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ON TWO-PHASE FLOW IN FRACTURED MEDIA

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A model describing two-phase, incompressible, immiscible flow in fractured media is discussed. A fractured medium is regarded as a porous medium consisting of two superimposed continua, a continuous fracture system and a discontinuous system of medium-sized matrix blocks. Transport of fluids through the medium is primarily within the fracture system. No flow is allowed between blocks, and only matrix-fracture flow is possible. Matrix block system plays the role of a global source distributed over the entire medium. Two-phase flow in a fractured medium is strongly related to phase mobilities and capillary pressures. In this work, four relations for these functions are presented, and the existence of weak solutions under each relation will also be shown.

Keywords: Two-phase flow; fractured media; capillary pressure.

AMS Subject Classification: 35B27, 35B45, 35K65, 76S05

1. Introduction

A dual-porosity model describing two-phase, incompressible, immiscible flow in fractured media is discussed. The phases are the nonwetting "o" (oil) phase and the wetting "w" (water) phase. Within a fractured medium there is an interconnected system of fracture planes dividing the porous medium into a collection of matrix blocks. The fracture planes, while very thin, form paths of high permeability. Most of the fluids reside in matrix blocks, where they move very slow. For model considered here, a fractured medium is regarded as a porous medium consisting of two superimposed continua, a continuous fracture system and a discontinuous system of medium-sized matrix blocks. Fracture system has a lower storage and higher conductivity than matrix block system. Transport of fluids through the medium is primarily within the fracture system. No flow is allowed between blocks, and fluids that reside in matrix blocks must enter the fractures before shifting. Essentially, matrix block system plays the role of a global source distributed over the entire medium. As a consequence, two sets of equations are obtained for the flow. One contains macroscopic equations for fracture flow, and the other consists of microscopic equations for flow in matrix blocks. The two sets of equations are coupled through locally defined macroscopic matrix-fracture sources, one for each phase. For more description of flow in the medium, readers are referred to Refs. 5, 7, 10, 12 and 13 and references therein.

If $\Omega \subset \mathbb{R}^3$ is a fractured medium, equations for fracture flow Refs. 5, 10 are, for $x \in \Omega$, $t > 0$,

$$\partial_t S - \nabla_x \cdot (\Lambda_{\rm w}(S)\nabla_x(P_{\rm w} - E_{\rm w})) = q_{\rm w}\,, \tag{1.1}$$

$$-\partial_t S - \nabla_x \cdot (\Lambda_{\rm o}(S)\nabla_x(P_{\rm o} - E_{\rm o})) = q_{\rm o}\,, \tag{1.2}$$

$$\Upsilon(S) = P_{\rm o} - P_{\rm w}\,. \tag{1.3}$$

$S \in [0,1]$ is water saturation; Λ_α ($\alpha = {\rm w}, {\rm o}$) is phase mobility of α-phase, a non-negative monotone function of S (see Fig. 1); P_α denotes pressure; E_α is a function depending on density, gravity, and position; q_α is the matrix-fracture source; and Υ is capillary pressure, a non-negative decreasing function of S (see Fig. 1). Porosity and permeability field have been set to 1 for convenience. Incompressibility implies $q_{\rm o} + q_{\rm w} = 0$.

A matrix block $\Omega_x \subset \mathbb{R}^3$ is suspended topologically above each point $x \in \Omega$. Equations for flow in a matrix block are, for $x \in \Omega$, $y \in \Omega_x$, $t > 0$,

$$\partial_t s - \nabla_y \cdot (\lambda_{\rm w}(s)\nabla_y p_{\rm w}) = 0\,, \tag{1.4}$$

$$-\partial_t s - \nabla_y \cdot (\lambda_{\rm o}(s)\nabla_y p_{\rm o}) = 0\,, \tag{1.5}$$

$$\upsilon(s) = p_{\rm o} - p_{\rm w}\,. \tag{1.6}$$

Each lower case symbol denotes the quantity on Ω_x corresponding to that denoted by an upper case symbol in the fracture system equations. S, P_α, q_α for $\alpha \in \{{\rm w}, {\rm o}\}$ in (1.1)–(1.3) are functions on $\Omega \times [0,T]$, and s, p_α in (1.4)–(1.6) are on $\prod_{x\in\Omega} \Omega_x \times [0,T]$. p_α ($\alpha = {\rm w}, {\rm o}$) in (1.4) and (1.5) only takes derivative with respect to variable y.

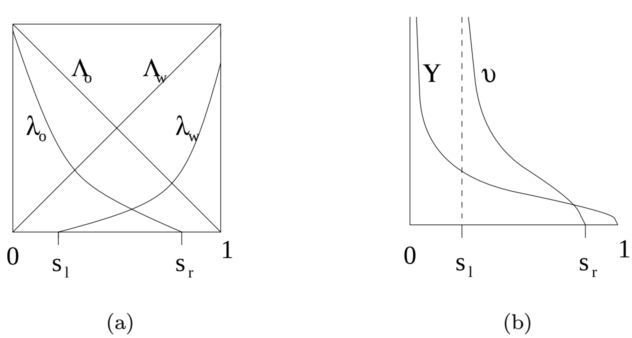

Fig. 1. (a) Phase mobilities and (b) capillary pressures of fracture system and matrix blocks.

The matrix-fracture sources are given by, for $x \in \Omega$, $t > 0$,

$$q_{\mathrm{w}}(x,t) = \frac{-1}{|\Omega_x|} \int_{\Omega_x} \partial_t s(x,y,t)dy = -q_{\mathrm{o}}(x,t)\,, \tag{1.7}$$

where $|\Omega_x|$ is the volume of Ω_x. Boundary $\partial\Omega$ of Ω includes $\partial_1\Omega$, $\partial_2\Omega$, which satisfying $\partial_1\Omega \cap \partial_2\Omega = \emptyset$, $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega$. Boundary conditions for fracture system are, for $t > 0$, $\alpha \in \{\mathrm{w}, \mathrm{o}\}$,

$$P_\alpha = P_{\alpha,b}\,, \qquad \text{for } x \in \partial_1\Omega\,, \tag{1.8}$$

$$\Lambda_\alpha(S)\nabla_x(P_\alpha - E_\alpha)\cdot\vec{n} = 0\,, \quad \text{for } x \in \partial_2\Omega\,, \tag{1.9}$$

where $\vec{n}$ is the unit vector outward normal to $\partial\Omega$. Boundary conditions for each matrix block require continuity of pressures, i.e. for $t > 0$, $x \in \Omega$, $y \in \partial\Omega_x$, $\alpha \in \{\mathrm{w}, \mathrm{o}\}$,

$$p_\alpha(x,y,t) = P_\alpha(x,t)\,. \tag{1.10}$$

Initial equilibrium gives

$$S(x,0) = S_0(x)\,, \qquad \text{for } x \in \Omega\,, \tag{1.11}$$

$$s(x,y,0) = s_0(x)\,, \quad \text{for } x \in \Omega\,, \quad y \in \Omega_x\,. \tag{1.12}$$

