Smith Form of FIR Pseudocirculants

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***Abstract*—The pseudocirculant matrices have been found to be a very useful tool in the analysis and design of communication systems, e.g., precoding systems and discrete multitone transceivers. In these systems, a scalar channel $P(z)$ is recast into a pseudocirculant channel matrix. Many important channel properties have been derived from the Smith form and the decomposition of pseudocirculants. In this letter, we will show that the Smith form of a pseudocirculant matrix can be given in terms of the zeros of the underlying finite-impulse response (FIR) channel $P(z)$. Once the zeros of $P(z)$ are known, the Smith form of the corresponding pseudocirculant matrix can be obtained in closed form.**

***Index Terms*—Block filtering, congruous zeros, pseudocirculant, Smith form, Smith form decomposition.**

I. Introduction

PSEUDOCIRCULANT matrices have found many applications in signal processing and communication systems [1]–[7]. These matrices arise from a block filtering implementation of scalar linear time invariant filters [1]. Fig. 1 shows the block filtering representation of $P(z)$ in terms of an $N \times N$ pseudocirculant matrix $\mathbf{C}(z)$. A first detailed study of pseudocirculants is made in [2]. One very useful property shown in [2] is that pseudocirculants can be diagonalized using simple orthonormal matrices. More recently, there has been growing interest in pseudocirculants due to their applications in communication systems such as precoding systems, transceivers, or transmultiplexers [3]. In these systems, the outputs of the transmitter are in blocks of size N, and a scalar finite-impulse response (FIR) channel $P(z)$ is recast into an FIR pseudocirculant matrix $\mathbf{C}(z)$ of $N \times N$ dimensions. The matrix formulation greatly facilitates the analysis of the transmitting and receiving systems.

It is well known that polynomial matrices in z^{-1} can be diagonalized using unimodular matrices, called Smith form decomposition. The decomposition has been demonstrated to be a very useful tool for the equalization of FIR channels [4]–[7]. By including a few redundant samples in every output block of the transmitter, the FIR channel $P(z)$ can be equalized using FIR transceivers [4]. It is shown in [5] that the minimum number of redundant samples in each block can be directly linked to the Smith form of the pseudocirculant channel matrix $\mathbf{C}(z)$. In particular, the minimum redundancy for the existence of FIR transceivers is equal to the number of nontrivial terms in the Smith form. As the decomposition is given in terms of FIR matrices, it can be directly incorporated in the design of FIR transceivers such that the intersymbol-interference-free property is achieved [4]–[7].

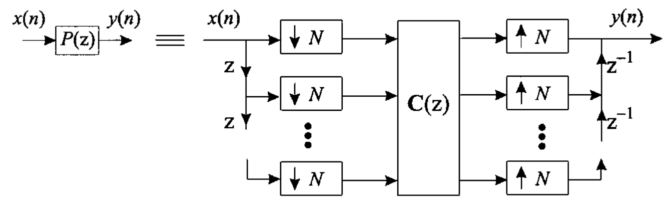

Fig. 1. Block filtering representation of a scalar filter $P(z)$.

In this letter, we show that given the scalar filter $P(z)$, the Smith form of its corresponding $N \times N$ pseudocirculant matrix $\mathbf{C}(z)$ can be given in closed form. In particular, the zeros of $P(z)$ can be grouped into sets of the so-called *congruous zeros*. Using congruous sets, we will see that the Smith form of $\mathbf{C}(z)$ can be determined by inspection.

II. Pseudocirculants and Decompositions

Suppose we are given a scalar filter $P(z)$ of order L. We can obtain the polyphase representation of $P(z)$ with respect to an integer N, where N is not necessarily larger than L. The polyphase representation is given by $P(z) = \sum_{\ell=0}^{N} P_\ell(z^N) z^{-\ell}$. The scalar filter $P(z)$ can be represented using block filtering of block size N as shown in Fig. 1. The corresponding $N \times N$ block filter is given by (1) as shown in the equation at the bottom of the next page. Matrices in the form shown in (1) are known as pseudocirculant matrices [1]. In what follows, we briefly review some known results of pseudocirculant matrices and Smith form decomposition for polynomial matrices.

