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Rotational perturbations of Friedmann–Robertson–Walker type brane-world cosmological models

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Abstract

First order rotational perturbations of the Friedmann–Robertson–Walker metric are considered in the framework of the brane-world cosmological models. A rotation equation, relating the perturbations of the metric tensor to the angular velocity of the matter on the brane is derived under the assumption of slow rotation. The mathematical structure of the rotation equation imposes strong restrictions on the temporal and spatial dependence of the brane matter angular velocity. The study of the integrable cases of the rotation equation leads to three distinct models, which are considered in detail. As a general result we find that, similarly to the general relativistic case, the rotational perturbations decay due to the expansion of the matter on the brane. One of the obtained consistency conditions leads to a particular, purely inflationary brane-world cosmological model, with the cosmological fluid obeying a non-linear barotropic equation of state. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

On an astronomical scale rotation is a basic property of cosmic objects. The rotation of planets, stars and galaxies inspired Gamow to suggest that the Universe is rotating and the angular momentum of stars and galaxies could be a result of the cosmic vorticity [1]. But

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even that observational evidences of cosmological rotation have been reported [2–5], they are still subject of controversy.

From the analysis of microwave background anisotropy Collins and Hawking [6] and Barrow, Juszkiewicz and Sonoda [7] have found some very tight limits of the cosmological vorticity, $T_{\text{obs}} > 3 \times 10^5 T_{\text{H}}$, where T_{obs} is the actual rotation period of our Universe and $T_{\text{H}} = (1 \sim 2) \times 10^{10}$ years is the Hubble time. Therefore, our present-day Universe is rotating very slowly, if at all.

From a theoretical point of view in 1949 Gödel [8] gave his famous example of a rotating cosmological solution to the Einstein gravitational field equations. The Gödel metric, describing a dust Universe with energy density ρ in the presence of a negative cosmological constant Λ is

$$ds^2 = \frac{1}{2\omega^2} \left[-(dt + e^x dz)^2 + dx^2 + dy^2 + \frac{1}{2} e^{2x} dz^2 \right]. \quad (1)$$

In this model the angular velocity of the cosmic rotation is given by $\omega^2 = 4\pi\rho = -\Lambda$. Gödel also discussed the possibility of a cosmic explanation of the galactic rotation [8]. This rotating solution has attracted considerable interest because the corresponding Universes possess the property of closed time-like curves.

The investigation of rotating and rotating-expanding Universes generated a large amount of literature in the field of general relativity, the combination of rotation with expansion in realistic cosmological models being one of the most difficult tasks in cosmology (see [9] for a recent review of the expansion–rotation problem in general relativity). Hence rotating solutions of the gravitational field equations cannot be excluded a priori. But this raises the question of why the Universe rotates so slowly. This problem can also be naturally solved in the framework of the inflationary model. Ellis and Olive [10] and Grøn and Soleng [11] pointed out that if the Universe came into being as a mini-universe of Planck dimensions and went directly into an inflationary epoch driven by a scalar field with a flat potential, due to the non-rotation of the false vacuum and the exponential expansion during inflation the cosmic vorticity has decayed by a factor of about 10^{-145} . The most important diluting effect of the order of 10^{-116} is due to the relative density of the rotating fluid compared to the non-rotating decay products of the false vacuum [11]. Inflationary cosmology also ruled out the possibility that the vorticity of galaxies and stars be of cosmic origin.

The possibility of incorporating a slowly rotating Universe into the framework of Friedmann–Robertson–Walker (FRW) type metrics has been considered by Bayin and Cooperstock [12], who obtained the restrictions imposed by the field equations on the matter angular velocity. They also shown that uniform rotation is incompatible with the dust filled (zero pressure) and with the radiation dominated Universe. Bayin [13] has also shown that the field equations admit solutions for a special class of nonseparable rotation functions of the matter distribution. The investigation of the first order rotational perturbations of flat FRW type Universes proved to be useful in the study of string cosmological models with dilaton and axion fields [14]. The form of the rotation equation imposes strong constraints on the form of the dilaton field potential U , restricting the allowed forms to two: the trivial case $U = 0$ and the exponential type potential.

Recently, Randall and Sundrum [15,16] have pointed out that a scenario with an infinite fifth dimension in the presence of a brane can generate a theory of gravity which mimics purely four-dimensional gravity, both with respect to the classical gravitational potential and with respect to gravitational radiation. The gravitational self-couplings are not significantly modified in this model. This result has been obtained from the study of a single 3-brane embedded in five dimensions, with the 5D metric given by $ds^2 = e^{-f(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2$, which can produce a large hierarchy between the scale of particle physics and gravity due to the appearance of the warp factor. Even if the fifth dimension is uncompactified, standard 4D gravity is reproduced on the brane. In contrast to the compactified case, this follows because the near-brane geometry traps the massless graviton. Hence this model allows the presence of large or even infinite non-compact extra dimensions. Our brane is identified to a domain wall in a 5-dimensional anti-de Sitter space–time.

The Randall–Sundrum (RS) model was inspired by superstring theory. The ten-dimensional $E_8 \times E_8$ heterotic string theory, which contains the standard model of elementary particle, could be a promising candidate for the description of the real Universe. This theory is connected with an eleven-dimensional theory compactified on the orbifold $R^{10} \times S^1/Z_2$ [17]. In this model we have two separated ten-dimensional manifolds.

The static RS solution has been extended to time-dependent solutions and their cosmological properties have been extensively studied [18–27] (for a review of dynamics and geometry of brane universes see [28]).

The effective gravitational field equations on the brane world, in which all the matter forces except gravity are confined on the 3-brane in a 5-dimensional space–time with Z_2 -symmetry have been obtained, by using an elegant geometric approach, by Shiromizu, Maeda and Sasaki [29,30]. The correct signature for gravity is provided by the brane with positive tension. If the bulk space–time is exactly anti-de Sitter, generically the matter on the brane is required to be spatially homogeneous. The electric part of the 5-dimensional Weyl tensor E_{IJ} gives the leading order corrections to the conventional Einstein equations on the brane. The effect of the dilaton field in the bulk can also be taken into account in this approach [31].

The linearized perturbation equations in the generalized RS model have been obtained, by using the covariant nonlinear dynamical equations for the gravitational and matter fields on the brane, by Maartens [32]. The behavior of an anisotropic Bianchi type I brane-world in the presence of inflationary scalar fields has been considered by Maartens, Sahni and Saini [33]. A systematic analysis, using dynamical systems techniques, of the qualitative behavior of the FRW, Bianchi type I and V cosmological models in the RS brane world scenario, with matter on the brane obeying a barotropic equation of state has been performed by Campos and Sopena [34,35]. In particular, they constructed the state spaces for these models and discussed what new critical points appear, the occurrence of bifurcations and the dynamics of the anisotropy. The general exact solution of the field equations for an anisotropic brane with Bianchi type I and V geometry, with perfect fluid and scalar fields as matter sources has been found in [36]. Expanding Bianchi type I and V brane-worlds always isotropize, although there could be intermediate stages in which the anisotropy grows. In spatially homogeneous brane world cosmological models the initial singularity is isotropic and hence the initial conditions problem is solved [37].