Two-phase flow in fractured media is strongly related to phase mobilities and capillary pressures.[10–13] For flow in a bundle of tubes, a mobility curve was measured to be a linear function of phase saturation. In general, phase mobility curves may be determined by being adjusted to history-match field data if all other data are known. Fracture capillary pressure would be near zero for most water saturation values. Matrix mobilities and matrix capillary pressure can be those measured on unfractured media. To maintain gravity/capillary equilibrium, capillary pressure endpoints in fracture system and matrix blocks must be set equal.[12,13] In reality, it is not easy to measure phase mobilities and capillary pressures accurately. Our intention is to look for proper relations for these functions. Some literatures related to this problem are listed below. For unfractured media case (that is, $q_{\mathrm{w}} = q_{\mathrm{o}} = 0$), existence of solutions of (1.1)–(1.3) were studied in Refs. 3, 4, 8 and 14 and references therein. If one linearizes matrix mobility λ_α ($\alpha = \mathrm{w}, \mathrm{o}$) or assumes matrix blocks are small, matrix-fracture source q_α is a function of phase saturation. Existence of weak solutions in these cases were considered in Refs. 6 and 9. Existence of solutions in a global pressure form of (1.1)–(1.12) could be found in Refs. 7 and 17. In this work, four relations for phase mobilities and capillary pressures are presented. Existence of weak solutions of (1.1)–(1.12) will be shown for each relation. To reach the goal, a global pressure is introduced to simplify system (1.1)–(1.12) first. Next, existence of solutions of the simplified system will be shown. Finally, we prove that a subsequence of these solutions converges to a weak solution of (1.1)–(1.12). Phase mobilities and capillary pressures in Refs. 10–13 satisfy one of the relations here.

The rest of the paper is organized as follows: notation is recalled and main result is stated in Sec. 2. An auxiliary system for (1.1)–(1.12) is derived and the procedure of proof for main result is described in Sec. 3. The main result is proved in Sec. 4 under the assumption of the existence of solution for auxiliary system, which is shown in Sec. 5.

2. Notation and Main Result

2.1. *Notation*

Let $\Omega \subset \mathbb{R}^3$ be open, bounded, and connected with Lipschitz boundary. For every $x \in \Omega$, $\Omega_x \subset \mathbb{R}^3$ is a bounded region. Identify the product space $\prod_{x\in\Omega} \Omega_x$ (denoted by $\mathcal{Q}$) as a subset of $\mathbb{R}^6$. For simplicity, all matrix blocks are assumed to be identical, volume 1, and smooth enough. That is, $\mathcal{Q} = \Omega \times \mathcal{M}$, $|\mathcal{M}| = 1$, and $\mathcal{M} \subset \mathbb{R}^3$ is assumed to be bounded with Lipschitz boundary $\partial\mathcal{M}$. $\Omega^t \overset{\text{def}}{=} \Omega \times [0,t]$ and $\mathcal{Q}^t \overset{\text{def}}{=} \mathcal{Q} \times [0,t]$.

$L^r(\mathcal{B})$, $H^m(\mathcal{B})$, $W^{m,r}(\mathcal{B})$, $L^r(\Omega, W^{m,r}(\mathcal{M}))$, $L^r(\Omega, L^r(\partial\mathcal{M}))$, $L^r(0,T;X)$, and $H^m(0,T;X)$ are Sobolev spaces[1] for $r > 1$, $m \in \mathbf{N}$, $\mathcal{B} \subset \mathcal{Q}^{\mathrm{T}}$, and a Banach space X.

$$\begin{cases} \mathcal{W}_0^{m,r}(\Omega) \overset{\text{def}}{=} \{f \in W^{m,r}(\Omega) : f|_{\partial_1\Omega} = 0\}\,, \\ \mathcal{V} \overset{\text{def}}{=} \mathcal{W}_0^{1,2}(\Omega)\,, \\ \mathcal{W}_y^{1,r}(\mathcal{Q}) \overset{\text{def}}{=} \{f \in L^r(\mathcal{Q}) : \nabla_y f \in L^r(\mathcal{Q})\}\,, \\ \mathcal{U} \overset{\text{def}}{=} \mathcal{W}_y^{1,2}(\mathcal{Q})\,. \end{cases}$$

Note that $\mathcal{W}_y^{1,r}(\mathcal{Q}) \subset L^r(\Omega, W^{1,r}(\mathcal{M}))$. Let $\mathcal{T}_x$ be the usual trace map of $W^{1,r}(\mathcal{M})$ into $L^r(\partial\mathcal{M})$. We define the distributed trace $\mathcal{T} : \mathcal{W}_y^{1,r}(\mathcal{Q}) \to L^r(\Omega, L^r(\partial\mathcal{M}))$ by $\mathcal{T}f(x,y) = (\mathcal{T}_x f(x))(y)$.

$$\begin{cases} \mathcal{W}_{y,0}^{1,r}(\mathcal{Q}) \overset{\text{def}}{=} \{f \in \mathcal{W}_y^{1,r}(\mathcal{Q}) : \mathcal{T}f = 0\}\,, \\ \mathcal{U}_0 \overset{\text{def}}{=} \mathcal{W}_{y,0}^{1,2}(\mathcal{Q})\,, \\ \mathcal{W}_1 \overset{\text{def}}{=} \mathcal{V} \times \mathcal{V} \times \mathcal{U}_0\,, \\ \mathcal{W}_2 \overset{\text{def}}{=} \mathcal{V} \times \mathcal{V} \times \mathcal{U}_0 \times \mathcal{U}_0\,, \\ \text{dual } X \overset{\text{def}}{=} \text{dual space of } X\,, \\ s_l \text{ (resp. } 1 - s_r) \overset{\text{def}}{=} \text{residual matrix water (resp. oil) saturation.} \end{cases}$$

$\mathbb{R}_0^+ \overset{\text{def}}{=} \mathbb{R}^+ \cup \{0\}$. If $\Upsilon : (0,1] \to \mathbb{R}_0^+$ (resp. $\upsilon : (s_l, s_r] \to \mathbb{R}_0^+$) is onto and a strictly decreasing function, let Υ^{-1} (resp. υ^{-1}) be the inverse function of Υ (resp. υ). We define $\mathcal{J} : (0,1] \to (s_l, s_r]$ by $\mathcal{J}(z) \overset{\text{def}}{=} \upsilon^{-1}(\Upsilon(z))$, and denote the inverse function of $\mathcal{J}$ by $\mathcal{J}^{-1}$. Let $\mathcal{J}(0.5) \in (s_l, s_r) \subset (0,1)$.

$$\begin{cases} \partial^h f(t) \stackrel{\text{def}}{=} \dfrac{f(t+h)-f(t)}{h}\,, \\ P_{c,b} \stackrel{\text{def}}{=} P_{\text{o},b} - P_{\text{w},b}\,, \\ \Lambda \stackrel{\text{def}}{=} \Lambda_{\text{w}} + \Lambda_{\text{o}}\,, \\ \lambda \stackrel{\text{def}}{=} \lambda_{\text{w}} + \lambda_{\text{o}}\,, \\ \mathbf{R}(z) \stackrel{\text{def}}{=} \displaystyle\int_{0.5}^{z} \frac{\Lambda_{\text{w}}\Lambda_{\text{o}}}{\Lambda}\left|\frac{d\Upsilon}{dS}\right|(\xi)d\xi\,, \qquad \text{for } z \in (0,1]\,, \\ \mathcal{D}(z) \stackrel{\text{def}}{=} \displaystyle\int_{\mathcal{J}(0.5)}^{z} \frac{\lambda_{\text{w}}\lambda_{\text{o}}}{\lambda}\left|\frac{dv}{ds}\right|(\xi)d\xi\,, \quad \text{for } z \in (s_l, s_r]\,. \end{cases} \tag{2.1}$$

We define $\mathcal{L} : L^r(\Omega) \to L^r(\Omega, L^\infty(\mathcal{M}))$ by $\mathcal{L}f(x,y) = f(x)1_y$, $x \in \Omega$, $y \in \mathcal{M}$, where $f(x)1_y$ is constant in $\mathcal{M}$ with value $f(x)$. $f \in L^r(\Omega)$ will be identified with $\mathcal{L}f \in L^r(\Omega, L^\infty(\mathcal{M}))$.

