega_2^2 + 4\Omega_1^2\Omega_2},$$

$$\begin{aligned} P^1_{3,112} = & -\{(-3\tilde{g}_{222} + 3\tilde{g}_{112} - \tilde{f}_{122} + 9\tilde{f}_{111})\Omega_2 \\ & \quad + (-9\tilde{g}_{222} + \tilde{g}_{112} - 3\tilde{f}_{122} + 3\tilde{f}_{111})\Omega_1\} \\ & /\{6\Omega_2^2 + 4\Omega_1\Omega_2 + 6\Omega_1^2\}, \end{aligned}$$

$$P^1_{3,113} = \frac{(2\tilde{g}_{223} + \tilde{f}_{123})\Omega_2 + \tilde{g}_{113}\Omega_1}{3\Omega_1\Omega_2},$$

$$P^1_{3,122} = 0,$$

$$P^1_{3,123} = \frac{2\tilde{f}_{223}\Omega_2 + (-\tilde{g}_{123} - 2\tilde{f}_{113})\Omega_1}{3\Omega_1\Omega_2},$$

$$P^1_{3,133} = 0,$$

$$P^1_{3,222} = -\frac{\tilde{f}_{122}\Omega_2^2 + (-2\tilde{g}_{222} + \tilde{g}_{112} + 3\tilde{f}_{111})\Omega_1\Omega_2 - 3\tilde{g}_{222}\Omega_1^2}{3\Omega_2^3 + 2\Omega_1\Omega_2^2 + 3\Omega_1^2\Omega_2},$$

$$P^1_{3,223} = \frac{(\tilde{g}_{223} - \tilde{f}_{123})\Omega_2 + 2\tilde{g}_{113}\Omega_1}{3\Omega_2^2},$$

$$P^1_{3,233} = -\frac{\tilde{f}_{133}\Omega_2 - \tilde{g}_{233}\Omega_1}{\Omega_2^2 + \Omega_1\Omega_2},$$

$$P^1_{3,333} = \frac{\tilde{g}_{333}}{\Omega_2},$$

$$\begin{aligned} P^2_{3,111} = & \{(-3\tilde{g}_{222} + 3\tilde{g}_{112} - \tilde{f}_{122} + 3\tilde{f}_{111})\Omega_2^2 \\ & \quad + (-3\tilde{g}_{222} + \tilde{g}_{112} - \tilde{f}_{122} - \tilde{f}_{111})\Omega_1\Omega_2 + 2\tilde{g}_{112}\Omega_1^2\} \\ & /\{6\Omega_1\Omega_2^2 + 4\Omega_1^2\Omega_2 + 6\Omega_1^3\}, \end{aligned}$$

$$P^2_{3,112} = \frac{(-3\tilde{f}_{222} + \tilde{f}_{112})\Omega_2 + (\tilde{g}_{122} - 3\tilde{g}_{111})\Omega_1}{2\Omega_1\Omega_2 + 2\Omega_1^2},$$

$$P^2_{3,113} = -\frac{2\tilde{f}_{223}\Omega_2 + (\tilde{f}_{113} - \tilde{g}_{123})\Omega_1}{3\Omega_1^2},$$

$$P^2_{3,122} = 0,$$

$$P^2_{3,123} = \frac{(2\tilde{g}_{223} + \tilde{f}_{123})\Omega_2 - 2\tilde{g}_{113}\Omega_1}{3\Omega_1\Omega_2},$$

$$P^2_{3,133} = 0,$$

$$P^2_{3,222} = \frac{-\tilde{f}_{222}\Omega_2^2 + (-\tilde{g}_{122} - 4\tilde{f}_{222} + \tilde{f}_{112})\Omega_1\Omega_2 - 3\tilde{g}_{111}\Omega_1^2}{4\Omega_1\Omega_2^2 + 4\Omega_1^2\Omega_2},$$

$$P^2_{3,223} = -\frac{\tilde{f}_{223}\Omega_2 + (\tilde{g}_{123} + 2\tilde{f}_{113})\Omega_1}{3\Omega_1\Omega_2},$$

$$P^2_{3,233} = \frac{-\tilde{f}_{233}\Omega_2 - \tilde{g}_{133}\Omega_1}{2\Omega_1\Omega_2},$$

$$P^2_{3,333} = -\frac{\tilde{f}_{333}}{\Omega_1},$$

$$P^3_{3,111} = \frac{2\Omega_2\tilde{r}_{222} + \Omega_1\tilde{r}_{112}}{3\Omega_1^2},$$

$$P^3_{3,112} = -\frac{\tilde{r}_{111}}{\Omega_2},$$

$$P^3_{3,113} = \frac{\tilde{r}_{123}}{2\Omega_1},$$

$$P^3_{3,122} = \frac{\tilde{r}_{222}}{\Omega_1}$$

$$P^3_{3,123} = \frac{\tilde{r}_{223} - \tilde{r}_{113}}{\Omega_2 + \Omega_1},$$

$$P^3_{3,133} = \frac{\tilde{r}_{233}}{\Omega_1},$$

$$P^3_{3,222} = -\frac{\Omega_2\tilde{r}_{122} + 2\Omega_1\tilde{r}_{111}}{3\Omega_2^2},$$

$$P^3_{3,223} = 0,$$

$$P^3_{3,233} = -\frac{\tilde{r}_{133}}{\Omega_2},$$

$$P^3_{3,333} = 0.$$

Appendix B

The polynomial functions P_2 and P_3 for deriving the normal form for the case in which A_{11} has exactly two distinct pairs of nonzero, pure imaginary eigenvalues are given below.

Let $P_2(x,y,z,w) = (P_2^1, P_2^2, P_2^3, P_2^4)'$, where $P_2^i(x,y,z,w)$ has the form as

$$\varphi = \varphi_{xx}x^2 + \varphi_{xy}xy + \varphi_{xz}xz + \varphi_{xw}xw + \varphi_{yy}y^2 + \varphi_{yz}yz + \varphi_{yw}yw + \varphi_{zz}z^2 + \varphi_{zw}zw + \varphi_{ww}w^2$$

for all $\varphi = P_2^i$, $i = 1,\ldots,4$. The coefficients are given as follows:

$$P^1_{2,xx} = \frac{(2g_{yy} + f_{xy})\Omega_2 + g_{xx}\Omega_1}{3\Omega_1\Omega_2},$$

$$P^1_{2,xy} = -\frac{(g_{xy} + 2f_{xx})\Omega_1 - 2f_{yy}\Omega_2}{3\Omega_1\Omega_2},$$

$$P^1_{2,xz} = \frac{f_{xw}\Omega_3\Omega_4 + \Omega_1((-2g_{yw} - 2f_{xw})\Omega_2 - g_{xz}\Omega_3) - f_{yz}\Omega_2\Omega_3}{\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3},$$

$$P^1_{2,xw} = -\frac{\Omega_1(g_{xw}\Omega_4 + (-2g_{yz} - 2f_{xz})\Omega_2) + f_{xz}\Omega_3\Omega_4 + f_{yw}\Omega_2\Omega_4}{\Omega_3\Omega_4^2 - 4\Omega_1\Omega_2\Omega_4},$$

$$P^1_{2,yy} = -\frac{(f_{xy} - g_{yy})\Omega_2 - 2g_{xx}\Omega_1}{3\Omega_2^2},$$

$$P^1_{2,yz} = \frac{\Omega_3(f_{yw}\Omega_4 + (f_{xz} - g_{yz})\Omega_1) - 2f_{yw}\Omega_1\Omega_2 + 2g_{xw}\Omega_1^2}{\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3},$$

$$P^1_{2,yw} = \frac{((f_{xw} - g_{yw})\Omega_1 - f_{yz}\Omega_3)\Omega_4 + 2f_{yz}\Omega_1\Omega_2 - 2g_{xz}\Omega_1^2}{\Omega_3\Omega_4^2 - 4\Omega_1\Omega_2\Omega_4},$$

$$P^1_{2,zz} = \frac{2g_{ww}\Omega_4^2 + (2g_{zz}\Omega_3 + f_{zw}\Omega_2)\Omega_4 - g_{zz}\Omega_1\Omega_2}{4\Omega_2\Omega_3\Omega_4 - \Omega_1\Omega_2^2},$$

$$P^1_{2,zw} = \frac{2f_{ww}\Omega_4 - 2f_{zz}\Omega_3 - g_{zw}\Omega_1}{4\Omega_3\Omega_4 - \Omega_1\Omega_2},$$

$$P^1_{2,ww} = \frac{2g_{ww}\Omega_3\Omega_4 + 2g_{zz}\Omega_3^2 - f_{zw}\Omega_2\Omega_3 - g_{ww}\Omega_1\Omega_2}{4\Omega_2\Omega_3\Omega_4 - \Omega_1\Omega_2^2},$$

$$P^2_{2,xx} = -\frac{2f_{yy}\Omega_2 + (f_{xx} - g_{xy})\Omega_1}{3\Omega_1^2},$$