1) *Diagonalizaion using orthonormal matrices* [2]. A pseudocirculant matrix $\mathbf{C}(z)$ of the form in (1) assumes the decomposition [1], [2]

$$\mathbf{C}(z^N) = \mathbf{D}(z)\mathbf{W}\mathbf{\Sigma}(z)\mathbf{W}^\dagger\mathbf{D}(z^{-1}) \quad (2)$$

where

$$\mathbf{\Sigma}(z) = \mathrm{diag}(P(z) \;\; P(zW^{-1}) \;\; \cdots \;\; P(zW^{-N+1}))$$

and

$$\mathbf{D}(z) = \mathrm{diag}(1 \;\; z^{-1} \;\; \cdots \;\; z^{-N+1}).$$

The matrix $\mathbf{W}$ is the $N \times N$ discrete Fourier transform matrix given by $[\mathbf{W}]_{kn} = (1/\sqrt{N})W^{kn}$ where $W = e^{-j2\pi/N}$.

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2) *Zeros of pseudocirculants* [5]. Suppose the underlying scalar filter $P(z)$ is a causal FIR filter of order L. Let the zeros be α_ℓ, for $\ell = 1, 2, \ldots, L$ and $P(z) = p_0 \prod_{\ell=1}^{L}(1-\alpha_\ell z^{-1})$. Then we have

$$\det \mathbf{C}(z) = p_0^N \prod_{\ell=1}^{L} \left(1 - \alpha_\ell^N z^{-1}\right). \qquad (3)$$

3) *Smith form decomposition.* An $N \times N$ polynomial matrix $\mathbf{C}(z)$ in z^{-1} can be represented using the Smith form decomposition [1]

$$\mathbf{C}(z) = \mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{V}(z) \qquad (4)$$

where all three matrices in the decomposition are matrix polynomials in the variable z^{-1}. The matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ are unimodular matrices, i.e., matrices with determinants equal to a constant; $\mathbf{\Gamma}(z)$ is a diagonal matrix $\mathbf{\Gamma}(z) = \mathrm{diag}(\gamma_0(z)\ \gamma_1(z)\ \cdots\ \gamma_{N-1}(z))$. Moreover, the unimodular matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ can be so chosen that the polynomials $\gamma_k(z)$ are monic (i.e., highest power has unity coefficient) and such that $\gamma_k(z)$ is a factor of $\gamma_{k+1}(z)$, i.e., $\gamma_k(z)$ divides $\gamma_{k+1}(z)$. The matrix $\mathbf{\Gamma}(z)$—called the *Smith form* of $\mathbf{C}(z)$—is unique. But the matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ are not unique. The decomposition can be obtained using a finite number of elementary row and column operations [1].

Note that from (2) we have

$$\mathbf{C}(z) = \mathbf{D}(z^{1/N})\mathbf{W}\mathbf{\Sigma}(z^{1/N})\mathbf{W}^{\dagger}\mathbf{D}(z^{-1/N}).$$

The expression contains $z^{-1/N}$, a fraction of a delay, which cannot be realized with a finite cost. On the other hand, matrices in the Smith decomposition are all FIR, and they can be used directly in FIR transceiver designs.

III. Smith Form of Pseudocirculants

In this section, we will derive the Smith form $\mathbf{\Gamma}(z)$ of the pseudocirculant matrix $\mathbf{C}(z)$. Given the zeros $\{\alpha_\ell\}_{\ell=1}^{L}$ of the scalar filter $P(z)$, we will see that the diagonal terms $\gamma_k(z)$ of the Smith form can be given in terms of α_ℓ. Consider the Smith form decomposition in (4). Since $\det \mathbf{U}(z)$ and $\det \mathbf{V}(z)$ are both constants, we have

$$\det \mathbf{C}(z) = c \det \mathbf{\Gamma}(z) = c\Pi_{k=0}^{N-1}\gamma_k(z)$$

where $c = \det \mathbf{U}(z) \det \mathbf{V}(z)$. This means that the zeros of $\gamma_k(z)$ are those of $\mathbf{C}(z)$.

Definition 1: Congruous Zeros: A set of zeros $\{\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_q}\}$ of $P(z)$ are congruous with respect to N if 1) $\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_q}$ are distinct and 2) $\alpha_{k_1}^N = \alpha_{k_2}^N = \cdots = \alpha_{k_q}^N$.