2.2. *Main result*

Taking $(\zeta_{\text{w}}, \zeta_{\text{o}}, \eta_{\text{w}}, \eta_{\text{o}}) \in L^2(0,T;\mathcal{W}_2)$, multiplying (1.1), (1.2), (1.4), (1.5) by $\zeta_{\text{w}}, \zeta_{\text{o}}$, $\eta_{\text{w}}, \eta_{\text{o}}$ respectively, and integrating these functions over $\mathcal{Q}^{\text{T}}$, one obtains a weak formulation for Eqs. (1.1)–(1.2) and (1.4)–(1.5), by (1.8)–(1.10),

$$\int_{\Omega^{\text{T}}} \partial_t S\ \zeta_{\text{w}} + \int_{\Omega^{\text{T}}} \Lambda_{\text{w}}(S)\nabla_x(P_{\text{w}} - E_{\text{w}})\nabla_x\zeta_{\text{w}} = -\int_{\mathcal{Q}^{\text{T}}} \partial_t s\ \zeta_{\text{w}}\,, \tag{2.2}$$

$$-\int_{\Omega^{\text{T}}} \partial_t S\ \zeta_{\text{o}} + \int_{\Omega^{\text{T}}} \Lambda_{\text{o}}(S)\nabla_x(P_{\text{o}} - E_{\text{o}})\nabla_x\zeta_{\text{o}} = \int_{\mathcal{Q}^{\text{T}}} \partial_t s\ \zeta_{\text{o}}\,, \tag{2.3}$$

$$\int_{\mathcal{Q}^{\text{T}}} \partial_t s\ \eta_{\text{w}} + \int_{\mathcal{Q}^{\text{T}}} \lambda_{\text{w}}(s)\nabla_y p_{\text{w}}\nabla_y\eta_{\text{w}} = 0\,, \tag{2.4}$$

$$-\int_{\mathcal{Q}^{\text{T}}} \partial_t s\ \eta_{\text{o}} + \int_{\mathcal{Q}^{\text{T}}} \lambda_{\text{o}}(s)\nabla_y p_{\text{o}}\nabla_y\eta_{\text{o}} = 0\,. \tag{2.5}$$

Definition 2.1. $\{S, P_{\text{w}}, P_{\text{o}}, s, p_{\text{w}}, p_{\text{o}}\}$ is a weak solution of Eqs. (1.1)–(1.12) if there is a number $r \in (1,2)$ such that, for $\alpha \in \{\text{w},\text{o}\}$,

1. $P_\alpha - P_{\alpha,b} \in L^r(0,T;\mathcal{W}_0^{1,r}(\Omega))$, $p_\alpha - P_\alpha \in L^r(0,T;\mathcal{W}_{y,0}^{1,r}(\mathcal{Q}))$,
2. $\partial_t S + \int_{\mathcal{M}} \partial_t s\ dy \in$ dual $L^2(0,T;\mathcal{V})$, $\partial_t s \in$ dual $L^2(0,T;\mathcal{U}_0)$,
3. $\Lambda_\alpha \nabla_x P_\alpha \in L^2(\Omega^{\text{T}})$, $\lambda_\alpha \nabla_y p_\alpha \in L^2(\mathcal{Q}^{\text{T}})$,

4. $\Upsilon(S) = P_{\rm o} - P_{\rm w}\,, \quad \upsilon(s) = p_{\rm o} - p_{\rm w}$,
5. (2.2)–(2.5) hold for any $\zeta_\alpha \in L^2(0,T;\mathcal{V})$, $\eta_\alpha \in L^2(0,T;\mathcal{U}_0)$,
6. $0 < S < 1$, $s_l < s < s_r$,
7. For $\zeta \in L^2(0,T;\mathcal{V}) \cap H^1(\Omega^{\rm T})$, $\eta \in L^2(0,T;\mathcal{U}) \cap H^1(0,T;L^2(\mathcal{Q}))$, $\zeta(T) = \eta(T) = 0$,

$$\int_{\Omega^{\rm T}} \partial_t S\, \zeta + \int_{\mathcal{Q}^{\rm T}} \partial_t s\, \eta = -\int_{\Omega^{\rm T}} (S - S_0)\partial_t \zeta - \int_{\mathcal{Q}^{\rm T}} (s - s_0)\partial_t \eta\,. \tag{2.6}$$

Theorem 2.1. *A weak solution of Eqs.* (1.1)–(1.12) *exists if the following conditions hold*:

A1. $\partial_1 \Omega \neq \emptyset$.

A2. $\Lambda_{\rm w}, \lambda_{\rm w}$ (*resp.* $\Lambda_{\rm o}, \lambda_{\rm o}$) : $[0,1] \to [0,1]$ *are continuous and increasing* (*resp. decreasing*), $\Lambda_{\rm w}(0) = \Lambda_{\rm o}(1) = \lambda_{\rm w}(s_l) = \lambda_{\rm o}(s_r) = 0$, $\Lambda_{\rm w}\Lambda_{\rm o}(z)|_{z\in(0,1)} \neq 0$, $\lambda_{\rm w}\lambda_{\rm o}(z)|_{z\in(s_l,s_r)} \neq 0$, $\inf_{z\in(0,1]}\{\Lambda(z), \lambda(z)\} > 0$.

A3. $\Upsilon : (0,1] \to \mathbb{R}_0^+$ ($\upsilon : (s_l, s_r] \to \mathbb{R}_0^+$) *is onto, decreasing, and a locally Lipschitz continuous function, and* $\inf_{z\in(0,1]} \left|\frac{d\Upsilon}{dS}\right| \times \inf_{z\in(s_l,s_r]} \left|\frac{d\upsilon}{ds}\right| > 0$.

A4. $\partial_t P_{c,b} \in L^1(\Omega^{\rm T})$, $E_\alpha \in L^\infty(0,T;W^{1,\infty}(\Omega))$, $P_{\alpha,b} \in L^2(0,T;H^1(\Omega))$, $\alpha = {\rm w}, {\rm o}$.

A5. $\mathbf{k}_1 \le \Upsilon^{-1}(P_{c,b}) \le 1 - \mathbf{k}_1$, $\mathbf{k}_1 \le S_0(x) \le 1 - \mathbf{k}_1$, $\upsilon(s_0(x)) = \Upsilon(S_0(x))$, $x \in \Omega$.

A6. $\min\{\overline{\lim}_{\xi\to 0} \frac{\Lambda_{\rm w}(\xi)}{\lambda_{\rm w}(\mathcal{J}(\xi))}, \overline{\lim}_{\xi\to 0} \frac{\lambda_{\rm w}(\mathcal{J}(\xi))}{\Lambda_{\rm w}(\xi)}\} < \infty$,
$\min\{\overline{\lim}_{\xi\to 1} \frac{\Lambda_{\rm o}(\xi)}{\lambda_{\rm o}(\mathcal{J}(\xi))}, \overline{\lim}_{\xi\to 1} \frac{\lambda_{\rm o}(\mathcal{J}(\xi))}{\Lambda_{\rm o}(\xi)}\} < \infty$.