$$P^2_{2,xy} = \frac{(2g_{yy} + f_{xy})\Omega_2 - 2g_{xx}\Omega_1}{3\Omega_1\Omega_2},$$

$$P^2_{2,xz} = \frac{\Omega_3(g_{xw}\Omega_4 + (f_{xz} - g_{yz})\Omega_2) + 2f_{yw}\Omega_2^2 - 2g_{xw}\Omega_1\Omega_2}{\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3},$$

$$P^2_{2,xw} = -\frac{(g_{xz}\Omega_3 + (g_{yw} - f_{xw})\Omega_2)\Omega_4 + 2f_{yz}\Omega_2^2 - 2g_{xz}\Omega_1\Omega_2}{\Omega_3\Omega_4^2 - 4\Omega_1\Omega_2\Omega_4},$$

$$P^2_{2,yy} = -\frac{f_{yy}\Omega_2 + (g_{xy} + 2f_{xx})\Omega_1}{3\Omega_1\Omega_2},$$

$$P^2_{2,yz} = \frac{g_{yw}\Omega_3\Omega_4 + \Omega_1(g_{xz}\Omega_3 + (-2g_{yw} - 2f_{xw})\Omega_2) + f_{yz}\Omega_2\Omega_3}{\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3},$$

$$P^2_{2,yw} = -\frac{\Omega_1((-2g_{yz} - 2f_{xz})\Omega_2 - g_{xw}\Omega_4) + g_{yz}\Omega_3\Omega_4 - f_{yw}\Omega_2\Omega_4}{\Omega_3\Omega_4^2 - 4\Omega_1\Omega_2\Omega_4},$$

$$P^2_{2,zz} = -\frac{2f_{ww}\Omega_4^2 + (2f_{zz}\Omega_3 - g_{zw}\Omega_1)\Omega_4 - f_{zz}\Omega_1\Omega_2}{4\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2},$$

$$P^2_{2,zw} = \frac{2g_{ww}\Omega_4 - 2g_{zz}\Omega_3 + f_{zw}\Omega_2}{4\Omega_3\Omega_4 - \Omega_1\Omega_2},$$

$$P^2_{2,ww} = -\frac{2f_{ww}\Omega_3\Omega_4 + 2f_{zz}\Omega_3^2 + \Omega_1(g_{zw}\Omega_3 - f_{ww}\Omega_2)}{4\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2},$$

$$P^3_{2,xx} = -\frac{2\Omega_2^2 s_{yy} - \Omega_3\Omega_4 s_{xx} + 2\Omega_1\Omega_2 s_{xx} + \Omega_2\Omega_4 r_{xy}}{\Omega_3\Omega_4^2 - 4\Omega_1\Omega_2\Omega_4},$$

$$P^3_{2,xy} = -\frac{-\Omega_3 s_{xy} + 2\Omega_2 r_{yy} - 2\Omega_1 r_{xx}}{\Omega_3\Omega_4 - 4\Omega_1\Omega_2},$$

$$P^3_{2,xz} = \frac{\Omega_4(2\Omega_3 s_{yw} + \Omega_1 r_{xw}) + \Omega_1\Omega_3 s_{xz} + (2\Omega_3\Omega_4 - \Omega_1\Omega_2) r_{yz}}{4\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2},$$

$$P^3_{2,xw} = \frac{-2\Omega_3^2 s_{yz} + \Omega_1\Omega_3(s_{xw} - r_{xz}) + (2\Omega_3\Omega_4 - \Omega_1\Omega_2) r_{yw}}{4\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2},$$

$$P^3_{2,yy} = \frac{\Omega_3\Omega_4 s_{yy} - 2\Omega_1\Omega_2 s_{yy} - 2\Omega_1^2 s_{xx} + \Omega_1\Omega_4 r_{xy}}{\Omega_3\Omega_4^2 - 4\Omega_1\Omega_2\Omega_4},$$

$$P^3_{2,yz} = \frac{\Omega_3(\Omega_2 s_{yz} - 2\Omega_4 s_{xw} - 2\Omega_4 r_{xz}) + \Omega_2\Omega_4 r_{yw} + \Omega_1\Omega_2 r_{xz}}{4\Omega_2\Omega_3\Omega_4 - \Omega_1\Omega_2^2},$$

$$P^3_{2,yw} = -\frac{-\Omega_2\Omega_3 s_{yw} - 2\Omega_3^2 s_{xz} + \Omega_2\Omega_3 r_{yz} + 2\Omega_3\Omega_4 r_{xw} - \Omega_1\Omega_2 r_{xw}}{4\Omega_2\Omega_3\Omega_4 - \Omega_1\Omega_2^2},$$

$$P^3_{2,zz} = \frac{2\Omega_4 s_{ww} + \Omega_3 s_{zz} + \Omega_4 r_{zw}}{3\Omega_3\Omega_4},$$

$$P^3_{2,zw} = -\frac{\Omega_3 s_{zw} - 2\Omega_4 r_{ww} + 2\Omega_3 r_{zz}}{3\Omega_3\Omega_4},$$

$$P^3_{2,ww} = -\frac{-\Omega_4 s_{ww} - 2\Omega_3 s_{zz} + \Omega_4 r_{zw}}{3\Omega_4^2},$$

$$P^4_{2,xx} = \frac{\Omega_3(-\Omega_2 s_{xy} - \Omega_4 r_{xx}) + 2\Omega_2^2 r_{yy} + 2\Omega_1\Omega_2 r_{xx}}{\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3},$$

$$P^4_{2,xy} = -\frac{2\Omega_2 s_{yy} - 2\Omega_1 s_{xx} + \Omega_4 r_{xy}}{\Omega_3\Omega_4 - 4\Omega_1\Omega_2},$$

$$P^4_{2,xz} = -\frac{\Omega_1(\Omega_2 s_{yz} - \Omega_4 s_{xw} + \Omega_4 r_{xz}) - 2\Omega_3\Omega_4 s_{yz} + 2\Omega_4^2 r_{yw}}{4\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2},$$
$$P^4_{2,xw} = \frac{\Omega_4(2\Omega_3 s_{yw} - \Omega_1 r_{xw}) - \Omega_1\Omega_2 s_{yw} - \Omega_1\Omega_3 s_{xz} + 2\Omega_3\Omega_4 r_{yz}}{4\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2},$$
$$P^4_{2,yy} = -\frac{-\Omega_1\Omega_3 s_{xy} + (\Omega_3\Omega_4 - 2\Omega_1\Omega_2) r_{yy} - 2\Omega_1^2 r_{xx}}{\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3},$$
$$P^4_{2,yz} = -\frac{\Omega_4(2\Omega_3 s_{xz} - \Omega_2 s_{yw}) - \Omega_1\Omega_2 s_{xz} + \Omega_2\Omega_4 r_{yz} - 2\Omega_4^2 r_{xw}}{4\Omega_2\Omega_3\Omega_4 - \Omega_1\Omega_2^2},$$
$$P^4_{2,yw} = -\frac{\Omega_3(\Omega_2 s_{yz} + 2\Omega_4 s_{xw} + 2\Omega_4 r_{xz}) - \Omega_1\Omega_2 s_{xw} + \Omega_2\Omega_4 r_{yw}}{4\Omega_2\Omega_3\Omega_4 - \Omega_1\Omega_2^2},$$
$$P^4_{2,zz} = -\frac{-\Omega_3 s_{zw} + 2\Omega_4 r_{ww} + \Omega_3 r_{zz}}{3\Omega_3^2},$$
$$P^4_{2,zw} = \frac{2\Omega_4 s_{ww} - 2\Omega_3 s_{zz} + \Omega_4 r_{zw}}{3\Omega_3\Omega_4},$$
$$P^4_{2,ww} = -\frac{\Omega_3 s_{zw} + \Omega_4 r_{ww} + 2\Omega_3 r_{zz}}{3\Omega_3\Omega_4}.$$

Let $\zeta := (z_1, z_2, z_3, z_4)'$ and $P_3(z_1, z_2, z_3, z_4) = (P_3^1, P_3^2, P_3^3, P_3^4)'$, where $P_3^i(z_1, z_2, z_3, z_4)$ has the form

$$\begin{aligned}\varphi = {} & \varphi_{111} z_1^3 + (\varphi_{112} z_2 + \varphi_{113} z_3 + \varphi_{114} z_4) z_1^2 \\ & + (\varphi_{122} z_1 + \varphi_{222} z_2 + \varphi_{223} z_3 + \varphi_{224} z_4) z_2^2 \\ & + \varphi_{123} z_1 z_2 z_3 + \varphi_{124} z_1 z_2 z_4 + \varphi_{134} z_3 z_1 z_4 + \varphi_{234} z_2 z_3 z_4 \\ & + (\varphi_{133} z_1 + \varphi_{233} z_2 + \varphi_{333} z_3 + \varphi_{334} z_4) z_3^2 \\ & + (\varphi_{144} z_1 + \varphi_{244} z_2 + \varphi_{344} z_3 + \varphi_{444} z_4) z_4^2\end{aligned}$$

for all $\varphi = P_3^i$, $i = 1, \ldots, 4$. The coefficients of P_3^i are given as follows.