The zeros that are congruous are distinct; however, their magnitudes are the same, and their angles differ by an integer multiple of $2\pi/N$. They can be expressed as the rotation of each other:

$$\alpha_{k_j} = \alpha_{k_1} W^{-n_j}, \qquad \text{where } W = e^{-j2\pi/N}, \quad 1 \le n_j < N, \quad j = 1, 2, \ldots, q. \qquad (5)$$

The numbers n_j are distinct.

Lemma 1: Let $\mathbf{C}(z)$ be a pseudocirculant matrix with diagonalization and Smith form decomposition as given in (2) and (4). Then, $\mathrm{rank}(\mathbf{C}(z^N)) = \mathrm{rank}(\mathbf{\Sigma}(z)) = \mathrm{rank}(\mathbf{\Gamma}(z^N))$.

Proof: The rank of $\mathbf{C}(z^N)$ is the same as the rank of $\mathbf{\Sigma}(z)$ because $\mathbf{W}$ and $\mathbf{D}(z)$ in (2) are nonsingular. On the other hand, the matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ in the Smith form decomposition (4) are unimodular; they are nonsingular for all z. Therefore, the rank of $\mathbf{C}(z^N)$, $\mathbf{\Sigma}(z)$, and $\mathbf{\Gamma}(z^N)$ are the same for all z.

Lemma 2: Let $B = \{\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_q}\}$ be a set of congruous zeros with respect to N. Suppose no other zeros can be included in B to form a larger congruous set. Then

1) $\mathrm{rank}(\mathbf{C}(\alpha_{k_1}^N)) = N - q$.
2) Diagonal terms $\gamma_k(z)$ in the Smith form of $\mathbf{C}(z)$ satisfy the property that exactly q terms have the factor $(1 - \alpha_{k_1}^N z^{-1})$. The q terms are $\gamma_{N-q}(z), \gamma_{N-q+1}(z), \ldots, \gamma_{N-1}(z)$.

Proof: As the zeros $\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_q}$ are congruous, they can be expressed as in (5). Consider the terms on the diagonal of $\mathbf{\Sigma}(z)$ in (2). Observe that $P(zW^{-n_j})|_{z=\alpha_{k_1}} = P(\alpha_{k_1}W^{-n_j}) = P(\alpha_{k_j}) = 0, j = 1, 2, \ldots, q$. Therefore, we have $\mathrm{rank}(\mathbf{\Sigma}(\alpha_{k_1})) = N - q$. By Lemma 1, this implies that $\mathrm{rank}(\mathbf{C}(\alpha_{k_1}^N)) = \mathrm{rank}(\mathbf{\Gamma}(\alpha_{k_1}^N)) = N - q$. This, in terms, means that the diagonal terms of the Smith form contain the factor $(1 - \alpha_{k_1}^N z^{-1})$. As $\gamma_k(z)$ divides $\gamma_{k+1}(z)$, we arrive at the second result of the lemma.

Lemma 2 provides us with the link between congruous zeros of the scalar filter $P(z)$ and the Smith form of $\mathbf{C}(z)$. Let us partition the zeros of $P(z)$ into sets of congruous zeros. Each set contains either congruous zeros or a single zero, and no two sets can be combined to form a larger congruous set. In this case, the number of congruous sets is minimum. Suppose there are a total of s congruous sets $B_1, B_2, \ldots, B_s$. Denoting the cardinality of B_j as ℓ_j, we have $\sum_{j=1}^{s} \ell_j = L$. Without loss of generality, we assume $\ell_1 \le \ell_2 \cdots \le \ell_s$. Let

$$B_j = \{\alpha_{j,1}, \alpha_{j,2}, \ldots, \alpha_{j,\ell_j}\}, \qquad j = 1, 2, \ldots, s. \qquad (6)$$

Using Lemma 2, we see that $\gamma_{N-k}(z)$, for $k = 1, 2, \ldots, \ell_j$ contains the factor $(1 - \alpha_{j,1}^N z^{-1})$. In other words, $\gamma_{N-k}(z)$ contains

$$\mathbf{C}(z) = \begin{pmatrix} P_0(z) & z^{-1}P_{N-1}(z) & z^{-1}P_{N-2}(z) & \cdots & z^{-1}P_1(z) \\ P_1(z) & P_0(z) & z^{-1}P_{N-1}(z) & \cdots & z^{-1}P_2(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{N-1}(z) & P_{N-2}(z) & P_{N-3}(z) & \cdots & P_0(z) \end{pmatrix}_{N \times N} \qquad (1)$$

every factor $(1-\alpha_{j,1}^N z^{-1})$ with $\ell_j \geq k$. For $k > \ell_s$, we have $\gamma_{N-k}(z) = 1$. Summarizing, we have the following theorem.