A7. *One of the following conditions is satisfied*:

$$\text{(a)} \begin{cases} \overline{\lim}_{z\to 0} \dfrac{\Lambda_{\rm w}(z)}{|z|^2 \left|\frac{d\Upsilon}{dS}(z)\right|} + \overline{\lim}_{z\to 1} \dfrac{\Lambda_{\rm o}(z)}{|1-z|^2 \left|\frac{d\Upsilon}{dS}(z)\right|} + \sup_{z\in(0,1]} \dfrac{\frac{d\upsilon}{ds}(\mathcal{J}(z))}{\frac{d\Upsilon}{dS}(z)} < \infty\,, \\ \inf_{z\in(0,1]} \dfrac{\Lambda_{\rm w}(z)\Lambda_{\rm o}(z)}{|z(1-z)|^{\mathbf{k}_2}} \times \inf_{z\in(0,1]} \dfrac{\lambda_{\rm w}(\mathcal{J}(z))\lambda_{\rm o}(\mathcal{J}(z))}{|z(1-z)|^{\mathbf{k}_2}} > 0\,, \\ \underline{\lim}_{z\to 0} \dfrac{1}{|z|^{\mathbf{k}_2}|\mathcal{D}(\mathcal{J}(z))|} > 0\,, \end{cases}$$

$$\text{(b)} \begin{cases} \overline{\lim}_{z\to s_l} \dfrac{\Lambda_{\rm w}(\mathcal{J}^{-1}(z))}{|z-s_l|^2 \left|\frac{d\upsilon}{ds}(z)\right|} + \overline{\lim}_{z\to s_r} \dfrac{\Lambda_{\rm o}(\mathcal{J}^{-1}(z))}{|s_r-z|^2 \left|\frac{d\upsilon}{ds}(z)\right|} + \sup_{z\in(s_l,s_r]} \dfrac{\frac{d\Upsilon}{dS}(\mathcal{J}^{-1}(z))}{\frac{d\upsilon}{ds}(z)} < \infty\,, \\ \inf_{z\in(s_l,s_r]} \dfrac{\Lambda_{\rm w}(\mathcal{J}^{-1}(z))\Lambda_{\rm o}(\mathcal{J}^{-1}(z))}{|(z-s_l)(s_r-z)|^{\mathbf{k}_2}} \times \inf_{z\in(s_l,s_r]} \dfrac{\lambda_{\rm w}(z)\lambda_{\rm o}(z)}{|(z-s_l)(s_r-z)|^{\mathbf{k}_2}} > 0\,, \\ \underline{\lim}_{z\to s_l} \dfrac{1}{|z-s_l|^{\mathbf{k}_2}|\mathcal{D}(z)|} > 0\,, \end{cases}$$

$$\text{(c)} \begin{cases} \overline{\lim_{z\to 0}} \dfrac{\Lambda_{\rm w}(z)}{|z|^2 \left|\frac{d\Upsilon}{dS}(z)\right|} + \overline{\lim_{z\to 0}} \dfrac{\frac{dv}{ds}(\mathcal{J}(z))}{\frac{d\Upsilon}{dS}(z)} < \infty\,, \\ \overline{\lim_{z\to s_r}} \dfrac{\Lambda_{\rm o}(\mathcal{J}^{-1}(z))}{|s_r - z|^2 \left|\frac{dv}{ds}(z)\right|} + \overline{\lim_{z\to s_r}} \dfrac{\frac{d\Upsilon}{dS}(\mathcal{J}^{-1}(z))}{\frac{dv}{ds}(z)} < \infty\,, \\ \underline{\lim_{z\to 0}} \dfrac{\Lambda_{\rm w}(z)}{|z|^{\mathbf{k}_2}} \times \underline{\lim_{z\to 0}} \dfrac{\lambda_{\rm w}(\mathcal{J}(z))}{|z|^{\mathbf{k}_2}} > 0\,, \\ \underline{\lim_{z\to s_r}} \dfrac{\Lambda_{\rm o}(\mathcal{J}^{-1}(z))}{|s_r - z|^{\mathbf{k}_2}} \times \underline{\lim_{z\to s_r}} \dfrac{\lambda_{\rm o}(z)}{|s_r - z|^{\mathbf{k}_2}} > 0\,, \\ \underline{\lim_{z\to 0}} \dfrac{1}{|z|^{\mathbf{k}_2} |\mathcal{D}(\mathcal{J}(z))|} > 0\,, \end{cases}$$

$$\text{(d)} \begin{cases} \overline{\lim_{z\to s_l}} \dfrac{\Lambda_{\rm w}(\mathcal{J}^{-1}(z))}{|z - s_l|^2 \left|\frac{dv}{ds}(z)\right|} + \overline{\lim_{z\to s_l}} \dfrac{\frac{d\Upsilon}{dS}(\mathcal{J}^{-1}(z))}{\frac{dv}{ds}(z)} < \infty\,, \\ \overline{\lim_{z\to 1}} \dfrac{\Lambda_{\rm o}(z)}{|1 - z|^2 \left|\frac{d\Upsilon}{dS}(z)\right|} + \overline{\lim_{z\to 1}} \dfrac{\frac{dv}{ds}(\mathcal{J}(z))}{\frac{d\Upsilon}{dS}(z)} < \infty\,, \\ \underline{\lim_{z\to s_l}} \dfrac{\Lambda_{\rm w}(\mathcal{J}^{-1}(z))}{|z - s_l|^{\mathbf{k}_2}} \times \underline{\lim_{z\to s_l}} \dfrac{\lambda_{\rm w}(z)}{|z - s_l|^{\mathbf{k}_2}} > 0\,, \\ \underline{\lim_{z\to 1}} \dfrac{\Lambda_{\rm o}(z)}{|1 - z|^{\mathbf{k}_2}} \times \underline{\lim_{z\to 1}} \dfrac{\lambda_{\rm o}(\mathcal{J}(z))}{|1 - z|^{\mathbf{k}_2}} > 0\,, \\ \underline{\lim_{z\to s_l}} \dfrac{1}{|z - s_l|^{\mathbf{k}_2} |\mathcal{D}(z)|} > 0\,, \end{cases}$$

where $\mathbf{k}_1$, $\mathbf{k}_2$ *are positive constants. See* (2.1) *for* Λ, λ, $P_{c,b}$, $\mathcal{D}$.

Remark 2.1. 1. By A2 and A3, $\mathcal{D}$ is a strictly increasing function on $(s_l, s_r]$, so it has a bounded and strictly increasing inverse function $\mathcal{D}^{-1}$. Let us extend $\mathcal{D}^{-1}$ to $\mathbb{R}$ continuously and linearly with slope 1. The new function will not be relabelled.

2. A7 sets restrictions on phase mobilities and capillary pressures around endpoints only. Roughly speaking, A7(a) corresponds to that fracture capillary pressure decreases faster than matrix capillary pressure around endpoints. A7(b) is the inverse case of A7(a). By proper combinations of the restrictions in A7(a) and A7(b), we obtain A7(c) and A7(d). A7(c) is the case that fracture capillary pressure drops faster around 0 (resp. slower around 1) than matrix capillary pressure around s_l (resp. around s_r). A7(d) is the inverse case of A7(c).

3. If $\frac{\lambda_{\rm w}\lambda_{\rm o}}{\lambda}\left|\frac{dv}{ds}\right| \in L^1(s_l, s_r]$ (assumption in Refs. 7 and 17), $\mathcal{D}$ is bounded. If $\mathcal{D}$ is a bounded function on $(s_l, s_r]$, then A7(a)$_3$, A7(b)$_3$, A7(c)$_5$, and A7(d)$_5$ obviously hold.