$$P^1_{3,111} = \frac{\tilde{f}_{222}\Omega_2^2 + (\tilde{g}_{122} + \tilde{f}_{112})\Omega_1\Omega_2 + \tilde{g}_{111}\Omega_1^2}{4\Omega_1^2\Omega_2},$$
$$\begin{aligned}P^1_{3,112} = {} & \{2\tilde{f}_{122}\Omega_2^2 + (-\tilde{g}_{222} - \tilde{g}_{112} + \tilde{f}_{122} - 3\tilde{f}_{111})\Omega_1\Omega_2 \\ & + (3\tilde{g}_{222} - \tilde{g}_{112} + 3\tilde{f}_{122} - 3\tilde{f}_{111})\Omega_1^2\}/\{6\Omega_1\Omega_2^2 + 4\Omega_1^2\Omega_2 + 6\Omega_1^3\},\end{aligned}$$
$$\begin{aligned}P^1_{3,113} = {} & \{\tilde{f}_{114}\Omega_3\Omega_4^2 + (\Omega_2((-2\tilde{g}_{124} - 7\tilde{f}_{114})\Omega_1 - \tilde{f}_{123}\Omega_3) - \tilde{g}_{113}\Omega_1\Omega_3, \\ & - 2\tilde{f}_{224}\Omega_2^2)\Omega_4 + (6\tilde{g}_{223} + 3\tilde{f}_{123})\Omega_1\Omega_2^2 + 3\tilde{g}_{113}\Omega_1^2\Omega_2\} \\ & /\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},\end{aligned}$$

$$P^1_{3,114} = -\{(\tilde{f}_{113}\Omega_3^2 + \tilde{f}_{124}\Omega_2\Omega_3 + \tilde{g}_{114}\Omega_1\Omega_3)\Omega_4 + \Omega_2(\Omega_1(-2\tilde{g}_{123}\Omega_3 - 7\tilde{f}_{113}\Omega_3) - 3\tilde{g}_{114}\Omega_1^2) + \Omega_2^2((-6\tilde{g}_{224} - 3\tilde{f}_{124})\Omega_1 - 2\tilde{f}_{223}\Omega_3))\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},$$

$$P^1_{3,122} = \frac{\tilde{f}_{222}\Omega_2^2 + (\tilde{g}_{122} + 4\tilde{f}_{222} - \tilde{f}_{112})\Omega_1\Omega_2 + 3\tilde{g}_{111}\Omega_1^2}{4\Omega_1\Omega_2^2 + 4\Omega_1^2\Omega_2},$$

$$P^1_{3,123} = \{\tilde{f}_{124}\Omega_3\Omega_4^2 + (\Omega_1(2\tilde{f}_{113}\Omega_3 - \tilde{g}_{123}\Omega_3) + \Omega_2((-4\tilde{g}_{224} - 5\tilde{f}_{124})\Omega_1 - 2\tilde{f}_{223}\Omega_3) + 4\tilde{g}_{114}\Omega_1^2)\Omega_4 + 6\tilde{f}_{223}\Omega_1\Omega_2^2 + (-3\tilde{g}_{123} - 6\tilde{f}_{113})\Omega_1^2\Omega_2\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},$$

$$P^1_{3,124} = -\{(\tilde{f}_{123}\Omega_3^2 + 2\tilde{f}_{224}\Omega_2\Omega_3 + (\tilde{g}_{124} - 2\tilde{f}_{114})\Omega_1\Omega_3)\Omega_4 + \Omega_2(\Omega_1(-4\tilde{g}_{223}\Omega_3 - 5\tilde{f}_{123}\Omega_3) + (3\tilde{g}_{124} + 6\tilde{f}_{114})\Omega_1^2) + 4\tilde{g}_{113}\Omega_1^2\Omega_3 - 6\tilde{f}_{224}\Omega_1\Omega_2^2\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},$$

$$P^1_{3,133} = \frac{(\tilde{f}_{244}\Omega_2 + \tilde{g}_{144}\Omega_1)\Omega_4 + (\tilde{f}_{233}\Omega_2 + \tilde{g}_{133}\Omega_1)\Omega_3 + (\tilde{g}_{234} + \tilde{f}_{134})\Omega_1\Omega_2}{4\Omega_1\Omega_2\Omega_3},$$

$$P^1_{3,134} = \{(2\tilde{f}_{144}\Omega_2 - 2\tilde{g}_{244}\Omega_1)\Omega_4^2 + (((2\tilde{f}_{144} - 2\tilde{f}_{133})\Omega_2 + (2\tilde{g}_{244} - 2\tilde{g}_{233} + 4\tilde{f}_{144} - 4\tilde{f}_{133})\Omega_1)\Omega_3 - \tilde{f}_{234}\Omega_2^2 + (-\tilde{g}_{134} - \tilde{f}_{234})\Omega_1\Omega_2 - \tilde{g}_{134}\Omega_1^2)\Omega_4 + (2\tilde{g}_{233}\Omega_1 - 2\tilde{f}_{133}\Omega_2)\Omega_3^2 + (-\tilde{f}_{234}\Omega_2^2 + (-\tilde{g}_{134} - \tilde{f}_{234})\Omega_1\Omega_2 - \tilde{g}_{134}\Omega_1^2)\Omega_3 + (-2\tilde{g}_{244} + 2\tilde{g}_{233} - 2\tilde{f}_{144} + 2\tilde{f}_{133})\Omega_1\Omega_2^2 + (-2\tilde{g}_{244} + 2\tilde{g}_{233} - 2\tilde{f}_{144} + 2\tilde{f}_{133})\Omega_1^2\Omega_2\}/\{(4\Omega_2 + 4\Omega_1)\Omega_3\Omega_4^2 + ((4\Omega_2 + 4\Omega_1)\Omega_3^2 - 4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_4 + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3\},$$

$$P^1_{3,144} = \{(\tilde{f}_{244}\Omega_2 + \tilde{g}_{144}\Omega_1)\Omega_3\Omega_4 + (\tilde{f}_{233}\Omega_2 + \tilde{g}_{133}\Omega_1)\Omega_3^2 + (\tilde{g}_{234} - \tilde{f}_{134})\Omega_1\Omega_2\Omega_3 - 2\tilde{f}_{244}\Omega_1\Omega_2^2 - 2\tilde{g}_{144}\Omega_1^2\Omega_2\}/\{4\Omega_1\Omega_2\Omega_3\Omega_4 - 4\Omega_1^2\Omega_2^2\},$$

$$P^1_{3,222} = 0,$$

$$P^1_{3,223} = \{\tilde{f}_{224}\Omega_3\Omega_4^2 + (\Omega_1(\tilde{f}_{123}\Omega_3 - \tilde{g}_{223}\Omega_3) - 7\tilde{f}_{224}\Omega_1\Omega_2 + (2\tilde{g}_{124} - 2\tilde{f}_{114})\Omega_1^2)\Omega_4 + (3\tilde{g}_{223} - 3\tilde{f}_{123})\Omega_1^2\Omega_2 + 6\tilde{g}_{113}\Omega_1^3\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},$$

$$P^1_{3,224} = -\{(\tilde{f}_{223}\Omega_3^2 + (\tilde{g}_{224} - \tilde{f}_{124})\Omega_1\Omega_3)\Omega_4 + \Omega_2((3\tilde{f}_{124} - 3\tilde{g}_{224})\Omega_1^2 - 7\tilde{f}_{223}\Omega_1\Omega_3) + \Omega_1^2(2\tilde{g}_{123}\Omega_3 - 2\tilde{f}_{113}\Omega_3) - 6\tilde{g}_{114}\Omega_1^3\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$

$$P^1_{3,233} = \{(2\tilde{f}_{234}(\Omega_2 + \Omega_1)\Omega_3 - 2\tilde{f}_{144}\Omega_1\Omega_2 + 2\tilde{g}_{244}\Omega_1^2)\Omega_4^2 + (2\tilde{f}_{234}(\Omega_2 + \Omega_1)\Omega_3^2 + ((2\tilde{g}_{244} - 2\tilde{g}_{233})\Omega_1\Omega_2 + (2\tilde{f}_{133} - 2\tilde{f}_{144})\Omega_1^2)\Omega_3 - \tilde{f}_{234}\Omega_1\Omega_2^2 + (\tilde{g}_{134} - \tilde{f}_{234})\Omega_1^2\Omega_2 + \tilde{g}_{134}\Omega_1^3)\Omega_4 + (2\tilde{f}_{133}\Omega_1\Omega_2 - 2\tilde{g}_{233}\Omega_1^2)\Omega_3^2 + (-\tilde{f}_{234}\Omega_1\Omega_2^2 + (\tilde{g}_{134} - \tilde{f}_{234})\Omega_1^2\Omega_2 + \tilde{g}_{134}\Omega_1^3)\Omega_3\}/\{(4\Omega_2 + 4\Omega_1)\Omega_3^2\Omega_4^2 + ((4\Omega_2 + 4\Omega_1)\Omega_3^3 + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3)\Omega_4 + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3^2\},$$