Theorem 1: Let the zeros of $P(z)$ be partitioned into a minimum number of congruous sets $B_1, B_2, \ldots, B_s$, where B_j are as given in (6). Then, the Smith form $\mathbf{\Gamma}(z)$ of the pseudocirculant $\mathbf{C}(z)$ has diagonal terms given by

$$\gamma_{N-k}(z) = \begin{cases} \prod_{\substack{j=1,\\ \ell_j \geq k}}^{s} (1-\alpha_{j,1}^N z^{-1}), & k = 1, 2, \ldots, \ell_s \\ 1, & \text{otherwise.} \end{cases} \tag{7}$$

Using this above theorem, one can determine the Smith form of any $N \times N$ FIR pseudocirculant matrix $\mathbf{C}(z)$ by inspection once the zeros of the corresponding scalar filter $P(z)$ are known. Theorem 1 implies that the number of nontrivial terms in the Smith form is ℓ_s, i.e., the cardinality of the largest congruous set. This property has an important application in transceiver design, where the minimum redundancy for the existence of FIR transceivers is equal to the number of nontrivial terms in the Smith form [5].

Example: Suppose the scalar filter $P(z)$ has the order $L = 4$, and let the zeros of the scalar filter $P(z)$ be $\{1, 1, e^{j\pi/2}, e^{-j\pi/2}\}$. Let us consider the Smith form $\mathbf{\Gamma}(z)$ of the pseudocirculant matrix for $N = 2, 3, 4$.

1) $N = 2$. In this case, $e^{j\pi/2}$ and $e^{-j\pi/2}$ are congruous, and no other zero is congruous with another. The minimum number of congruous sets is three. The congruous sets are $\{1\}, \{1\}$, and $\{e^{j\pi/2}, e^{-j\pi/2}\}$. The zeros of $\det \mathbf{C}(z)$ are $\{1\}, \{1\}$, and $\{-1, -1\}$. We have $\gamma_0(z) = (1+z^{-1})$ and $\gamma_1(z) = (1+z^{-1})(1-z^{-1})^2$.
2) $N = 3$. In this case, no two zeros are congruous. Each congruous set has only one entry: $\{e^{j\pi/2}\}$, $\{e^{-j\pi/2}\}$, $\{1\}$, or $\{1\}$, and the zeros of $\det \mathbf{C}(z)$ are $\{e^{j3\pi/2}\}$, $\{e^{-3j\pi/2}\}$, $\{1\}$, and $\{1\}$. We have $\gamma_0(z) = \gamma_1(z) = 1$ and $\gamma_2(z) = (1+z^{-2})(1-z^{-1})^2$.
3) $N = 4$. There are two congruous sets: $\{1\}$ and $\{1, e^{j\pi/2}, ^{-j\pi/2}\}$. The zeros of $\det \mathbf{C}(z)$ are $\{1\}$ and $\{1, 1, 1\}$. Therefore, $\gamma_0(z) = 1$, $\gamma_1(z) = \gamma_2(z) = 1 - z^{-1}$, and $\gamma_3(z) = (1-z^{-1})^2$.

IV. Concluding Remarks

In the application of FIR precoding systems or transceivers, an FIR channel $P(z)$ is often recast as a pseudocirculant matrix $\mathbf{C}(z)$. With the matrix formulation, Smith form decomposition can be employed for the design of FIR transceivers that use redundant samples to equalize the FIR channel $P(z)$. It has also been shown that for FIR transceiver solutions the minimum redundancy is equal to the number of nontrivial terms in the Smith form of $\mathbf{C}(z)$. In this letter, we show that given the FIR channel $P(z)$, the Smith form of the pseudocirculant matrix $\mathbf{C}(z)$ can be given in closed form, and as a result, the minimum redundancy for designing FIR transceivers can be determined by inspection.

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