Consequently, these models do not exhibit Mixmaster or chaotic-like behavior close to the initial singularity [38].

Realistic brane-world cosmological models require the consideration of more general matter sources to describe the evolution and dynamics of the very early Universe. Hence the effect of the bulk viscosity of the matter on the brane have been analyzed in [39]. Limits on the initial anisotropy induced by the 5-dimensional Kaluza–Klein graviton stresses by using the CMB anisotropies have been obtained by Barrow and Maartens [40]. Anisotropic Bianchi type I brane-worlds with a pure magnetic field and a perfect fluid have also been analyzed [41].

It is the purpose of the present paper to investigate the effects of the rotational perturbations on a brane world with FRW type geometry. A similar case was also considered in [42]. Assuming the rotation is slow and by keeping only the first order rotational terms in the field equations, a rotation equation describing the time and space evolution of the metric perturbations is obtained. This equation also contains the angular velocity of the matter rotating on the brane. By assuming that the metric perturbation and the matter angular velocity are separable functions of the variables t and r , the mathematical consistency of the rotation equation leads to some restrictions on the functional form of the angular velocity. In particular, a class of solutions of the field equations leads to a barotropic brane-world cosmological model, with a non-linear pressure-energy density dependence. But generally, similar to the general relativistic case, the rotational perturbations will rapidly decay due to the expansion of the Universe. However, this general result is valid only in the presence of the dark energy term, describing the influence of the five-dimensional bulk on the brane. If this term is zero, rotational perturbations in a very high density (stiff) cosmological fluid decay only in the so-called case of the “perfect dragging”.

The present paper is organized as follows. The field equations for a slowly rotating brane-world are written down and the basic rotation equation is obtained in Section 2. The integrable cases of the rotation equation are considered in Section 3. A brane world cosmological model, derived from the mathematical consistency requirement of the rotation equation, is obtained in Section 4. In Section 5 we discuss and conclude our results.

2. Geometry, brane-world field equations and consequences

In the 5D space–time the brane-world is located as $Y(X^I) = 0$, where X^I , $I = 0, 1, 2, 3, 4$ are 5-dimensional coordinates. The effective action in five dimensions is [31]

$$S = \int d^5 X \sqrt{-g_5} \left(\frac{1}{2k_5^2} R_5 - \Lambda_5 \right) + \int_{Y=0} d^4 x \sqrt{-g} \left(\frac{1}{k_5^2} K^\pm - \lambda + L^{\text{matter}} \right), \quad (2)$$

with $k_5^2 = 8\pi G_5$ the 5-dimensional gravitational coupling constant and where x^μ , $\mu = 0, 1, 2, 3$ are the induced 4-dimensional brane world coordinates. We use a system of units so that the speed of light $c = 1$. R_5 is the 5D intrinsic curvature in the bulk and K^\pm is the extrinsic curvature on either side of the brane.

On the 5-dimensional space–time (the bulk), with the negative vacuum energy Λ_5 as only source of the gravitational field the Einstein field equations are given by

$$G_{IJ} = k_5^2 T_{IJ}, \quad T_{IJ} = -\Lambda_5 g_{IJ} + \delta(Y) [-\lambda g_{IJ} + T_{IJ}^{\text{matter}}]. \tag{3}$$

In this space–time a brane is a fixed point of the Z_2 symmetry. In the following capital Latin indices run in the range $0, \dots, 4$ while Greek indices take the values $0, \dots, 3$.

Assuming a metric of the form $ds^2 = (n_I n_J + g_{IJ}) dx^I dx^J$, with $n_I dx^I = d\chi$ the unit normal to the $\chi = \text{constant}$ hypersurfaces and g_{IJ} the induced metric on $\chi = \text{constant}$ hypersurfaces, the effective four-dimensional gravitational equations on the brane take the form [29,30]:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu} + k_5^4 S_{\mu\nu} - E_{\mu\nu}, \tag{4}$$

where

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_{\mu}{}^{\alpha} T_{\nu\alpha} + \frac{1}{24} g_{\mu\nu} (3T^{\alpha\beta} T_{\alpha\beta} - T^2), \tag{5}$$

and $\Lambda = k_5^2(\Lambda_5 + k_5^2 \lambda^2/6)/2$, $k_4^2 = k_5^4 \lambda/6$ and $E_{IJ} = C_{IAJB} n^A n^B$. C_{IAJB} is the 5-dimensional Weyl tensor in the bulk and λ is the vacuum energy on the brane. $T_{\mu\nu}$ is the matter energy–momentum tensor on the brane and $T = T^{\mu}{}_{\mu}$ is the trace of the energy–momentum tensor.

For any matter fields (scalar field, perfect fluids, kinetic gases, dissipative fluids, etc.) the general form of the brane energy–momentum tensor can be covariantly given as

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + ph_{\mu\nu} + \pi_{\mu\nu} + 2q_{(\mu}u_{\nu)}. \tag{6}$$

The decomposition is irreducible for any chosen 4-velocity u^{μ} . Here ρ and p are the energy density and isotropic pressure, and $h_{\mu\nu} = g_{\mu\nu} + u_{\mu}u_{\nu}$ projects orthogonal to u^{μ} . The energy flux obeys $q_{\mu} = q_{\langle\mu}$, and the anisotropic stress obeys $\pi_{\mu\nu} = \pi_{\langle\mu\nu\rangle}$, where angular brackets denote the projected, symmetric and tracefree part:

$$V_{\langle\mu\rangle} = h_{\mu}{}^{\nu} V_{\nu}, \quad W_{\langle\mu\nu\rangle} = \left[h_{(\mu}{}^{\alpha} h_{\nu)}{}^{\beta} - \frac{1}{3} h^{\alpha\beta} h_{\mu\nu} \right] W_{\alpha\beta}. \tag{7}$$

The symmetric properties of $E_{\mu\nu}$ imply that in general we can decompose it irreducibly with respect to a chosen 4-velocity field u^{μ} as

$$E_{\mu\nu} = -\frac{6}{\lambda k_4^2} \left[\mathcal{U} \left(u_{\mu}u_{\nu} + \frac{1}{3} h_{\mu\nu} \right) + \mathcal{P}_{\mu\nu} + 2\mathcal{Q}_{(\mu}u_{\nu)} \right], \tag{8}$$

where \mathcal{U} is a scalar, \mathcal{Q}_{μ} a spatial vector and $\mathcal{P}_{\mu\nu}$ a spatial, symmetric and trace-free tensor. For a FRW model $\mathcal{Q}_{\mu} = 0$, $\mathcal{P}_{\mu\nu} = 0$ [35] and hence the only non-zero contribution from the 5-dimensional Weyl tensor from the bulk is given by the scalar term \mathcal{U} .