$$P^1_{3,234} = \{(\tilde{f}_{244}\Omega_2 + \tilde{g}_{144}\Omega_1)\Omega_4^2 + (((3\tilde{f}_{244} - 3\tilde{f}_{233})\Omega_2 - \tilde{g}_{144}\Omega_1 + \tilde{g}_{133}\Omega_1)\Omega_3 + (\tilde{f}_{134} - \tilde{g}_{234})\Omega_1\Omega_2)\Omega_4 + (-\tilde{f}_{233}\Omega_2 - \tilde{g}_{133}\Omega_1)\Omega_3^2 + (\tilde{f}_{134} - \tilde{g}_{234})\Omega_1\Omega_2\Omega_3 + (2\tilde{f}_{233} - 2\tilde{f}_{244})\Omega_1\Omega_2^2 + (2\tilde{g}_{144}\Omega_1^2 - 2\tilde{g}_{133}\Omega_1^2)\Omega_2\} /\{4\Omega_2\Omega_3\Omega_4^2 + (4\Omega_2\Omega_3^2 - 4\Omega_1\Omega_2^2)\Omega_4 - 4\Omega_1\Omega_2^2\Omega_3\},$$

$$P^1_{3,244} = 0,$$

$$P^1_{3,333} = \{6\tilde{f}_{444}\Omega_4^3 + (3\tilde{f}_{334}\Omega_3 - 2\tilde{g}_{344}\Omega_1)\Omega_4^2 + (-7\tilde{g}_{333}\Omega_1\Omega_3 - \tilde{f}_{334}\Omega_1\Omega_2)\Omega_4 + \tilde{g}_{333}\Omega_1^2\Omega_2\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^1_{3,334} = -\{6\tilde{g}_{444}\Omega_1\Omega_4^2 + (9\tilde{f}_{333}\Omega_3^2 + \Omega_1(3\tilde{g}_{334}\Omega_3 + 2\tilde{f}_{344}\Omega_2))\Omega_4 - 3\tilde{f}_{333}\Omega_1\Omega_2\Omega_3 - \tilde{g}_{334}\Omega_1^2\Omega_2\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\},$$

$$P^1_{3,344} = \{9\tilde{f}_{444}\Omega_3\Omega_4^2 + (-3\tilde{g}_{344}\Omega_1\Omega_3 - 3\tilde{f}_{444}\Omega_1\Omega_2)\Omega_4 - 6\tilde{g}_{333}\Omega_1\Omega_3^2 + \Omega_2(2\tilde{f}_{334}\Omega_1\Omega_3 + \tilde{g}_{344}\Omega_1^2)\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^1_{3,444} = -\{(3\tilde{f}_{344}\Omega_3^2 + 7\tilde{g}_{444}\Omega_1\Omega_3)\Omega_4 + 6\tilde{f}_{333}\Omega_3^3 + \Omega_1(2\tilde{g}_{334}\Omega_3^2 - \tilde{f}_{344}\Omega_2\Omega_3) - \tilde{g}_{444}\Omega_1^2\Omega_2\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^2_{3,111} = -\{2\tilde{f}_{122}\Omega_2^3 + ((-\tilde{g}_{222} - \tilde{g}_{112} + \tilde{f}_{122} + 3\tilde{f}_{111})\Omega_1)\Omega_2^2 + (-3\tilde{g}_{222} - \tilde{g}_{112} + \tilde{f}_{122} + \tilde{f}_{111})\Omega_1^2\Omega_2 - 2\tilde{g}_{112}\Omega_1^3\} /\{6\Omega_1^2\Omega_2^2 + 4\Omega_1^3\Omega_2 + 6\Omega_1^4\},$$

$$P^2_{3,112} = \{\tilde{f}_{222}\Omega_2^2 + (\tilde{g}_{122} - 2\tilde{f}_{222} + \tilde{f}_{112})\Omega_1\Omega_2 + (2\tilde{g}_{122} - 3\tilde{g}_{111})\Omega_1^2\}/\{4\Omega_1^2\Omega_2 + 4\Omega_1^3\},$$

$$
\begin{aligned}
P^2_{3,113} = {} & \{\tilde{g}_{114}\Omega_3\Omega_4^2 + (\Omega_2(-\tilde{g}_{123}\Omega_3 + \tilde{f}_{113}\Omega_3 - 7\tilde{g}_{114}\Omega_1) \\
& + (2\tilde{f}_{124} - 2\tilde{g}_{224})\Omega_2^2)\Omega_4 - 6\tilde{f}_{223}\Omega_2^3 + (3\tilde{g}_{123} - 3\tilde{f}_{113})\Omega_1\Omega_2^2\} \\
& /\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},
\end{aligned}
$$

$$
\begin{aligned}
P^2_{3,114} = {} & \{((\tilde{f}_{114} - \tilde{g}_{124})\Omega_2\Omega_3 - \tilde{g}_{113}\Omega_3^2)\Omega_4 + \Omega_2^2(2\tilde{g}_{223}\Omega_3 - 2\tilde{f}_{123}\Omega_3 \\
& + (3\tilde{g}_{124} - 3\tilde{f}_{114})\Omega_1) + 7\tilde{g}_{113}\Omega_1\Omega_2\Omega_3 - 6\tilde{f}_{224}\Omega_2^3\} \\
& /\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},
\end{aligned}
$$

$$
P^2_{3,122} = -\frac{\tilde{f}_{122}\Omega_2^2 + (-2\tilde{g}_{222} + \tilde{g}_{112} + 3\tilde{f}_{111})\Omega_1\Omega_2 - 3\tilde{g}_{222}\Omega_1^2}{3\Omega_1\Omega_2^2 + 2\Omega_1^2\Omega_2 + 3\Omega_1^3},
$$

$$
\begin{aligned}
P^2_{3,123} = {} & \{\tilde{g}_{124}\Omega_3\Omega_4^2 + (\Omega_2(-2\tilde{g}_{223}\Omega_3 + \tilde{f}_{123}\Omega_3 \\
& + (-5\tilde{g}_{124} - 4\tilde{f}_{114})\Omega_1) + 2\tilde{g}_{113}\Omega_1\Omega_3 + 4\tilde{f}_{224}\Omega_2^2)\Omega_4 \\
& + (6\tilde{g}_{223} + 3\tilde{f}_{123})\Omega_1\Omega_2^2 - 6\tilde{g}_{113}\Omega_1^2\Omega_2\} \\
& /\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},
\end{aligned}
$$

$$
\begin{aligned}
P^2_{3,124} = {} & -\{(\tilde{g}_{123}\Omega_3^2 + (2\tilde{g}_{224} - \tilde{f}_{124})\Omega_2\Omega_3 - 2\tilde{g}_{114}\Omega_1\Omega_3)\Omega_4 \\
& + \Omega_2(\Omega_1(-5\tilde{g}_{123}\Omega_3 - 4\tilde{f}_{113}\Omega_3) + 6\tilde{g}_{114}\Omega_1^2) \\
& + \Omega_2^2(4\tilde{f}_{223}\Omega_3 + (-6\tilde{g}_{224} - 3\tilde{f}_{124})\Omega_1)\} \\
& /\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},
\end{aligned}
$$

$$
\begin{aligned}
P^2_{3,133} = {} & -\{(2\tilde{f}_{144}\Omega_2 - 2\tilde{g}_{244}\Omega_1)\Omega_4 + (2\tilde{f}_{133}\Omega_2 - 2\tilde{g}_{233}\Omega_1)\Omega_3 \\
& + \tilde{f}_{234}\Omega_2^2 + (\tilde{f}_{234} - \tilde{g}_{134})\Omega_1\Omega_2 - \tilde{g}_{134}\Omega_1^2\} \\
& /\{(4\Omega_1\Omega_2 + 4\Omega_1^2)\Omega_3\},
\end{aligned}
$$