4. If $\overline{\lim}_{z\to 0}|z|^{\mathbf{k}_2}|\Upsilon(z)| < \infty$, A7(a)$_3$ and A7(c)$_5$ hold. If $\overline{\lim}_{z\to s_l}|z-s_l|^{\mathbf{k}_2}|\upsilon(z)| < \infty$, A7(b)$_3$ and A7(d)$_5$ hold. So, if $\mathcal{D}(\mathcal{J}(z))$ (resp. $\mathcal{D}(z)$) grows slower than $\frac{1}{|z|^{\mathbf{k}_2}}$ (resp. $\frac{1}{|z-s_l|^{\mathbf{k}_2}}$) as z approaches 0 (resp. as z approaches s_l), then A7(a)$_3$ and A7(c)$_5$ (resp. A7(b)$_3$ and A7(d)$_5$) hold.

3. Procedure of Proof

Now we derive an auxiliary system for (1.1)–(1.12), and describe procedure of proof for Theorem 2.1. Global pressure[8] is defined as

$$P \stackrel{\text{def}}{=} \frac{1}{2}\left(P_{\text{o}} + P_{\text{w}} + \int_0^{\Upsilon(S)} \left(\frac{\Lambda_{\text{o}}}{\Lambda}(\Upsilon^{-1}(\xi)) - \frac{\Lambda_{\text{w}}}{\Lambda}(\Upsilon^{-1}(\xi))\right) d\xi\right). \tag{3.1}$$

See (2.1) for Λ. Then $\nabla_x P = \frac{\Lambda_{\text{w}}}{\Lambda}\nabla_x P_{\text{w}} + \frac{\Lambda_{\text{o}}}{\Lambda}\nabla_x P_{\text{o}}$. Let $\zeta_{\text{w}} = \zeta_{\text{o}} = \zeta$ in (2.2) and (2.3), and add the two equations to obtain

$$\int_{\Omega^{\text{T}}} \Lambda(S)\nabla_x P \nabla_x \zeta - \sum_{\alpha\in\{\text{w},\text{o}\}} \int_{\Omega^{\text{T}}} \Lambda_\alpha(S)\nabla_x E_\alpha \nabla_x \zeta = 0\,. \tag{3.2}$$

If we define

$$\mathcal{G} \stackrel{\text{def}}{=} \mathcal{J}(S)\,, \tag{3.3}$$

(2.2) can be written as

$$\int_{\Omega^{\text{T}}} \partial_t S\ \zeta_{\text{w}} + \int_{\Omega^{\text{T}}} \left(\Lambda_{\text{w}}(S)\nabla_x(P - E_{\text{w}}) - \frac{\Lambda_{\text{w}}(S)\Lambda_{\text{o}}(S)\frac{d\upsilon}{ds}(\mathcal{G})}{\Lambda(S)}\nabla_x \mathcal{G}\right)\nabla_x \zeta_{\text{w}}$$
$$= -\int_{\mathcal{Q}^{\text{T}}} \partial_t s\ \zeta_{\text{w}}\,. \tag{3.4}$$

If we repeat the process (3.1)–(3.4) in each matrix block, (2.4) can be written as

$$\int_{\mathcal{Q}^{\text{T}}} \partial_t s\ \eta_{\text{w}} - \int_{\mathcal{Q}^{\text{T}}} \frac{\lambda_{\text{w}}(s)\lambda_{\text{o}}(s)\frac{d\upsilon}{ds}(s)}{\lambda(s)}\nabla_y s \nabla_y \eta_{\text{w}} = 0\,. \tag{3.5}$$

Let ε be a small number satisfying

$$0 < \varepsilon < \mathbf{k}_1/4\,, \tag{3.6}$$

where $\mathbf{k}_1$ is the one in A5. Let us extend mobility functions Λ_α, λ_α ($\alpha = \text{w}, \text{o}$) constantly and continuously to $\mathbb{R}$, and find continuous monotone functions $\Lambda_\alpha^\varepsilon$, $\lambda_\alpha^\varepsilon$ in $\mathbb{R}$ such that

$$\begin{cases} \varepsilon \le \inf\limits_{z\in\mathbb{R}}\{\Lambda_\alpha^\varepsilon(z), \lambda_\alpha^\varepsilon(z)\} \le \sup\limits_{z\in\mathbb{R}}\{\Lambda_\alpha^\varepsilon(z), \lambda_\alpha^\varepsilon(z)\} \le 1\,, \\ \Lambda_\alpha^\varepsilon(z) = \Lambda_\alpha(z) \text{ and } \lambda_\alpha^\varepsilon(\mathcal{J}(z)) = \lambda_\alpha(\mathcal{J}(z)) \quad \text{for } z \in [\varepsilon, 1-\varepsilon]\,. \end{cases} \tag{3.7}$$

Next we define, for $z \in \mathbb{R}$,

$$\begin{cases} \Lambda^\varepsilon(z) \stackrel{\text{def}}{=} \Lambda^\varepsilon_{\mathrm{w}}(z) + \Lambda^\varepsilon_{\mathrm{o}}(z)\,, \\ \lambda^\varepsilon(z) \stackrel{\text{def}}{=} \lambda^\varepsilon_{\mathrm{w}}(z) + \lambda^\varepsilon_{\mathrm{o}}(z)\,, \\ \tilde{\Lambda}^\varepsilon_\alpha(z) \stackrel{\text{def}}{=} \Lambda_\alpha\left(0.5\left(\dfrac{z-\varepsilon}{0.5-\varepsilon}\right)\right), \quad \alpha \in \{\mathrm{w},\mathrm{o}\}\,, \\ \tilde{\Lambda}^\varepsilon(z) \stackrel{\text{def}}{=} \tilde{\Lambda}^\varepsilon_{\mathrm{w}}(z) + \tilde{\Lambda}^\varepsilon_{\mathrm{o}}(z)\,. \end{cases} \tag{3.8}$$

By A3, one may find decreasing and Lipschitz functions Υ^ε, υ^ε in $\mathbb{R}$ so that

$$\begin{cases} 0 < \mathbf{k}_3 \le \inf\limits_{z\in\mathbb{R}}\left\{\left|\dfrac{d\Upsilon^\varepsilon}{dS}\right|(z), \left|\dfrac{d\upsilon^\varepsilon}{ds}\right|(z)\right\} \le \sup\limits_{z\in\mathbb{R}}\left\{\left|\dfrac{d\Upsilon^\varepsilon}{dS}\right|(z), \left|\dfrac{d\upsilon^\varepsilon}{ds}\right|(z)\right\} < \infty\,, \\ \Upsilon^\varepsilon(z) = \Upsilon(z) \text{ and } \upsilon^\varepsilon(\mathcal{J}(z)) = \upsilon(\mathcal{J}(z)) \quad \text{for } z \in [\varepsilon, 1-\varepsilon]\,, \\ \Upsilon^\varepsilon \text{ (resp. } \upsilon^\varepsilon\text{) has inverse function } \Upsilon^{\varepsilon,-1} \text{ (resp. } \upsilon^{\varepsilon,-1}\text{) in } \mathbb{R}\,, \\ \mathcal{J}^\varepsilon(\stackrel{\text{def}}{=} \upsilon^{\varepsilon,-1}(\Upsilon^\varepsilon)) \text{ is linear in } \mathbb{R}\setminus[\varepsilon,1-\varepsilon] \text{ and has inverse } \mathcal{J}^{\varepsilon,-1}\,, \end{cases} \tag{3.9}$$