The Einstein equation in the bulk imply the conservation of the energy momentum tensor of the matter on the brane,

$$T_{\mu}{}^{\nu}{}_{; \nu} \Big|_{\chi=0} = 0. \tag{9}$$

The rotationally perturbed metric can be expressed in terms of the usual coordinates in the form [43]

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right] - 2\Omega(t, r)a^2(t)r^2 \sin^2 \theta dt d\varphi, \tag{10}$$

where $\Omega(t, r)$ is the metric rotation function. Although Ω plays a role in the “dragging” of local inertial frames, it is not the angular velocity of these frames, except for the special case when it coincides with the angular velocity of the matter fields. $k = 1$ corresponds to closed Universes, with $0 \leq r \leq 1$. $k = -1$ corresponds to open Universes, while the case $k = 0$ describes a flat geometry, where the range of r is $0 \leq r < \infty$. In all models, the time-like variable t ranges from 0 to ∞ .

For the matter energy–momentum tensor on the brane we restrict our analysis to the case of the perfect fluid energy–momentum tensor,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}. \tag{11}$$

The components of the four-velocity vector are $u^0 = 1$, $u^1 = u^2 = 0$ and $u^3 = \omega(t, r)$. $\omega = d\varphi/dt$ is the angular velocity of the matter distribution. Consequently, for the rotating brane the energy–momentum tensor has a supplementary component

$$T_{03} = \{[\Omega(t, r) - \omega(t, r)]\rho - \omega(t, r)p\}r^2a^2(t) \sin^2 \theta. \tag{12}$$

We assume that rotation is sufficiently slow so that deviations from spherical symmetry can be neglected. Then to first order in Ω the gravitational and field equations become

$$3\frac{\dot{a}^2}{a^2} + \frac{3k}{a^2} = \Lambda + k_4^2\rho + \frac{k_4^2}{2\lambda}\rho^2 + \frac{6}{\lambda k_4^2}\mathcal{U}, \tag{13}$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \Lambda - k_4^2p - \frac{k_4^2}{\lambda}\rho p - \frac{k_4^2}{2\lambda}\rho^2 - \frac{2}{\lambda k_4^2}\mathcal{U}, \tag{14}$$

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0, \tag{15}$$

$$\dot{\mathcal{U}} + \frac{4\dot{a}}{a}\mathcal{U} = 0, \tag{16}$$

$$3\frac{\dot{a}}{a}\frac{\partial\Omega(t, r)}{\partial r} + \frac{\partial^2\Omega(t, r)}{\partial t\partial r} = 0, \tag{17}$$

$$(1 - kr^2)\frac{\partial^2\Omega(t, r)}{\partial r^2} + \left(\frac{4}{r} - 5kr\right)\frac{\partial\Omega(t, r)}{\partial r} - 4[\Omega(t, r) - \omega(t, r)](k - a\ddot{a} + \dot{a}^2) = 0. \tag{18}$$

The last two equations follows from the R_{13} and R_{03} components of the field equations, respectively. Generally, we shall assume that the thermodynamic pressure p and the energy density ρ are related by a barotropic equation of state $p = p(\rho)$.

From a mathematical point of view the field equations (13)–(14) and (16)–(18) represent a system of five equations in five unknowns $a(t)$, $\rho(t)$, \mathcal{U} , $\Omega(t, r)$ and $\omega(t, r)$ (the Bianchi

identity (15) is a consequence of the field equations and the pressure can be eliminated via the equation of state of the matter on the brane). Therefore, to find the general solution of this system is a well-posed problem. Since the Bianchi identity and the evolution equation for \mathcal{U} can be generally integrated, giving the energy density ρ and the dark matter term as some functions of the scale factor a , the basic field equations describing the first order rotational perturbations of brane world cosmologies are Eqs. (13) and (17)–(18), three equations in three unknowns $a(t)$, $\Omega(t, r)$ and $\omega(t, r)$.

Mathematically, Eqs. (17)–(18) represent a system of partial differential equations. To obtain the solution of these equations we shall use the standard technique of the separation of variables [44]. Therefore, we generally assume that both the rotation metric function and the angular velocity of the matter on the brane are the product of two functions, the first one depending on the cosmological time t only and the second one on the radial variable r only. When these representations of the unknown functions are substituted back to Eqs. (17)–(18), the field equations become identities in the independent variable, which can be separated in two independent equations, giving the time evolution and the spatial behavior of the considered physical parameters. If the values of the separation constants are properly chosen, then the combination of the solutions of the separated equations gives the solution of the initial partial differential equation for any boundary or initial conditions [44].

Eq. (16) can be immediately integrated to give the following general expression for the “dark energy” \mathcal{U} :

$$\mathcal{U} = \frac{\mathcal{U}_0}{a^4}, \tag{19}$$

with $\mathcal{U}_0 > 0$ a constant of integration.

The temporal and spatial dependence of $\Omega(t, r)$ is determined by Eqs. (17) and (18). The general solution of Eq. (17) can be immediately found and is given by

$$\Omega(t, r) = \frac{A(r)}{a^3(t)}, \tag{20}$$

where $A(r)$ is a function to be determined from the field equations and an arbitrary time dependent function has been set to zero, without altering the physical structure of the model. Substituting Eq. (20) into Eq. (18) gives the following equation describing the evolution of the rotational perturbations on the brane:

$$(1 - kr^2) \frac{d^2 A(r)}{dr^2} + \left(\frac{4}{r} - 5kr \right) \frac{dA(r)}{dr} - 4[A(r) - \omega(t, r)a^3](k - a\ddot{a} + \dot{a}^2) = 0. \tag{21}$$

By introducing a new variable $\eta = r^2$, Eq. (21) can be transformed into the following form:

$$\eta(1 - k\eta) \frac{d^2 A(\eta)}{d\eta^2} + \left(\frac{5}{2} - 3k\eta \right) \frac{dA(\eta)}{d\eta} = [A(\eta) - \omega(t, \eta)a^3](k - a\ddot{a} + \dot{a}^2). \tag{22}$$

Eq. (22), the basic rotation equation, governs the evolution of the first order rotational perturbations of brane world models. For a given equation of state of the matter, $a(t)$ is

obtained from the field equations (13)–(16). Hence Eq. (22) places some restrictions on the possible form of the angular velocity of the matter $\omega(t, r)$. Indeed, since the left-hand side of the Eq. (22) is a function of r alone, the right-hand side must be either a function of r alone or a function of t alone (and hence set equal to a constant). As a first consequence of the mathematical structure of Eq. (22) it follows that the angular velocity of the rotating brane world must also be a separable function and we assume it to be of the form $\omega(t, r) = f(t)g(r)$, with $f(t)$ and $g(r)$ functions to be determined from Eq. (22).