$$
\begin{aligned}
P^2_{3,134} = {} & \{(\tilde{f}_{244}\Omega_2 + \tilde{g}_{144}\Omega_1)\Omega_4^2 + (((\tilde{f}_{233} - \tilde{f}_{244})\Omega_2 \\
& + 3\tilde{g}_{144}\Omega_1 - 3\tilde{g}_{133}\Omega_1)\Omega_3 + (\tilde{f}_{134} - \tilde{g}_{234})\Omega_1\Omega_2)\Omega_4 \\
& + (-\tilde{f}_{233}\Omega_2 - \tilde{g}_{133}\Omega_1)\Omega_3^2 + (\tilde{f}_{134} - \tilde{g}_{234})\Omega_1\Omega_2\Omega_3 \\
& + (2\tilde{f}_{244} - 2\tilde{f}_{233})\Omega_1\Omega_2^2 + (2\tilde{g}_{133}\Omega_1^2 - 2\tilde{g}_{144}\Omega_1^2)\Omega_2\} \\
& /\{4\Omega_1\Omega_3\Omega_4^2 + (4\Omega_1\Omega_3^2 - 4\Omega_1^2\Omega_2)\Omega_4 - 4\Omega_1^2\Omega_2\Omega_3\},
\end{aligned}
$$

$$
\begin{aligned}
P^2_{3,144} = {} & -\{(2\tilde{f}_{144}\Omega_2 - 2\tilde{g}_{244}\Omega_1)\Omega_3\Omega_4^2 + (((2\tilde{f}_{144} + 2\tilde{f}_{133})\Omega_2 \\
& + (-2\tilde{g}_{244} - 2\tilde{g}_{233})\Omega_1)\Omega_3^2 + (\tilde{f}_{234}\Omega_2^2 + (\tilde{g}_{134} \\
& + \tilde{f}_{234})\Omega_1\Omega_2 + \tilde{g}_{134}\Omega_1^2)\Omega_3 - 4\tilde{f}_{144}\Omega_1\Omega_2^2 + 4\tilde{g}_{244}\Omega_1^2\Omega_2)\Omega_4
\end{aligned}
$$

$$+ (2\tilde{f}_{133}\Omega_2 - 2\tilde{g}_{233}\Omega_1)\Omega_3^3 + (\tilde{f}_{234}\Omega_2^2 + (\tilde{g}_{134} + \tilde{f}_{234})\Omega_1\Omega_2 + \tilde{g}_{134}\Omega_1^2)\Omega_3^2 + ((2\tilde{g}_{244} - 2\tilde{g}_{233} - 2\tilde{f}_{144} - 2\tilde{f}_{133})\Omega_1\Omega_2^2 + (2\tilde{g}_{244} + 2\tilde{g}_{233} - 2\tilde{f}_{144} + 2\tilde{f}_{133})\Omega_1^2\Omega_2)\Omega_3\}/\{(4\Omega_1\Omega_2 + 4\Omega_1^2)\Omega_3\Omega_4^2 + ((4\Omega_1\Omega_2 + 4\Omega_1^2)\Omega_3^2 - 4\Omega_1^2\Omega_2^2 - 4\Omega_1^3\Omega_2)\Omega_4 + (-4\Omega_1^2\Omega_2^2 - 4\Omega_1^3\Omega_2)\Omega_3\},$$

$$P^2_{3,222} = 0,$$

$$P^2_{3,223} = \{\tilde{g}_{224}\Omega_3\Omega_4^2 + (\Omega_2(\tilde{f}_{223}\Omega_3 + (-7\tilde{g}_{224} - 2\tilde{f}_{124})\Omega_1) + \tilde{g}_{123}\Omega_1\Omega_3 - 2\tilde{g}_{114}\Omega_1^2)\Omega_4 - 3\tilde{f}_{223}\Omega_1\Omega_2^2 + (-3\tilde{g}_{123} - 6\tilde{f}_{113})\Omega_1^2\Omega_2\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$

$$P^2_{3,224} = \{(-\tilde{g}_{223}\Omega_3^2 + \tilde{f}_{224}\Omega_2\Omega_3 + \tilde{g}_{124}\Omega_1\Omega_3)\Omega_4 + \Omega_2(\Omega_1(7\tilde{g}_{223}\Omega_3 + 2\tilde{f}_{123}\Omega_3) + (-3\tilde{g}_{124} - 6\tilde{f}_{114})\Omega_1^2) + 2\tilde{g}_{113}\Omega_1^2\Omega_3 - 3\tilde{f}_{224}\Omega_1\Omega_2^2\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$

$$P^2_{3,233} = \{(2\tilde{g}_{234}\Omega_3 - \tilde{f}_{244}\Omega_2 - \tilde{g}_{144}\Omega_1)\Omega_4 + (\tilde{f}_{233}\Omega_2 + \tilde{g}_{133}\Omega_1)\Omega_3 + (-\tilde{g}_{234} - \tilde{f}_{134})\Omega_1\Omega_2\}/\{4\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3\},$$

$$P^2_{3,234} = -\{(2\tilde{f}_{144}\Omega_2 - 2\tilde{g}_{244}\Omega_1)\Omega_4^2 + (((-4\tilde{g}_{244} + 4\tilde{g}_{233} - 2\tilde{f}_{144} + 2\tilde{f}_{133})\Omega_2 + (2\tilde{g}_{233} - 2\tilde{g}_{244})\Omega_1)\Omega_3 - \tilde{f}_{234}\Omega_2^2 + (-\tilde{g}_{134} - \tilde{f}_{234})\Omega_1\Omega_2 - \tilde{g}_{134}\Omega_1^2)\Omega_4 + (2\tilde{g}_{233}\Omega_1 - 2\tilde{f}_{133}\Omega_2)\Omega_3^2 + (-\tilde{f}_{234}\Omega_2^2 + (-\tilde{g}_{134} - \tilde{f}_{234})\Omega_1\Omega_2 - \tilde{g}_{134}\Omega_1^2)\Omega_3 + (2\tilde{g}_{244} - 2\tilde{g}_{233} + 2\tilde{f}_{144} - 2\tilde{f}_{133})\Omega_1\Omega_2^2 + (2\tilde{g}_{244} - 2\tilde{g}_{233} + 2\tilde{f}_{144} - 2\tilde{f}_{133})\Omega_1^2\Omega_2\}/\{(4\Omega_2 + 4\Omega_1)\Omega_3\Omega_4^2 + ((4\Omega_2 + 4\Omega_1)\Omega_3^2 - 4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_4 + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3\},$$

$$P^2_{3,244} = 0,$$

$$P^2_{3,333} = \{6\tilde{g}_{444}\Omega_4^3 + (3\tilde{g}_{334}\Omega_3 + 2\tilde{f}_{344}\Omega_2)\Omega_4^2 + (7\tilde{f}_{333}\Omega_2\Omega_3 - \tilde{g}_{334}\Omega_1\Omega_2)\Omega_4 - \tilde{f}_{333}\Omega_1\Omega_2^2\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\},$$

$$P^2_{3,334} = \{6\tilde{f}_{444}\Omega_2\Omega_4^2 + (\Omega_2(3\tilde{f}_{334}\Omega_3 - 2\tilde{g}_{344}\Omega_1) - 9\tilde{g}_{333}\Omega_3^2)\Omega_4 + 3\tilde{g}_{333}\Omega_1\Omega_2\Omega_3 - \tilde{f}_{334}\Omega_1\Omega_2^2\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\},$$

$$P^2_{3,344} = \{9\tilde{g}_{444}\Omega_3\Omega_4^2 + (3\tilde{f}_{344}\Omega_2\Omega_3 - 3\tilde{g}_{444}\Omega_1\Omega_2)\Omega_4 + 6\tilde{f}_{333}\Omega_2\Omega_3^2 + \Omega_1(2\tilde{g}_{334}\Omega_2\Omega_3 - \tilde{f}_{344}\Omega_2^2)\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\},$$

$$P^2_{3,444} = \{(7\tilde{f}_{444}\Omega_2\Omega_3 - 3\tilde{g}_{344}\Omega_3^2)\Omega_4 - 6\tilde{g}_{333}\Omega_3^3 + \Omega_2(2\tilde{f}_{334}\Omega_3^2 + \tilde{g}_{344}\Omega_1\Omega_3) - \tilde{f}_{444}\Omega_1\Omega_2^2\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}.$$

$$P^3_{3,111} = \{\Omega_2^2(3\Omega_1\tilde{r}_{112} - 2\Omega_3\tilde{s}_{122}) + (\Omega_3^2\Omega_4 - 7\Omega_1\Omega_2\Omega_3)\tilde{s}_{111} + 6\Omega_2^3\tilde{r}_{222} - \Omega_2\Omega_3\Omega_4\tilde{r}_{112}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},$$

$$P^3_{3,112} = \{\Omega_3(\Omega_2(-3\Omega_1\tilde{s}_{112} - 2\Omega_4\tilde{r}_{122}) - 6\Omega_2^2\tilde{s}_{222}) + \Omega_3^2\Omega_4\tilde{s}_{112} + (3\Omega_1\Omega_3\Omega_4 - 9\Omega_1^2\Omega_2)\tilde{r}_{111}\} /\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},$$