where $\mathbf{k}_3$ is a constant independent of ε. By A4 and A5, there exist smooth functions S^ε_0, s^ε_0, $P^\varepsilon_{c,b}$, $P^\varepsilon_{\alpha,b}$ ($\alpha = \mathrm{w},\mathrm{o}$) such that

$$\begin{cases} P^\varepsilon_{c,b} = P^\varepsilon_{\mathrm{o},b} - P^\varepsilon_{\mathrm{w},b}\,, \\ 0 < \dfrac{\mathbf{k}_1}{2} \le \inf\limits_{(x,t)\in\Omega^{\mathrm{T}}}\{S^\varepsilon_0, \Upsilon^{-1}(P^\varepsilon_{c,b})\} \le \sup\limits_{(x,t)\in\Omega^{\mathrm{T}}}\{S^\varepsilon_0, \Upsilon^{-1}(P^\varepsilon_{c,b})\} \le 1 - \dfrac{\mathbf{k}_1}{2}\,, \\ s^\varepsilon_0 = \mathcal{J}(S^\varepsilon_0)\,, \\ s^\varepsilon_0 - \upsilon^{-1}(P^\varepsilon_{c,b})(x,0) \in \mathcal{V}\,, \end{cases} \tag{3.10}$$

and, as $\varepsilon \to 0$,

$$\begin{cases} S^\varepsilon_0 \to S_0\,, & \text{in } L^2(\Omega)\,, \\ P^\varepsilon_{\alpha,b} \to P_{\alpha,b}\,, & \text{in } L^2(0,T;H^1(\Omega))\,, \\ \partial_t P^\varepsilon_{c,b} \to \partial_t P_{c,b}\,, & \text{in } L^1(\Omega^{\mathrm{T}})\,. \end{cases} \tag{3.11}$$

Auxiliary initial and boundary conditions are defined as

$$\begin{cases} \mathcal{G}^\varepsilon_0 \stackrel{\text{def}}{=} \mathcal{J}(S^\varepsilon_0)\,, \\ \mathcal{G}^\varepsilon_b \stackrel{\text{def}}{=} \upsilon^{-1}(P^\varepsilon_{c,b})\,, \\ P^\varepsilon_b \stackrel{\text{def}}{=} \dfrac{1}{2}\left(P^\varepsilon_{\mathrm{o},b} + P^\varepsilon_{\mathrm{w},b} + \displaystyle\int_0^{P^\varepsilon_{c,b}}\left(\dfrac{\Lambda^\varepsilon_{\mathrm{o}}}{\Lambda^\varepsilon}(\Upsilon^{\varepsilon,-1}(\xi)) - \dfrac{\Lambda^\varepsilon_{\mathrm{w}}}{\Lambda^\varepsilon}(\Upsilon^{\varepsilon,-1}(\xi))\right)d\xi\right)\,. \end{cases} \tag{3.12}$$

Auxiliary system of (1.1)–(1.12) for each ε is to find $\{S^\varepsilon, \mathcal{G}^\varepsilon, P^\varepsilon, s^\varepsilon\}$ such that

$$\partial_t S^\varepsilon + \int_{\mathcal{M}} \partial_t s^\varepsilon dy \in \text{dual } L^2(0,T;\mathcal{V}), \quad \partial_t s^\varepsilon \in \text{dual } L^2(0,T;\mathcal{U}_0), \tag{3.13}$$

$$\varepsilon \le S^\varepsilon \le 1-\varepsilon, \quad \mathcal{J}(\varepsilon) \le s^\varepsilon \le \mathcal{J}(1-\varepsilon), \tag{3.14}$$

$$\mathcal{G}^\varepsilon = \mathcal{J}(S^\varepsilon), \quad (\mathcal{G}^\varepsilon - \mathcal{G}^\varepsilon_b,\ P^\varepsilon - P^\varepsilon_b,\ s^\varepsilon - \mathcal{G}^\varepsilon) \in L^2(0,T;\mathcal{W}_1), \tag{3.15}$$

$$\int_{\Omega^{\mathrm{T}}} \partial_t S^\varepsilon \zeta_1 + \int_{\Omega^{\mathrm{T}}} \left(\tilde{\Lambda}^\varepsilon_{\mathrm{w}}(S^\varepsilon)\nabla_x(P^\varepsilon - E_{\mathrm{w}}) - \frac{\Lambda_{\mathrm{w}}\Lambda_{\mathrm{o}}}{\Lambda}(S^\varepsilon)\nabla_x \Upsilon(S^\varepsilon)\right)\nabla_x\zeta_1 + \int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s^\varepsilon \zeta_1 = 0, \tag{3.16}$$

$$\int_{\Omega^{\mathrm{T}}} \tilde{\Lambda}^\varepsilon(S^\varepsilon)\nabla_x P^\varepsilon \nabla_x \zeta_2 - \sum_{\alpha\in\{\mathrm{w},\mathrm{o}\}} \int_{\Omega^{\mathrm{T}}} \tilde{\Lambda}^\varepsilon_\alpha(S^\varepsilon)\nabla_x E_\alpha \nabla_x \zeta_2 = 0, \tag{3.17}$$

$$\int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s^\varepsilon \eta - \int_{\mathcal{Q}^{\mathrm{T}}} \frac{\lambda_{\mathrm{w}}\lambda_{\mathrm{o}}}{\lambda}(s^\varepsilon)\nabla_y \upsilon(s^\varepsilon)\nabla_y \eta = 0, \tag{3.18}$$

$$\mathcal{G}^\varepsilon(x,0) = \mathcal{G}^\varepsilon_0, \quad s^\varepsilon(x,y,0) = s^\varepsilon_0, \tag{3.19}$$

for any $(\zeta_1, \zeta_2, \eta) \in L^2(0,T;\mathcal{W}_1)$. See (3.8) for $\tilde{\Lambda}^\varepsilon$, $\tilde{\Lambda}^\varepsilon_\alpha$ ($\alpha = \mathrm{w},\mathrm{o}$). Later the following result will be proved:

Theorem 3.1. *Under* A1–A5, *for each* ε, *there is* $\{S^\varepsilon, \mathcal{G}^\varepsilon, P^\varepsilon, s^\varepsilon, P^\varepsilon_\alpha, p^\varepsilon_\alpha\ (\alpha = \mathrm{w},\mathrm{o})\}$ *such that* (3.13)–(3.19) *hold. Moreover*,

$$(P^\varepsilon_{\mathrm{w}} - P^\varepsilon_{\mathrm{w},b},\ P^\varepsilon_{\mathrm{o}} - P^\varepsilon_{\mathrm{o},b},\ p^\varepsilon_{\mathrm{w}} - P^\varepsilon_{\mathrm{w}},\ p^\varepsilon_{\mathrm{o}} - P^\varepsilon_{\mathrm{o}}) \in L^2(0,T;\mathcal{W}_2), \tag{3.20}$$

$$\Upsilon(S^\varepsilon) = P^\varepsilon_{\mathrm{o}} - P^\varepsilon_{\mathrm{w}}, \quad \upsilon(s^\varepsilon) = p^\varepsilon_{\mathrm{o}} - p^\varepsilon_{\mathrm{w}}, \tag{3.21}$$