3. Consistency conditions for the rotation equation

The first and simplest case in which Eq. (22) has a solution corresponds to the case $\omega(t, r) = \Omega(t, r)$. Physically, this situation corresponds to the so-called “perfect dragging”, in which the function $\Omega(t, r)$ appearing in the rotationally perturbed line element Eq. (10) is the angular velocity of the rotating matter. Then the spatial evolution of the angular velocity is determined from the equation

$$\eta(1 - k\eta) \frac{d^2 A(\eta)}{d\eta^2} + \left(\frac{5}{2} - 3k\eta\right) \frac{dA(\eta)}{d\eta} = 0. \tag{23}$$

Eq. (23) has the following solutions:

$$A(r) = \begin{cases} C_1^{(1)}(1 - r^2)^{1/2}(2r^{-1} + r^{-3}) + C_2^{(1)}, & k = +1, \\ C_1^{(-1)}(1 + r^2)^{1/2}(2r^{-1} - r^{-3}) + C_2^{(-1)}, & k = -1, \\ C_1^{(0)}r^{-3} + C_2^{(0)}, & k = 0, \end{cases} \tag{24}$$

where $C_1^{(i)}$ and $C_2^{(i)}$, $i = -1, 0, 1$ are arbitrary constants of integration.

These solutions are not regular at the origin. They could be used in regions away from the origin and joined continuously to other solutions which are regular at the origin, which could be obtained by taking into account second order or higher terms in Ω in the field equations. The temporal behavior of the angular velocity is given by $a^{-3}(t)$. Thus for an expanding Universe this type of dependence guarantees the decay of the rotational perturbations in the limit of large cosmological times.

The second case in which the rotation equation can be solved corresponds to the choice $g(r) = A(r)$. In this case both $\omega(t, r)$ and $\Omega(t, r)$ have the same spatial dependence, but their time dependence is different. Therefore, Eq. (22) can be decoupled into the following two simple ordinary differential equations:

$$\eta(1 - k\eta) \frac{d^2 A(\eta)}{d\eta^2} + \left(\frac{5}{2} - 3k\eta\right) \frac{dA(\eta)}{d\eta} - CA(\eta) = 0, \tag{25}$$

$$[1 - f(t)a^3](k - a\ddot{a} + \dot{a}^2) = C, \tag{26}$$

where C is a separation constant.

Consider first the time dependence of the angular velocity. With the use of the field equations (13)–(16), Eq. (26) gives

$$f(a) = \frac{a^2[k_4^2(\rho + p) + (k_4^2/\lambda)\rho^2 + (k_4^2/\lambda)\rho p + (8\mathcal{U}_0/\lambda k_4^2)a^{-4}] - 2C}{a^5[k_4^2(\rho + p) + (k_4^2/\lambda)\rho^2 + (k_4^2/\lambda)\rho p + (8\mathcal{U}_0/\lambda k_4^2)a^{-4}]} \tag{27}$$

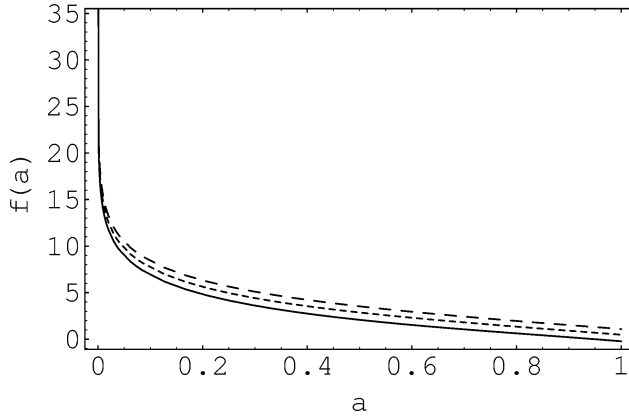


Fig. 1. Dynamics of the temporal part of the angular velocity $f(a)$ (in a logarithmic scale) as a function of the scale factor a for different equations of state of the brane matter: $\gamma = 2$ (solid curve), $\gamma = 4/3$ (dotted curve) and $\gamma = 1$ (dashed curve). We have normalized the parameters so that $\rho_0 = \lambda$, $8\mathcal{U}_0 = k_4^4 \rho_0^2$ and $2C = k_4^2 \rho_0$.

In the very early stages of the evolution of the Universe it is natural to assume that the very high density matter obeys a barotropic equation of state of the form $p = (\gamma - 1)\rho$, with γ a constant and $1 \leq \gamma \leq 2$. $\gamma = 2$ corresponds to the extreme limit of very high densities in which the speed of sound equals the speed of light (stiff case) [45]. For a barotropic equation of state the conservation equation (15) can immediately be integrated to give $\rho = \rho_0 a^{-3\gamma}$, where ρ_0 is a constant of integration.

Therefore, for a cosmological fluid obeying a barotropic equation of state the time dependence of the angular velocity of the slowly rotating brane world is given by

$$f(a) = \frac{\gamma \rho_0 k_4^2 a^2 [a^{3\gamma} + (8\mathcal{U}_0 / \gamma \lambda k_4^4 \rho_0) a^{6\gamma-4} + (\rho_0 / \lambda)] - 2C a^{6\gamma}}{\gamma \rho_0 k_4^2 a^5 [a^{3\gamma} + (8\mathcal{U}_0 / \gamma \lambda k_4^4 \rho_0) a^{6\gamma-4} + (\rho_0 / \lambda)]}. \tag{28}$$

The variation of the function $f(a)$ for different values of γ is represented in Fig. 1.

For all values of γ and for an expanding Universe the rotational perturbations tend, in the large time limit, to zero. This result is also independent on the numerical value of the separation constant C .

In the particular case of the vanishing dark matter, $\mathcal{U}_0 = 0$, and in the limit of large times $f(a)$ tends to the limit

$$f(a) = a^{-3} - \frac{2C}{\gamma \rho_0 k_4^2} a^{3\gamma-5}. \tag{29}$$

Therefore, in the absence of dark energy the rotational perturbations on the brane would decay in time only if the barotropic fluid satisfies the condition $\gamma < 5/3$. In particular, this condition is not satisfied by the stiff cosmological fluid with $\gamma = 2$. Hence in this case the observational evidence of a non-rotating Universe requires the condition $C = 0$ and, consequently, a “perfect dragging” of the cosmological fluid. But in a radiation fluid with $\gamma = 4/3$ the rotational perturbations will decay in time.

For this model the spatial dependence of the angular velocity is described by Eq. (25). For $k = +1$ this equation is of the form $x(1 - x)F_{xx} + [\delta - (1 + \alpha + \beta)x]F_x - \alpha\beta F = 0$

(the hypergeometric equation), with $\delta \neq 0, 1, 2, \dots$, and α, β are constants. The general solution of the hypergeometric equation is given by [46]

$$F = C_1 F(\alpha, \beta; \delta; x) + C_2 x^{1-\delta} F(1 - \delta + \alpha, 1 - \delta + \beta; 2 - \delta; x),$$

where C_1 and C_2 are arbitrary constants of integration, and

$$F(\alpha, \beta; \delta; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! (\delta)_n} x^n,$$

where

$$(\alpha)_0 = 1, \quad (\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \alpha(\alpha + 1) \cdots (\alpha + n - 1), \quad n = 1, 2, \dots$$

The radius of convergence of the solution is unity and if one of the constants $\alpha, \beta, \delta - \alpha, \delta - \beta$ is a negative integer, then the series terminates [46].