$$P^3_{3,113} = -\{\Omega_3(\Omega_2\tilde{s}_{223} + \Omega_4(-\tilde{s}_{124} - \tilde{r}_{123}) - \Omega_1\tilde{s}_{113}) + \Omega_2\Omega_4\tilde{r}_{224} + 2\Omega_1\Omega_2\tilde{r}_{123} - \Omega_1\Omega_4\tilde{r}_{114}\}/\{4\Omega_1\Omega_3\Omega_4 - 4\Omega_1^2\Omega_2\},$$

$$P^3_{3,114} = -\{\Omega_3(-2\Omega_2\tilde{s}_{224} - 2\Omega_1\tilde{s}_{114}) + \Omega_3\Omega_4(\tilde{s}_{123} - \tilde{r}_{124}) + \Omega_3^2\tilde{s}_{123} + 2\Omega_2\Omega_4\tilde{r}_{223} - \Omega_4^2\tilde{r}_{124} + 2\Omega_1\Omega_4\tilde{r}_{113}\} /\{4\Omega_1\Omega_4^2 + 4\Omega_1\Omega_3\Omega_4\},$$

$$P^3_{3,122} = -\{\Omega_4(-\Omega_3^2\tilde{s}_{122} + 3\Omega_2\Omega_3\tilde{r}_{222} - 2\Omega_1\Omega_3\tilde{r}_{112}) + 3\Omega_1\Omega_2\Omega_3\tilde{s}_{122} + 6\Omega_1^2\Omega_3\tilde{s}_{111} - 9\Omega_1\Omega_2^2\tilde{r}_{222}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},$$

$$P^3_{3,123} = \{\Omega_4(\Omega_3^2(2\tilde{s}_{224} - 2\tilde{s}_{114}) + \Omega_3(\Omega_2(\tilde{s}_{123} + \tilde{r}_{124}) + \Omega_1(\tilde{s}_{123} + \tilde{r}_{124}))) + \Omega_4^2(\Omega_3(2\tilde{s}_{224} - 2\tilde{s}_{114}) + \Omega_2\tilde{r}_{124} + \Omega_1\tilde{r}_{124}) + \Omega_3(2\Omega_2^2\tilde{s}_{224} + \Omega_1\Omega_2(2\tilde{s}_{114} - 2\tilde{s}_{224}) - 2\Omega_1^2\tilde{s}_{114}) + \Omega_3^2(\Omega_2\tilde{s}_{123} + \Omega_1\tilde{s}_{123}) + (2\Omega_3\Omega_4^2 + (2\Omega_3^2 - 2\Omega_2^2 - 2\Omega_1\Omega_2)\Omega_4 - 4\Omega_1\Omega_2\Omega_3)\tilde{r}_{223} + (-2\Omega_3\Omega_4^2 + (-2\Omega_3^2 + 2\Omega_1\Omega_2 + 2\Omega_1^2)\Omega_4 + 4\Omega_1\Omega_2\Omega_3)\tilde{r}_{113}\} /\{(4\Omega_2 + 4\Omega_1)\Omega_3\Omega_4^2 + ((4\Omega_2 + 4\Omega_1)\Omega_3^2 - 4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_4 + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3\},$$

$$P^3_{3,124} = \{\Omega_3(\Omega_2(\Omega_1(\tilde{s}_{223} - \tilde{s}_{113}) + \Omega_4(\tilde{s}_{124} - \tilde{r}_{123})) - \Omega_2^2\tilde{s}_{223} + \Omega_1\Omega_4(\tilde{s}_{124} - \tilde{r}_{123}) + \Omega_1^2\tilde{s}_{113}) + \Omega_3^2(2\Omega_4\tilde{s}_{113} - 2\Omega_4\tilde{s}_{223}) + (2\Omega_3\Omega_4^2 - \Omega_2^2\Omega_4 - 3\Omega_1\Omega_2\Omega_4)\tilde{r}_{224} + (2\Omega_3\Omega_4^2 + 3\Omega_1\Omega_2\Omega_4 + \Omega_1^2\Omega_4)\tilde{r}_{114}\}/\{\Omega_3(4\Omega_2\Omega_4^2 + 4\Omega_1\Omega_4^2) - 4\Omega_1\Omega_2^2\Omega_4 - 4\Omega_1^2\Omega_2\Omega_4\},$$

$$
\begin{aligned}
P^3_{3,133} = {} & \{\Omega_2\Omega_4(\Omega_3(-2\tilde{s}_{234} - 7\tilde{r}_{233}) - \Omega_1\tilde{r}_{134}) + \Omega_4^2(6\Omega_3\tilde{s}_{144} \\
& - 2\Omega_2\tilde{r}_{244} + 3\Omega_3\tilde{r}_{134}) + (3\Omega_3^2\Omega_4 - \Omega_1\Omega_2\Omega_3)\tilde{s}_{133} + \Omega_1\Omega_2^2\tilde{r}_{233}\} \\
& /\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}, \\
P^3_{3,134} = {} & -\{\Omega_4(\Omega_2\Omega_3(4\tilde{s}_{244} + 5\tilde{r}_{234}) + 3\Omega_3^2\tilde{s}_{134} + 2\Omega_1\Omega_2\tilde{r}_{144}) \\
& - 4\Omega_2\Omega_3^2\tilde{s}_{233} + \Omega_1\Omega_2\Omega_3\tilde{s}_{134} - \Omega_1\Omega_2^2\tilde{r}_{234} - 6\Omega_3\Omega_4^2\tilde{r}_{144} \\
& + (6\Omega_3^2\Omega_4 - 2\Omega_1\Omega_2\Omega_3)\tilde{r}_{133}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}, \\
P^3_{3,144} = {} & \{\Omega_2(\Omega_3^2(2\tilde{s}_{234} - 2\tilde{r}_{233}) - \Omega_1\Omega_3\tilde{s}_{144} + \Omega_1\Omega_3\tilde{r}_{134}) \\
& + \Omega_4(3\Omega_3^2\tilde{s}_{144} - 7\Omega_2\Omega_3\tilde{r}_{244} - 3\Omega_3^2\tilde{r}_{134}) + 6\Omega_3^3\tilde{s}_{133} \\
& + \Omega_1\Omega_2^2\tilde{r}_{244}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}, \\
P^3_{3,222} = {} & -\{\Omega_3(7\Omega_1\Omega_2\tilde{s}_{222} + 2\Omega_1^2\tilde{s}_{112} - \Omega_1\Omega_4\tilde{r}_{122}) - \Omega_3^2\Omega_4\tilde{s}_{222} \\
& + 3\Omega_1^2\Omega_2\tilde{r}_{122} + 6\Omega_1^3\tilde{r}_{111}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}, \\
P^3_{3,223} = {} & 0, \\
P^3_{3,224} = {} & -\{\Omega_4(\Omega_3^2(\Omega_1(-2\tilde{s}_{224} - 2\tilde{s}_{114}) - 4\Omega_2\tilde{s}_{224}) - \Omega_3\Omega_1(\Omega_1 + \Omega_2)(\tilde{s}_{123} \\
& + \tilde{r}_{124})) + \Omega_3(\Omega_1^2\Omega_2(2\tilde{s}_{224} + 2\tilde{s}_{114}) + 2\Omega_1\Omega_2^2\tilde{s}_{224} \\
& + 2\Omega_1^3\tilde{s}_{114}) + \Omega_4^2(\Omega_1\Omega_3(2\tilde{s}_{114} - 2\tilde{s}_{224}) - \Omega_1\Omega_2\tilde{r}_{124}, \\
& - \Omega_1^2\tilde{r}_{124}) - \Omega_3^2(\Omega_2 + \Omega_1)\Omega_1\tilde{s}_{123} + ((4\Omega_2 + 2\Omega_1)\Omega_3\Omega_4^2 \\
& + (2\Omega_1\Omega_3^2 - 2\Omega_1\Omega_2^2 - 2\Omega_1^2\Omega_2)\Omega_4)\tilde{r}_{223} + (2\Omega_1\Omega_3\Omega_4^2 \\
& + (-2\Omega_1\Omega_3^2 - 2\Omega_1^2\Omega_2 - 2\Omega_1^3)\Omega_4)\tilde{r}_{113}\}/\{(4\Omega_2 + 4\Omega_1)\Omega_3\Omega_4^3 \\
& + ((4\Omega_2 + 4\Omega_1)\Omega_3^2 - 4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_4^2 + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3\Omega_4\}, \\
P^3_{3,233} = {} & \{\Omega_4^2(\Omega_3(6\tilde{s}_{244} + 3\tilde{r}_{234}) + 2\Omega_1\tilde{r}_{144}) + \Omega_4(3\Omega_3^2\tilde{s}_{233} \\
& + 2\Omega_1\Omega_3\tilde{s}_{134} - \Omega_1\Omega_2\tilde{r}_{234}) - \Omega_1\Omega_2\Omega_3\tilde{s}_{233} + (7\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2)\tilde{r}_{133}\} \\
& /\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}, \\
P^3_{3,234} = {} & -\{\Omega_2(\Omega_3(\Omega_1\tilde{s}_{234} - 2\Omega_1\tilde{r}_{233}) + \Omega_1^2\tilde{r}_{134}) + \Omega_4(\Omega_3^2(3\tilde{s}_{234} \\
& + 6\tilde{r}_{233}) - 4\Omega_1\Omega_3\tilde{s}_{144} + 2\Omega_1\Omega_2\tilde{r}_{244} - 5\Omega_1\Omega_3\tilde{r}_{134}) \\
& + 4\Omega_1\Omega_3^2\tilde{s}_{133} - 6\Omega_3\Omega_4^2\tilde{r}_{244}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}, \\
P^3_{3,244} = {} & \{\Omega_4(\Omega_3^2(3\tilde{s}_{244} - 3\tilde{r}_{234}) + 7\Omega_1\Omega_3\tilde{r}_{144}) + \Omega_1\Omega_2\Omega_3(\tilde{r}_{234} \\
& - \tilde{s}_{244}) + 6\Omega_3^3\tilde{s}_{233} - 2\Omega_1\Omega_3^2\tilde{s}_{134} - \Omega_1^2\Omega_2\tilde{r}_{144} \\
& + 2\Omega_1\Omega_3^2\tilde{r}_{133}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}, \\
P^3_{3,333} = {} & -\{\Omega_3\Omega_4(2\tilde{s}_{333} - \tilde{s}_{344}) - \Omega_3^2\tilde{s}_{333} + 3\Omega_4^2\tilde{r}_{444} \\
& + (-2\Omega_4^2 - \Omega_3\Omega_4)\tilde{r}_{334}\}/\{4\Omega_3\Omega_4^2 + 4\Omega_3^2\Omega_4\},
\end{aligned}
$$