$$\int_{\Omega^{\mathrm{T}}} \partial_t S^\varepsilon \zeta_{\mathrm{w}} + \int_{\Omega^{\mathrm{T}}} (\Lambda_{\mathrm{w}}(S^\varepsilon)\nabla_x P^\varepsilon_{\mathrm{w}} - \tilde{\Lambda}^\varepsilon_{\mathrm{w}}(S^\varepsilon)\nabla_x E_{\mathrm{w}} + (\tilde{\Lambda}^\varepsilon_{\mathrm{w}} - \Lambda_{\mathrm{w}})\nabla_x P^\varepsilon)\nabla_x \zeta_{\mathrm{w}} + \int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s^\varepsilon \zeta_{\mathrm{w}} = 0, \tag{3.22}$$

$$-\int_{\Omega^{\mathrm{T}}} \partial_t S^\varepsilon \zeta_{\mathrm{o}} + \int_{\Omega^{\mathrm{T}}} (\Lambda_{\mathrm{o}}(S^\varepsilon)\nabla_x P^\varepsilon_{\mathrm{o}} - \tilde{\Lambda}^\varepsilon_{\mathrm{o}}(S^\varepsilon)\nabla_x E_{\mathrm{o}} + (\tilde{\Lambda}^\varepsilon_{\mathrm{o}} - \Lambda_{\mathrm{o}})\nabla_x P^\varepsilon)\nabla_x \zeta_{\mathrm{o}} - \int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s^\varepsilon \zeta_{\mathrm{o}} = 0, \tag{3.23}$$

$$\int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s^\varepsilon \eta_{\mathrm{w}} + \int_{\mathcal{Q}^{\mathrm{T}}} \lambda_{\mathrm{w}}(s^\varepsilon)\nabla_y p^\varepsilon_{\mathrm{w}} \nabla_y \eta_{\mathrm{w}} = 0, \tag{3.24}$$

$$-\int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s^\varepsilon \eta_{\mathrm{o}} + \int_{\mathcal{Q}^{\mathrm{T}}} \lambda_{\mathrm{o}}(s^\varepsilon)\nabla_y p^\varepsilon_{\mathrm{o}} \nabla_y \eta_{\mathrm{o}} = 0, \tag{3.25}$$

for all $\zeta_\alpha \in L^2(0,T;\mathcal{V})$, $\eta_\alpha \in L^2(0,T;\mathcal{U}_0)$.

Similar result as Theorem 3.1 had been considered in Ref. 17. For completeness, the proof of this theorem will be given in Sec. 5. In the next section one will see that a subsequence of the solutions of Theorem 3.1 converges weakly to a solution of (1.1)–(1.12) as ε approaches 0, which implies Theorem 2.1.

4. Existence of a Weak Solution

The objective of this section is to prove Theorem 2.1 if Theorem 3.1 holds. It is done as follows: First we show P^ε_α, p^ε_α for $\alpha \in \{\mathrm{w}, \mathrm{o}\}$ (solutions of Theorem 3.1) are bounded independently of ε (see Lemmas 4.1–4.3), next we prove $\{S^\varepsilon\}$ has a convergent subsequence in $L^2(\Omega^{\mathrm{T}})$ (see Lemmas 4.4–4.6), then show $\{s^\varepsilon\}$ has a convergent subsequence in $L^2(\mathcal{Q}^{\mathrm{T}})$ (see Lemmas 4.7–4.9), and finally conclude the existence of a weak solution of (1.1)–(1.12). We define

$$\begin{cases} \Theta(z) \overset{\text{def}}{=} \displaystyle\int_0^z (\Upsilon^{-1}(-z) - \Upsilon^{-1}(-\xi))d\xi\,, \quad \text{for } z \in (-\infty, 0]\,, \\ \Psi^\varepsilon \overset{\text{def}}{=} -\Upsilon(S^\varepsilon)\,, \\ \theta(z) \overset{\text{def}}{=} \displaystyle\int_0^z (\upsilon^{-1}(-z) - \upsilon^{-1}(-\xi))d\xi\,, \quad \text{for } z \in (-\infty, 0]\,, \\ \psi^\varepsilon \overset{\text{def}}{=} -\upsilon(s^\varepsilon)\,, \\ \rho^\varepsilon \overset{\text{def}}{=} \mathcal{J}^{-1}(s^\varepsilon)\,, \\ \Psi^\varepsilon_b \overset{\text{def}}{=} -P^\varepsilon_{c,b}\,. \end{cases} \tag{4.1}$$

$(4.1)_{2,4,5}$ are well-defined by (3.14). (3.15) implies $\rho^\varepsilon|_{\partial\mathcal{M}} = S^\varepsilon$ in Ω^{T}. $\Theta(z)$ and $\theta(z)$ are non-negative functions on $(-\infty, 0]$, and, for any $z_1, z_2 \le 0$,

$$\begin{cases} \Theta(z_1) - \Theta(z_2) \le (\Upsilon^{-1}(-z_1) - \Upsilon^{-1}(-z_2))z_1\,, \\ \theta(z_1) - \theta(z_2) \le (\upsilon^{-1}(-z_1) - \upsilon^{-1}(-z_2))z_1\,. \end{cases} \tag{4.2}$$

$\mathcal{X}_\mathcal{B}$, $\mathcal{B} \subset \mathcal{Q}^{\mathrm{T}}$, is a characteristic function defined as

$$\mathcal{X}_\mathcal{B}(z) = \begin{cases} 1\,, & \text{for } z \in \mathcal{B}\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{4.3}$$

Let us find two non-negative smooth functions $\mathbf{g}_1$ and $\mathbf{g}_2$ defined on $[0, 1]$ such that $\mathbf{g}_1$ (resp. $\mathbf{g}_2$) is decreasing (resp. increasing), $\mathbf{g}_1(0) = \mathbf{g}_2(1) = 1$, $\mathbf{g}_1(0.6) = \mathbf{g}_2(0.4) = 0$, and $\mathbf{g}_1 + \mathbf{g}_2 > 0$ in $[0, 1]$. Let $\tilde{\mathbf{g}}_1, \tilde{\mathbf{g}}_2$: $(s_l, s_r] \to \mathbb{R}$ by $\tilde{\mathbf{g}}_1(\xi) \overset{\text{def}}{=} \mathbf{g}_1(\mathcal{J}^{-1}(\xi))$, $\tilde{\mathbf{g}}_2(\xi) \overset{\text{def}}{=} \mathbf{g}_2(\mathcal{J}^{-1}(\xi))$. By A6, we define $\mathcal{E} : (0, 1] \to \mathbb{R}$ by