For Eq. (25) $\delta = 5/2$ and the constants α and β must satisfy the conditions $\alpha + \beta = 2$ and $\alpha\beta = C$. The separation constant C must satisfy the condition $C \leq 1$.

Therefore, the general solution of Eq. (25) is given by

$$g(r) = A(r) = C_1^{(1)} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! (\frac{5}{2})_n} r^{2n} + C_2^{(1)} r^{-3} \sum_{n=0}^{\infty} \frac{(\alpha - \frac{3}{2})_n (\beta - \frac{3}{2})_n}{n! (-\frac{1}{2})_n} r^{2n},$$

$$k = +1. \tag{30}$$

Since the second term is not regular at $r = 0$ we must take $C_2^{(1)} = 0$. Therefore, the resulting solution is also defined and is regular at the origin of the radial coordinate.

For open models $k = -1$ and the solution of Eq. (25) can be obtained from the previous one by replacing η by $-\eta$ and the separation constant C by $-C$. These solutions are convergent only for $\eta < 1$. For solutions regular outside $\eta = 1$, one must consider solutions found in the neighborhood of the remaining two singular points of the differential equation, namely 1 and ∞ [46].

For $k = 0$ Eq. (21) gives

$$\frac{d^2 A(r)}{dr^2} + \frac{4}{r} \frac{dA(r)}{dr} - 4CA(r) = 0, \tag{31}$$

with the general solution given by [14]

$$g(r) = A(r) = \frac{1}{r^2} \left[2\sqrt{C} (C_1^{(0)} e^{2\sqrt{C}r} + C_2^{(0)} e^{-2\sqrt{C}r}) - \frac{1}{r} (C_1^{(0)} e^{2\sqrt{C}r} - C_2^{(0)} e^{-2\sqrt{C}r}) \right], \quad k = 0, \tag{32}$$

and $C_1^{(0)}$ and $C_2^{(0)}$ arbitrary constants. Hence we have obtained in a closed form all the possible spatial distribution functions corresponding to the time dependence Eq. (27) of the angular velocity of the slowly rotating brane world.

The third case in which the variable in Eq. (21) can be separated is given by assuming that $\omega(t, r) = G(r)/a^3(t)$ and the time only dependent term in the right-hand side of the

equation is a constant. Therefore, we obtain:

$$\eta(1 - k\eta) \frac{d^2 A(\eta)}{d\eta^2} + \left(\frac{5}{2} - 3k\eta\right) \frac{dA(\eta)}{d\eta} - K[A(\eta) - G(\eta)] = 0, \tag{33}$$

$$k - a\ddot{a} + \dot{a}^2 = K, \tag{34}$$

where K is a separation constant. Generally Eq. (33) cannot be solved, due to a lack of exact knowledge of the mathematical form of the function $G(\eta)$. In the particular case $G(\eta) = G_0 = \text{constant}$, the solutions of Eq. (33) can be obtained by the substitution $A(r) \rightarrow A(r) + G_0$, where $A(r)$ is any of the solutions of the homogeneous equation (25).

Finally, we consider the possibility of the existence of an uniformly rotating brane world, with $\omega = \omega_0 = \text{constant}$. In this case the rotation equation takes the form

$$\begin{aligned} \eta(1 - k\eta) \frac{d^2 A(\eta)}{d\eta^2} + \left(\frac{5}{2} - 3k\eta\right) \frac{dA(\eta)}{d\eta} - A(\eta)(k - a\ddot{a} + \dot{a}^2) \\ = -\omega_0 a^3 (k - a\ddot{a} + \dot{a}^2). \end{aligned} \tag{35}$$

Therefore, the mathematical consistency of Eq. (35) requires either $A(\eta) = \text{constant}$, implying $a = \text{constant}$ or $(k - a\ddot{a} + \dot{a}^2) = \text{constant}$ and $a^3(k - a\ddot{a} + \dot{a}^2) = \text{constant}$, also leading to $a = \text{constant}$. Hence uniform rotation is possible only for a static brane, the expansion of the Universe making the angular velocity a complicated function of the temporal and spatial coordinates.

4. An inflationary brane world cosmological model

In the previous section we have shown that the third condition of integrability of the rotation equation leads to the Eq. (34), describing the evolution of the scale factor of the slowly rotating brane world. Hence in this case the time evolution of the brane Universe is strongly correlated with its rotational properties.

With the help of the substitutions $\dot{a} = u$, $u^2 = v$, Eq. (34) can be transformed into a first order linear differential equation of the form

$$\frac{dv}{da} = \frac{2}{a}v - \frac{2}{a}B, \tag{36}$$

where we denoted $B = K - k$. The general solution of Eq. (36) is $v = H_0^2 a^2 + B$, with H_0^2 a positive-definite arbitrary constant of integration. We will assume $H_0 > 0$ for later convenience.

Therefore, the general solution of Eq. (34) is given by:

$$a = \begin{cases} \sqrt{B} H_0^{-1} \sinh[H_0(t - t_0)], & B > 0, \\ \sqrt{|B|} H_0^{-1} \cosh[H_0(t - t_0)], & B < 0, \\ e^{H_0(t-t_0)}, & B = 0, \end{cases} \tag{37}$$

where t_0 is a constant of integration.

In this class of models the Hubble parameter is given by

$$H = \begin{cases} H_0 \coth[H_0(t - t_0)], & B > 0, \\ H_0 \tanh[H_0(t - t_0)], & B < 0, \\ H_0 = \text{constant}, & B = 0. \end{cases} \tag{38}$$

The energy density and pressure of the matter on the brane follow from the field equations (13) and (14), with the energy-density of the cosmological fluid given by

$$\rho = \lambda \left(\sqrt{\frac{6K}{\lambda k_4^2} \frac{1}{a^2} - \frac{12\mathcal{U}_0}{\lambda^2 k_4^4} \frac{1}{a^4} + 1 - \frac{2\Lambda}{\lambda k_4^2} + \frac{6H_0^2}{\lambda k_4^2} - 1} \right). \tag{39}$$

In order to obtain Eq. (39) we have used the identity

$$\frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} = \frac{3K}{a^2} + 3H_0^2.$$

Due to the presence of dark matter term, the energy density of the brane world has a maximum, $d\rho/da = 0$, corresponding to a value of the scale factor given by $a_{\max} = (2/k_4)\sqrt{\mathcal{U}_0/K\lambda}$. For this value of a the energy density satisfies the condition $d^2\rho/da^2 < 0$. The maximum value of the energy density is given by

$$\rho_{\max} = \lambda \left(\sqrt{\frac{3K^2}{4\mathcal{U}_0} + 1 - \frac{2\Lambda}{\lambda k_4^2} + \frac{6H_0^2}{\lambda k_4^2} - 1} \right).$$

If $\mathcal{U} = 0$ and the scale factor a is a monotonically increasing function of time, then the energy density of the brane world is a monotonically decreasing function for all times.