$$P^3_{3,334} = \{\Omega_3(9\tilde{s}_{444} - \tilde{s}_{334}) + \Omega_4(3\tilde{s}_{444} - 3\tilde{s}_{334}) + (\Omega_4 + 3\Omega_3)\tilde{r}_{344} + (-9\Omega_4 - 3\Omega_3)\tilde{r}_{333}\}/\{6\Omega_4^2 + 4\Omega_3\Omega_4 + 6\Omega_3^2\},$$

$$P^3_{3,344} = 0,$$

$$P^3_{3,444} = -\frac{-3\Omega_3^2\tilde{s}_{444} + \Omega_4(\Omega_3(\tilde{s}_{334} - 2\tilde{s}_{444}) + \Omega_4^2\tilde{r}_{344} + 3\Omega_3\Omega_4\tilde{r}_{333}}{3\Omega_4^3 + 2\Omega_3\Omega_4^2 + 3\Omega_3^2\Omega_4}.$$

$$P^4_{3,111} = -\{-6\Omega_2^3\tilde{s}_{222} + \Omega_2^2(-3\Omega_1\tilde{s}_{112} - 2\Omega_4\tilde{r}_{122}) + \Omega_2\Omega_3\Omega_4\tilde{s}_{112} + (\Omega_3\Omega_4^2 - 7\Omega_1\Omega_2\Omega_4)\tilde{r}_{111}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},$$

$$P^4_{3,112} = \{\Omega_4(\Omega_2(3\Omega_1\tilde{r}_{112} - 2\Omega_3\tilde{s}_{122}) + 6\Omega_2^2\tilde{r}_{222}) + (3\Omega_1\Omega_3\Omega_4 - 9\Omega_1^2\Omega_2)\tilde{s}_{111} - \Omega_3\Omega_4^2\tilde{r}_{112}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},$$

$$\begin{aligned} P^4_{3,113} = {} & \{\Omega_4(\Omega_3(2\Omega_1^2\tilde{s}_{114} - 2\Omega_2^2\tilde{s}_{224}) + (\Omega_3^2 - 2\Omega_1\Omega_2)(\Omega_1 + \Omega_2)\tilde{s}_{123}) \\ & + \Omega_4^2(\Omega_1\Omega_2(2\tilde{s}_{114} - 2\tilde{s}_{224}) + \Omega_3(\Omega_1 + \Omega_2)(\tilde{s}_{123} - \tilde{r}_{124})) \\ & + \Omega_3(\Omega_2^2(-2\Omega_1\tilde{s}_{123}) - 2\Omega_1^2\Omega_2\tilde{s}_{123}) + (2\Omega_2^2\Omega_4^2 + 2\Omega_1\Omega_2\Omega_3\Omega_4)\tilde{r}_{223} \\ & + \Omega_4^3(-\Omega_2\tilde{r}_{124} - \Omega_1\tilde{r}_{124}) + (-2\Omega_1^2\Omega_4^2 - 2\Omega_1\Omega_2\Omega_3\Omega_4)\tilde{r}_{113}\} \\ & /\{(4\Omega_1\Omega_2 + 4\Omega_1^2)\Omega_3\Omega_4^2 + ((4\Omega_1\Omega_2 + 4\Omega_1^2)\Omega_3^2 \\ & - 4\Omega_1^2\Omega_2^2 - 4\Omega_1^3\Omega_2)\Omega_4 + (-4\Omega_1^2\Omega_2^2 - 4\Omega_1^3\Omega_2)\Omega_3\}, \end{aligned}$$

$$P^4_{3,114} = -\{\Omega_3(\Omega_2\tilde{s}_{223} + \Omega_4(-\tilde{s}_{124} - \tilde{r}_{123}) + \Omega_1\tilde{s}_{113}) + \Omega_2\Omega_4\tilde{r}_{224} + \Omega_1\Omega_4\tilde{r}_{114}\}/\{4\Omega_1\Omega_3\Omega_4\},$$

$$P^4_{3,122} = \{\Omega_3(-3\Omega_2\Omega_4\tilde{s}_{222} + 2\Omega_1\Omega_4\tilde{s}_{112} - \Omega_4^2\tilde{r}_{122}) + 9\Omega_1\Omega_2^2\tilde{s}_{222} + 3\Omega_1\Omega_2\Omega_4\tilde{r}_{122} + 6\Omega_1^2\Omega_4\tilde{r}_{111}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},$$

$$\begin{aligned} P^4_{3,123} = {} & -\{\Omega_3(\Omega_2(\Omega_1(3\tilde{s}_{223} - 3\tilde{s}_{113}) + \Omega_4(\tilde{r}_{123} - \tilde{s}_{124})) \\ & + \Omega_2^2\tilde{s}_{223} + \Omega_1\Omega_4(\tilde{r}_{123} - \tilde{s}_{124}) - \Omega_1^2\tilde{s}_{113}) \\ & + \Omega_3^2(2\Omega_4\tilde{s}_{113} - 2\Omega_4\tilde{s}_{223}) + (2\Omega_3\Omega_4^2 + \Omega_2^2\Omega_4 - \Omega_1\Omega_2\Omega_4)\tilde{r}_{224} \\ & + (-2\Omega_3\Omega_4^2 + \Omega_1\Omega_2\Omega_4 - \Omega_1^2\Omega_4)\tilde{r}_{114}\}/\{\Omega_3^2(4\Omega_2\Omega_4 \\ & + 4\Omega_1\Omega_4) + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3\}, \end{aligned}$$

$$\begin{aligned} P^4_{3,124} = {} & \{\Omega_4(\Omega_3^2(2\tilde{s}_{224} - 2\tilde{s}_{114}) + \Omega_1\Omega_2(4\tilde{s}_{114} - 4\tilde{s}_{224}) \\ & + \Omega_3(\Omega_2(-\tilde{s}_{123} - \tilde{r}_{124}) + \Omega_1(-\tilde{s}_{123} - \tilde{r}_{124}))) \\ & + \Omega_4^2(\Omega_3(2\tilde{s}_{224} - 2\tilde{s}_{114}) - \Omega_2\tilde{r}_{124} - \Omega_1\tilde{r}_{124}) \\ & + \Omega_3(-2\Omega_2^2\tilde{s}_{224} + \Omega_1\Omega_2(2\tilde{s}_{114} - 2\tilde{s}_{224}) + 2\Omega_1^2\tilde{s}_{114}) \\ & + \Omega_3^2(-\Omega_2\tilde{s}_{123} - \Omega_1\tilde{s}_{123}) + (2\Omega_3\Omega_4^2 + (2\Omega_3^2 + 2\Omega_2^2 - 2\Omega_1\Omega_2)\Omega_4)\tilde{r}_{223} \\ & + ((-2\Omega_3^2 + 2\Omega_1\Omega_2 - 2\Omega_1^2)\Omega_4 - 2\Omega_3\Omega_4^2)\tilde{r}_{113}\} \\ & /\{(4\Omega_2 + 4\Omega_1)\Omega_3\Omega_4^2 + ((4\Omega_2 + 4\Omega_1)\Omega_3^2 \\ & - 4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_4 + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3\}, \end{aligned}$$