The condition of the non-negativity of the energy density $\rho \geq 0$ can be reformulated as a condition on the scale factor of the form $(\Lambda - 3H_0^2)\lambda k_4^2 a^4 - 3\lambda K k_4^2 a^2 + 6\mathcal{U}_0 < 0$, inequality that is satisfied for all a if $3H_0^2 > \Lambda$ and $3\lambda K^2 k_4^2 + 8(3H_0^2 - \Lambda)\mathcal{U}_0 < 0$. Since these two conditions cannot be satisfied simultaneously, it follows that generally the energy density is non-negative only for a finite time interval. For $\mathcal{U}_0 = 0$, $\rho \geq 0$ for $a \in (0, \sqrt{3K/(\Lambda - 3H_0^2)}]$.

The general pressure-energy density relation (the equation of state) is given by

$$p = \frac{(2K/k_4^2)a^{-2} - (8\mathcal{U}_0/\lambda k_4^4)a^{-4} - \rho - (\rho^2/\lambda)}{1 + \rho/\lambda}. \tag{40}$$

The condition of the non-negativity of the pressure requires the condition

$$\frac{2K}{k_4^2} \frac{1}{a^2} - \frac{8\mathcal{U}_0}{\lambda k_4^4} \frac{1}{a^4} \geq \rho \left(1 + \frac{\rho}{\lambda} \right), \tag{41}$$

be satisfied for all a . From Eq. (13) we obtain

$$\rho \left(1 + \frac{\rho}{\lambda} \right) = \frac{6K}{k_4^2} \frac{1}{a^2} - \frac{12\mathcal{U}_0}{\lambda k_4^4} \frac{1}{a^4} - \rho + \frac{2(3H_0^2 - \Lambda)}{k_4^2}, \tag{42}$$

and therefore the condition of the non-negativity of the pressure can be reformulated as the following condition which must be satisfied by the energy density of the matter on the

brane:

$$\rho \geq \frac{4K}{k_4^2} \frac{1}{a^2} - \frac{4\mathcal{U}_0}{\lambda k_4^4} \frac{1}{a^4} + \frac{2(3H_0^2 - \Lambda)}{k_4^2}. \tag{43}$$

Hence with the use of Eqs. (37) we obtain the following exact analytic representations for the energy density and pressure:

$$\rho(t) = \lambda \left\{ \sqrt{a_0 \sinh^{-2}[H_0(t - t_0)] - b_0 \sinh^{-4}[H_0(t - t_0)] + c_0} - 1 \right\}, \quad B > 0, \tag{44}$$

$$p(t) = \frac{k_0 \sinh^{-2}[H_0(t - t_0)] - u_0 \sinh^{-4}[H_0(t - t_0)]}{\sqrt{a_0 \sinh^{-2}[H_0(t - t_0)] - b_0 \sinh^{-4}[H_0(t - t_0)] + c_0}} - \lambda \left\{ \sqrt{a_0 \sinh^{-2}[H_0(t - t_0)] - b_0 \sinh^{-4}[H_0(t - t_0)] + c_0} - 1 \right\}, \quad B > 0, \tag{45}$$

$$\rho(t) = \lambda \left\{ \sqrt{a_0 \cosh^{-2}[H_0(t - t_0)] - b_0 \cosh^{-4}[H_0(t - t_0)] + c_0} - 1 \right\}, \quad B < 0, \tag{46}$$

$$p(t) = \frac{k_0 \cosh^{-2}[H_0(t - t_0)] - u_0 \cosh^{-4}[H_0(t - t_0)]}{\sqrt{a_0 \cosh^{-2}[H_0(t - t_0)] - b_0 \cosh^{-4}[H_0(t - t_0)] + c_0}} - \lambda \left\{ \sqrt{a_0 \cosh^{-2}[H_0(t - t_0)] - b_0 \cosh^{-4}[H_0(t - t_0)] + c_0} - 1 \right\}, \quad B < 0, \tag{47}$$

$$\rho(t) = \lambda \left\{ \sqrt{\frac{6K}{\lambda k_4^2} \exp[-2H_0(t - t_0)] - \frac{12\mathcal{U}_0}{\lambda^2 k_4^4} \exp[-4H_0(t - t_0)] + c_0} - 1 \right\}, \tag{48}$$

$B = 0,$

$$p(t) = \frac{(2K/k_4^2) \exp[-2H_0(t - t_0)] - (8\mathcal{U}_0/\lambda k_4^2) \exp[-4H_0(t - t_0)]}{\sqrt{(6K/\lambda k_4^2) \exp[-2H_0(t - t_0)] - (12\mathcal{U}_0/\lambda^2 k_4^4) \exp[-4H_0(t - t_0)] + c_0}} - \lambda \left\{ \sqrt{\frac{6K}{\lambda k_4^2} \exp[-2H_0(t - t_0)] - \frac{12\mathcal{U}_0}{\lambda^2 k_4^4} \exp[-4H_0(t - t_0)] + c_0} - 1 \right\}, \tag{49}$$

$B = 0,$

where we denoted

$$a_0 = \frac{6KH_0^2}{\lambda k_4^2 |B|}, \quad b_0 = \frac{12\mathcal{U}_0 H_0^4}{\lambda^2 k_4^4 |B|^2}, \quad c_0 = 1 - \frac{2\Lambda}{\lambda k_4^2} + \frac{6H_0^2}{\lambda k_4^2},$$

$$k_0 = \frac{2KH_0^2}{k_4^2 |B|}, \quad u_0 = \frac{8\mathcal{U}_0 H_0^4}{\lambda k_4^4 |B|^2}.$$

An important observational quantity is the deceleration parameter, defined as $q = (d/dt)H^{-1} - 1$, given by

$$q = \begin{cases} -\tanh^2[H_0(t - t_0)], & B > 0, \\ -\coth^2[H_0(t - t_0)], & B < 0, \\ -1, & B = 0. \end{cases} \quad (50)$$

There are two distinct types of behavior for the brane-world type cosmological models described by Eqs. (44)–(49). For the first model, with the scale factor having a singularity at the origin, with an appropriate choice of the parameters, the energy density can be scaled to zero at the beginning of the cosmological evolution, corresponding to $t = t_0$. Then, for times $t > t_0$, the energy density is an increasing function of time and reaches a maximum value at $t = t_{\max}$. For time intervals $t > t_{\max}$ the energy density is a decreasing function of time. Hence this model can be used to describe matter creation on the brane, a phenomenon which is entirely due to the presence of the dark matter term. If the dark energy term $\mathcal{U} = 0$, then, due to the singular behavior of the scale factor, the initial energy density and pressure of the cosmological fluid are infinite at the beginning of the cosmological evolution, $\rho, p \rightarrow \infty$ for $a \rightarrow 0$.