$$\begin{aligned}
P^4_{3,133} = {} & -\{\Omega_4^2(\Omega_2(2\tilde{s}_{244} - 2\tilde{r}_{234}) - 3\Omega_3\tilde{s}_{134}) + \Omega_4(7\Omega_2\Omega_3\tilde{s}_{233} \\
& + \Omega_1\Omega_2\tilde{s}_{134}) - \Omega_1\Omega_2^2\tilde{s}_{233} + 6\Omega_4^3\tilde{r}_{144} + (3\Omega_3\Omega_4^2 \\
& - \Omega_1\Omega_2\Omega_4)\tilde{r}_{133}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\},\\
P^4_{3,134} = {} & -\{\Omega_2\Omega_4(\Omega_3(5\tilde{s}_{234} + 4\tilde{r}_{233}) + 2\Omega_1\tilde{s}_{144} - \Omega_1\tilde{r}_{134}) \\
& - \Omega_1\Omega_2^2\tilde{s}_{234} + \Omega_4^2(-6\Omega_3\tilde{s}_{144} - 4\Omega_2\tilde{r}_{244} - 3\Omega_3\tilde{r}_{134}) \\
& + (6\Omega_3^2\Omega_4 - 2\Omega_1\Omega_2\Omega_3)\tilde{s}_{133}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\},\\
P^4_{3,144} = {} & -\{\Omega_4(\Omega_2\Omega_3(7\tilde{s}_{244} + 2\tilde{r}_{234}) + 3\Omega_3^2\tilde{s}_{134} - \Omega_1\Omega_2\tilde{r}_{144}) \\
& - \Omega_1\Omega_2^2\tilde{s}_{244} + 2\Omega_2\Omega_3^2\tilde{s}_{233} - \Omega_1\Omega_2\Omega_3\tilde{s}_{134} + 3\Omega_3\Omega_4^2\tilde{r}_{144} \\
& + 6\Omega_3^2\Omega_4\tilde{r}_{133}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\},\\
P^4_{3,222} = {} & -\{\Omega_4(-\Omega_1\Omega_3\tilde{s}_{122} - 7\Omega_1\Omega_2\tilde{r}_{222} - 2\Omega_1^2\tilde{r}_{112}) + 3\Omega_1^2\Omega_2\tilde{s}_{122} \\
& + 6\Omega_1^3\tilde{s}_{111} + \Omega_3\Omega_4^2\tilde{r}_{222}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\},\\
P^4_{3,223} = {} & 0,\\
P^4_{3,224} = {} & -\{2\Omega_3^2\Omega_4\tilde{s}_{223} + \Omega_3(-\Omega_1\Omega_2\tilde{s}_{223} + \Omega_1\Omega_4(\tilde{r}_{123} - \tilde{s}_{124}) \\
& - \Omega_1^2\tilde{s}_{113}) + (2\Omega_3\Omega_4^2 - \Omega_1\Omega_2\Omega_4)\tilde{r}_{224} - \Omega_1^2\Omega_4\tilde{r}_{114}\} \\
& /\{4\Omega_3^2\Omega_4^2 - 4\Omega_1\Omega_2\Omega_3\Omega_4\},\\
P^4_{3,233} = {} & \{\Omega_4^2(\Omega_3(3\tilde{s}_{234} - 3\tilde{r}_{233}) + 2\Omega_1\tilde{s}_{144} - 2\Omega_1\tilde{r}_{134}) \\
& + \Omega_2\Omega_4(\Omega_1\tilde{r}_{233} - \Omega_1\tilde{s}_{234}) + (7\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2)\tilde{s}_{133} \\
& - 6\Omega_4^3\tilde{r}_{244}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\},\\
P^4_{3,234} = {} & \{\Omega_4^2(\Omega_3(6\tilde{s}_{244} + 3\tilde{r}_{234}) - 4\Omega_1\tilde{r}_{144}) + \Omega_4(\Omega_1\Omega_2(\tilde{r}_{234} \\
& - 2\tilde{s}_{244}) - 6\Omega_3^2\tilde{s}_{233} + 5\Omega_1\Omega_3\tilde{s}_{134}) + 2\Omega_1\Omega_2\Omega_3\tilde{s}_{233} \\
& - \Omega_1^2\Omega_2\tilde{s}_{134} + 4\Omega_1\Omega_3\Omega_4\tilde{r}_{133}\}/\{9\Omega_3^2\Omega_4^2 \\
& - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\},
\end{aligned}$$

$$\begin{aligned}
P^4_{3,244} = {} & \{\Omega_2(\Omega_1\Omega_3\tilde{s}_{234} - \Omega_1^2\tilde{s}_{144}) + \Omega_4(\Omega_3^2(-3\tilde{s}_{234} - 6\tilde{r}_{233}) \\
& + 7\Omega_1\Omega_3\tilde{s}_{144} + \Omega_1\Omega_2\tilde{r}_{244} + 2\Omega_1\Omega_3\tilde{r}_{134}) + 2\Omega_1\Omega_3^2\tilde{s}_{133} \\
& - 3\Omega_3\Omega_4^2\tilde{r}_{244}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\},
\end{aligned}$$

$$\begin{aligned}
P^4_{3,333} = {} & -\{\Omega_4(\Omega_3(3\tilde{s}_{444} - \tilde{s}_{334})) + \Omega_4^2(3\tilde{s}_{444} - 3\tilde{s}_{334}) \\
& - 2\Omega_3^2\tilde{s}_{334} + (\Omega_4^2 + \Omega_3\Omega_4)\tilde{r}_{344} + (\Omega_3\Omega_4 - 3\Omega_4^2)\tilde{r}_{333}\} \\
& /\{6\Omega_3\Omega_4^2 + 4\Omega_3^2\Omega_4 + 6\Omega_3^3\},
\end{aligned}$$

$$P^4_{3,334} = -\frac{\Omega_3(3\tilde{s}_{333} - \tilde{s}_{344}) + 3\Omega_4\tilde{r}_{444} - \Omega_4\tilde{r}_{334}}{2\Omega_3\Omega_4 + 2\Omega_3^2},$$

$$P^4_{3,344} = 0,$$

$$P^4_{3,444} = -\frac{\Omega_4\Omega_3\tilde{s}_{344} + 3\Omega_3^2\tilde{s}_{333} + (\Omega_4^2 + 4\Omega_3\Omega_4)\tilde{r}_{444} - \Omega_3\Omega_4\tilde{r}_{334}}{4\Omega_3\Omega_4^2 + 4\Omega_3^2\Omega_4}.$$

Appendix C

The coefficients of the quadratic terms and cubic terms of functions $\hat{f}, \hat{g}, \hat{r}, \hat{s}$ are as follows:

$$\begin{aligned}
\hat{\varphi}_{ii} &= \varphi_{ii} + \varphi_{i\xi}E_{[i]} + E'_{[i]}\varphi_{\xi\xi}E_{[i]},\\
\hat{\varphi}_{ij} &= \varphi_{ij} + \varphi_{i\xi}E_{[j]} + \varphi_{j\xi}E_{[i]} + 2E'_{[i]}\varphi_{\xi\xi}E_{[j]},\\
\hat{\varphi}_{iii} &= \varphi_{iii} + \varphi_{ii\xi}E_{[i]} + E'_{[i]}\varphi_{i\xi\xi}E_{[i]} + \varphi_{\xi\xi\xi}(E_{[i]}, E_{[i]}, E_{[i]}) + \varphi_{i\xi}h_{ii} + 2E'_{[i]}\varphi_{\xi\xi}h_{ii},\\
\hat{\varphi}_{iij} &= \varphi_{j\xi}h_{ii} + \varphi_{i\xi}h_{ij} + 2E'_{[j]}\varphi_{\xi\xi}h_{ii} + 2E'_{[i]}\varphi_{\xi\xi}h_{ij} + \varphi_{iij} + \varphi_{ij\xi}E_{[i]}\\
&\quad + \varphi_{ii\xi}E_{[j]} + E'_{[i]}\varphi_{j\xi\xi}E_{[i]} + 2E'_{[i]}\varphi_{i\xi\xi}E_{[j]} + 3\varphi_{\xi\xi\xi}(E_{[i]}, E_{[i]}, E_{[j]}),\\
\hat{\varphi}_{ijk} &= \varphi_{ijk} + \varphi_{i\xi}h_{jk} + \varphi_{j\xi}h_{ik} + \varphi_{k\xi}h_{ij}\\
&\quad + 2E'_{[i]}\varphi_{\xi\xi}h_{jk} + 2E'_{[j]}\varphi_{\xi\xi}h_{ik} + 2E'_{[k]}\varphi_{\xi\xi}h_{ik}.
\end{aligned}$$

Here, i, j, k are distinct, $\varphi \in \{f, g, r, s\}$, and $i, j, k \in \{x, y, z, w\}$ with $E_{[x]} = E_1$, $E_{[y]} = E_2$, $E_{[z]} = E_3$, $E_{[w]} = E_4$, .

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