For the second and third class of models, the scale factor, energy density and pressure are all finite at $t = t_0$. Hence in this case the brane is filled with an initial cosmological fluid. Depending on the numerical choice of the parameters, there are also two types of cosmological behavior, with the energy density reaching its maximum value at the initial moment or at a time $t = t_{\max} > t_0$. In the first case for times larger than the initial time, the energy density and the pressure are monotonically decreasing functions for all times $t > t_0$, while in the second case ρ and p are monotonically decreasing functions of time only for $t > t_{\max}$.

The behavior of the energy density and pressure is represented, for all three cosmological models and for a particular choice of the numerical values of the parameters, in Figs. 2 and 3.

The equation of state of matter is presented in Fig. 4.

For the given choice of parameters, the pressure-energy density dependence is generally non-linear, with a dependence which can be approximated for small densities by a linear function, $p \sim \rho$. In this region the effects of the contribution from the extra-dimensions can be neglected. For high densities the non-linear effects become important, showing that the quadratic effects due to the effects of the five-dimensional bulk also modify the equation of state of the matter.

An important condition that must be satisfied by any realistic equation of state of dense matter is the requirement that the speed of sound $c_s = (dp/d\rho)^{1/2}$ be smaller or equal to the speed of light, $c_s \leq 1$. The time variation of the speed of sound in the cosmological fluid with equation of state given by Eq. (40) is represented in Fig. 5.

For the given range of parameters and in the considered time interval, the condition $c_s \leq 1$ is satisfied. During this period the speed of sound is a rapidly increasing function of time for the first solution and a slowly increasing function of time (almost a constant) for the second and third class of solutions. But for other choices of the numerical values of the constants or different time intervals this condition could be violated.

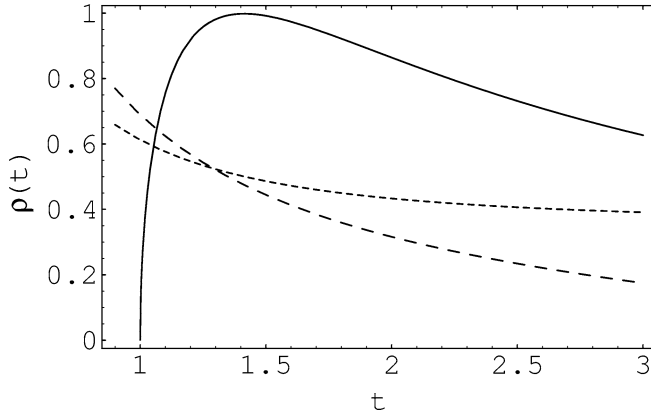


Fig. 2. Time evolution of the energy density $\rho(t)$ for the brane world cosmological model, for the three classes of solutions: first class ($a \sim \sinh[H_0(t - t_0)]$) (solid curve), second class ($a \sim \cosh[H_0(t - t_0)]$) (dotted curve) and third class ($a = \exp[H_0(t - t_0)]$) (dashed curve). We have normalized the parameters so that $2K = k_4^2$, $6K/\lambda k_4^2 = 1$, $12\mathcal{U}_0/\lambda^2 k_4^4 = 1$ and $2(3H_0^2 - \Lambda) = \lambda k_4^2$. For the sake of presentation the curves have been rescaled with different factors.

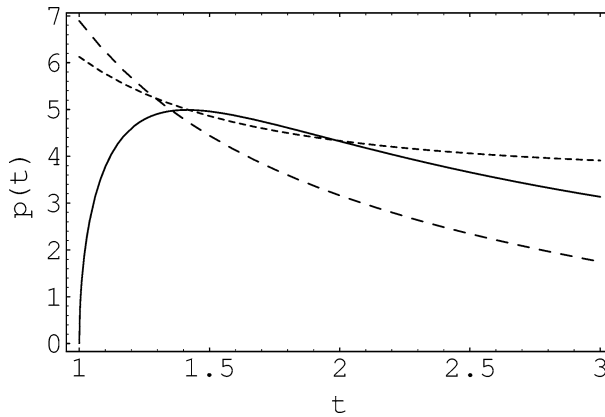


Fig. 3. Time evolution of the pressure $p(t)$ of the cosmological fluid for the brane world cosmological model, for the three classes of solutions: first class ($a \sim \sinh[H_0(t - t_0)]$) (solid curve), second class ($a \sim \cosh[H_0(t - t_0)]$) (dotted curve) and third class ($a = \exp[H_0(t - t_0)]$) (dashed curve). We have normalized the parameters so that $2K = k_4^2$, $6K/\lambda k_4^2 = 1$, $12\mathcal{U}_0/\lambda^2 k_4^4 = 1$ and $2(3H_0^2 - \Lambda) = \lambda k_4^2$. For the sake of presentation the curves have been rescaled with different factors.

Since $q < 0$ for all times, the evolution of the models is purely inflationary. However, due to the corrections from the extra-dimensions, the initial period of the inflationary phase can be associated to an increase in the energy-density of the Universe.

In all these cases, due to the rapid expansion of the brane-world, there is a fast time decay of the angular velocity $\omega \sim a^{-3}$. Since in the large time limit the scale factor is of the form $\exp(H_0 t)$, in this brane world model one obtains an exponential decay of the cosmic vorticity.

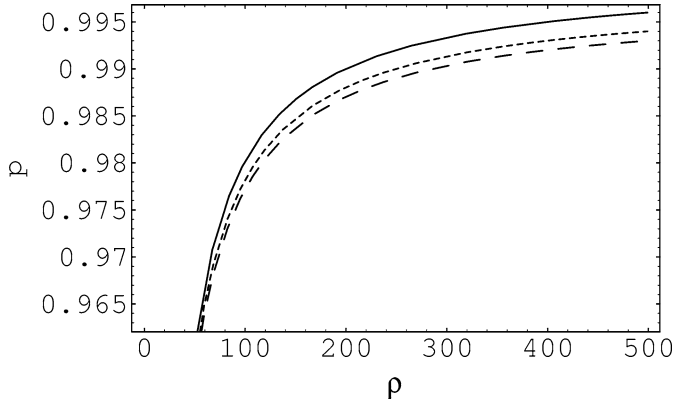


Fig. 4. Equation of state of the cosmological fluid for the brane world cosmological model, for the three classes of solutions: first class ($a \sim \sinh[H_0(t - t_0)]$) (solid curve), second class ($a \sim \cosh[H_0(t - t_0)]$) (dotted curve) and third class ($a = \exp[H_0(t - t_0)]$) (dashed curve). We have normalized the parameters so that $2K = k_4^2$, $6K/\lambda k_4^2 = 1$, $12\mathcal{U}_0/\lambda^2 k_4^4 = 1$ and $2(3H_0^2 - \Lambda) = \lambda k_4^2$. For the sake of presentation the curves have been rescaled with different factors.

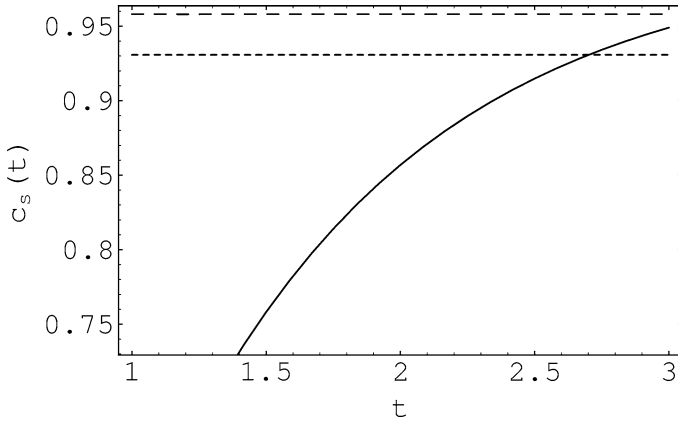


Fig. 5. Time variation of the speed of sound $c_s = (dp/d\rho)^{1/2}$ in the cosmological fluid for the three classes of solutions: first class ($a \sim \sinh[H_0(t - t_0)]$) (solid curve), second class ($a \sim \cosh[H_0(t - t_0)]$) (dotted curve) and third class ($a = \exp[H_0(t - t_0)]$) (dashed curve). We have normalized the parameters so that $2K = k_4^2$, $6K/\lambda k_4^2 = 1$, $12\mathcal{U}_0/\lambda^2 k_4^4 = 1$ and $2(3H_0^2 - \Lambda) = \lambda k_4^2$.

5. Discussions and final remarks

In the present paper we have considered the evolution of the rotational perturbations in the framework of brane world cosmology. As a first step we have obtained the basic dynamical equation governing the temporal variation and spatial distribution of the metric perturbation function $\Omega(t, r)$. This equation also contains the angular velocity of the slowly rotating matter. From the consistency condition of the rotation equation one can obtain some restrictions on the admissible mathematical form of ω . As a general result

we have shown that in the presence of dark energy, describing the effects of the bulk on the brane, rotational perturbations always decay for the slowly rotating brane. But for a vanishing dark energy term, rotational perturbations do not decay for very high density stiff matter with $\gamma = 2$, except for the case of the perfect dragging. Hence the dark energy term could have play an important role in suppressing the vorticity of the very early Universe.

In order to find the time evolution of the four-velocities of the particles on the brane we consider the dynamics of a test particle in the perturbed metric (10). The equations of motion are

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = 0,$$

where u^μ are the components of the four-velocity and the Christoffel symbols $\Gamma^\mu_{\nu\lambda}$ are computed from the metric. To simplify the calculations we consider only the first order corrections to the metric in Ω and assume that test particles have small velocities, thus retaining only terms which are linear in velocity. Consequently, we obtain

$$u^0 = 1, \quad \frac{du^1}{dt} = -\frac{2\dot{a}}{a}u^1, \quad \frac{du^2}{dt} = 0, \quad \frac{du^3}{dt} = \frac{2\dot{a}}{a}\Omega + \dot{\Omega} - \frac{2\dot{a}}{a}u^3. \quad (51)$$

By integrating Eqs. (51) we find

$$u^1 = \frac{u_0^1}{a^2(t)}, \quad u^2 = u_0^2, \quad u^3 = \frac{u_0^3}{a^2(t)} + \frac{A(r)}{a^3(t)}, \quad (52)$$

with u_0^i , $i = 1, 2, 3$ constants of integration. u_0^2 can be set to zero without altering the physical structure. In the limit of large t , $a(t) \rightarrow \infty$ and we have $u^1 = 0$ and $u^3 = 0$. At the time $t = t_\infty$ the test particle will have shifted in azimuth by $\Delta\varphi = \int_0^{t_\infty} u^3 dt$, having an angular velocity $\omega = \Delta\varphi/\Delta t$. Present-day observations impose a strong restriction on the numerical value of the angular velocity of the Universe of the form $\omega(t_\infty) < 10^{-15} \text{ year}^{-1}$. Hence in realistic cosmological models the angular velocity must tend to zero in the large time limit.

The condition of the mathematical consistency of the rotation equation also leads to a specific brane world cosmological model, with the equation of state of the matter given in a parametric form. In the high density regime the pressure-energy density dependence is nonlinear. The presence of the dark energy term has major implications on the time evolution of physical parameters. Due to the presence of \mathcal{U} , the energy density and pressure of the matter on the brane have a maximum. The increase in the energy density from zero to a maximum value can model the energy transfer from the bulk to the zero-density brane, thus leading to the possibility of a phenomenological description of matter creation on the brane. If the dark energy term is zero, the singularity in the scale factor is associated to a singular behavior of ρ and p . In the large time limit, $a \rightarrow \infty$ and

$$\rho \rightarrow \lambda \left(\sqrt{1 - \frac{2\Lambda}{\lambda k_4^2} + \frac{6H_0^2}{\lambda k_4^2}} - 1 \right),$$

with the pressure $p \rightarrow -\rho$. Since for ordinary matter p cannot be smaller than zero, it follows that in this limit ρ must be zero, condition which leads to $H_0 = \sqrt{\Lambda/3}$ and

$a = \exp(H_0 t)$. Therefore, the de Sitter solution is an attractor for this brane world model. Independently on the initial state, the Universe ends in an inflationary phase.

On the other hand, one must point out that the curvature of the brane, described by the parameter $k = -1, 0, +1$ does not play any significant role in the time evolution of this brane-world model. Since k is absorbed in the arbitrary constant B (which in fact is the most important parameter of the model), the dynamics of the matter on the brane is independent on the three-dimensional geometry of the Universe. However, the knowledge of the spatial distribution of the angular velocity could, at least in principle, allow for the determination of the separation constant K . If the numerical value and sign of K would be known, then the effects of the curvature of the brane on the temporal dynamics could also be obtained. But due to the exponential increase in the scale factor, the Universe ends in a non-rotating state with a flat geometry.

Since the equation of state of matter is unusual (even in the extreme limit of high densities the equation of state of ordinary matter is still linear with $\rho = p$), it is natural to assume that in this cosmological model the initial matter content of the Universe on the brane consists from a field rather than ordinary matter. The best candidate is a scalar field ϕ , with

$$\rho = \rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) \geq 0 \quad \text{and} \quad p = p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi),$$

where $V(\phi) \geq 0$ is the self-interaction potential. Then the time evolution of the scalar field can be immediately obtained from $\dot{\phi}(t) = \sqrt{2(\rho + p)}$, while the potential is given by $V(t) = (\rho - p)/2$. With the use of Eqs. (44)–(49) one can easily obtain the time dependence of the scalar field and scalar field potential for each class of solutions. The scalar field drives the Universe into an inflationary era. In this case solutions with negative pressure are also allowed. In the limit of large times, since the field satisfies the equation of state $\rho + p = 0$, we obtain $\dot{\phi} = 0$ and $\phi = \text{constant}$. For the scalar field potential we find $V \rightarrow \rho$, and, in the de Sitter limit, it follows $V = 0$. Hence V does not give any contribution to the cosmological constant Λ .

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