$$\mathcal{E}(z)=\begin{cases}\displaystyle\int_{0.5}^{z}\sqrt{\Lambda_{\rm w}\Lambda_{\rm o}}\left|\frac{d\Upsilon}{dS}\right|\,, & \text{if } \begin{cases}\overline{\lim}_{\xi\to 0}\frac{\Lambda_{\rm w}(\xi)}{\lambda_{\rm w}(\mathcal{J}(\xi))}<\infty\,,\\ \overline{\lim}_{\xi\to 1}\frac{\Lambda_{\rm o}(\xi)}{\lambda_{\rm o}(\mathcal{J}(\xi))}<\infty\,,\end{cases}\\ \displaystyle\int_{\mathcal{J}(0.5)}^{\mathcal{J}(z)}\sqrt{\lambda_{\rm w}\lambda_{\rm o}}\left|\frac{dv}{ds}\right|\,, & \text{if } \begin{cases}\overline{\lim}_{\xi\to 0}\frac{\lambda_{\rm w}(\mathcal{J}(\xi))}{\Lambda_{\rm w}(\xi)}<\infty\,,\\ \overline{\lim}_{\xi\to 1}\frac{\lambda_{\rm o}(\mathcal{J}(\xi))}{\Lambda_{\rm o}(\xi)}<\infty\,,\end{cases}\\ \displaystyle\int_{0.5}^{z}\sqrt{\Lambda_{\rm w}\Lambda_{\rm o}}\left|\frac{d\Upsilon}{dS}\right|\mathbf{g}_1+\int_{\mathcal{J}(0.5)}^{\mathcal{J}(z)}\sqrt{\lambda_{\rm w}\lambda_{\rm o}}\left|\frac{dv}{ds}\right|\tilde{\mathbf{g}}_2\,, & \text{if } \begin{cases}\overline{\lim}_{\xi\to 0}\frac{\Lambda_{\rm w}(\xi)}{\lambda_{\rm w}(\mathcal{J}(\xi))}<\infty\,,\\ \overline{\lim}_{\xi\to 1}\frac{\lambda_{\rm o}(\mathcal{J}(\xi))}{\Lambda_{\rm o}(\xi)}<\infty\,,\end{cases}\\ \displaystyle\int_{0.5}^{z}\sqrt{\Lambda_{\rm w}\Lambda_{\rm o}}\left|\frac{d\Upsilon}{dS}\right|\mathbf{g}_2+\int_{\mathcal{J}(0.5)}^{\mathcal{J}(z)}\sqrt{\lambda_{\rm w}\lambda_{\rm o}}\left|\frac{dv}{ds}\right|\tilde{\mathbf{g}}_1\,, & \text{if } \begin{cases}\overline{\lim}_{\xi\to 0}\frac{\lambda_{\rm w}(\mathcal{J}(\xi))}{\Lambda_{\rm w}(\xi)}<\infty\,,\\ \overline{\lim}_{\xi\to 1}\frac{\Lambda_{\rm o}(\xi)}{\lambda_{\rm o}(\mathcal{J}(\xi))}<\infty\,.\end{cases}\end{cases} \tag{4.4}$$

$\mathcal{E}$ in (4.4) may have more than two options. If so, one selects the foremost possible one in (4.4) so that $\mathcal{E}$ is well-defined. $\mathcal{E}$ is a strictly increasing function, so it has a bounded and strictly increasing inverse function $\mathcal{E}^{-1}$. We extend $\mathcal{E}^{-1}$ to $\mathbb{R}$ so that it is bounded, continuous, and strictly increasing in $\mathbb{R}$. Let us define

$$\begin{cases}\Phi^\varepsilon \stackrel{\text{def}}{=} \mathcal{E}(S^\varepsilon)\,,\\ \phi^\varepsilon \stackrel{\text{def}}{=} \mathcal{E}(\rho^\varepsilon)\,.\end{cases}$$

Lemma 4.1. *Solutions of Theorem* 3.1 *satisfy*

$$\sum_{\alpha\in\{\rm w,o\}}\left(\|\sqrt{\Lambda_\alpha(S^\varepsilon)}\nabla_x P_\alpha^\varepsilon\|_{L^2(\Omega^{\rm T})}+\|\sqrt{\lambda_\alpha(s^\varepsilon)}\nabla_y p_\alpha^\varepsilon\|_{L^2(\mathcal{Q}^{\rm T})}\right) + \|P^\varepsilon\|_{L^2(0,T;H^1(\Omega))}\le c\,, \tag{4.5}$$

$$\|\mathbf{R}(S^\varepsilon)\|_{L^2(0,T;H^1(\Omega))}+\|\nabla_y\mathcal{D}(s^\varepsilon)\|_{L^2(\mathcal{Q}^{\rm T})}+\|\Phi^\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|\phi^\varepsilon\|_{L^2(0,T;\mathcal{U})}\le c\,, \tag{4.6}$$

where c is a constant independent of ε. See (2.1) *for* $\mathbf{R}$, $\mathcal{D}$.

Proof. Set $\zeta_2 = P^\varepsilon - P_b^\varepsilon$ in (3.17) to obtain, by A4 and $(3.12)_3$,

$$\|P^\varepsilon\|_{L^2(0,T;H^1(\Omega))}\le c \text{ (independent of } \varepsilon)\,. \tag{4.7}$$

By $(4.2)_1$, for all t, $\varpi > 0$,

$$\Theta(\Psi^\varepsilon(t))-\Theta(\Psi^\varepsilon(t-\varpi))\le (S^\varepsilon(t)-S^\varepsilon(t-\varpi))\Psi^\varepsilon(t)\,, \tag{4.8}$$

where $\Psi^\varepsilon(t)=\Psi^\varepsilon(0)$ for $-\varpi<t<0$. Integrate (4.8) over Ω^τ to obtain

$$\frac{1}{\varpi}\int_{\tau-\varpi}^{\tau}\int_{\Omega}\Theta(\Psi^{\varepsilon}) \le \int_{\Omega^{\tau}}(\Psi^{\varepsilon}-\Psi_b^{\varepsilon})\partial^{-\varpi}S^{\varepsilon} + \int_{\Omega}\Theta(\Psi^{\varepsilon}(0))$$

$$-\int_0^{\tau-\varpi}\int_{\Omega}(S^{\varepsilon}-S^{\varepsilon}(0))\partial^{\varpi}\Psi_b^{\varepsilon} + \frac{1}{\varpi}\int_{\tau-\varpi}^{\tau}\int_{\Omega}(S^{\varepsilon}-S^{\varepsilon}(0))\Psi_b^{\varepsilon}\,. \quad (4.9)$$

See $(2.1)_1$ for time differentiation. Similarly, by $(4.2)_2$, one obtains

$$\frac{1}{\varpi}\int_{\tau-\varpi}^{\tau}\int_{\mathcal{Q}}\theta(\psi^{\varepsilon}) \le \int_{\mathcal{Q}^{\tau}}(\psi^{\varepsilon}-\Psi_b^{\varepsilon})\partial^{-\varpi}s^{\varepsilon} + \int_{\mathcal{Q}}\theta(\psi^{\varepsilon}(0))$$

$$-\int_0^{\tau-\varpi}\int_{\mathcal{Q}}(s^{\varepsilon}-s^{\varepsilon}(0))\partial^{\varpi}\Psi_b^{\varepsilon} + \frac{1}{\varpi}\int_{\tau-\varpi}^{\tau}\int_{\mathcal{Q}}(s^{\varepsilon}-s^{\varepsilon}(0))\Psi_b^{\varepsilon}\,. \quad (4.10)$$

Summing (4.9) and (4.10) as well as letting $\varpi \to 0$, by boundedness of S^{ε} and s^{ε}, we get, for almost all $\tau \in (0,T]$,

$$\int_{\Omega}\Theta(\Psi^{\varepsilon})(\tau) + \int_{\mathcal{Q}}\theta(\psi^{\varepsilon})(\tau) \le \int_{\Omega^{\tau}}(\Psi^{\varepsilon}-\Psi_b^{\varepsilon})\partial_t S^{\varepsilon} + \int_{\mathcal{Q}^{\tau}}(\psi^{\varepsilon}-\Psi_b^{\varepsilon})\partial_t s^{\varepsilon}$$

$$+\,c(\|\Psi^{\varepsilon}(0)\|_{L^1(\Omega)}, \|\Psi_b^{\varepsilon}\|_{L^{\infty}(0,T;L^1(\Omega))}, \|\partial_t\Psi_b^{\varepsilon}\|_{L^1(\Omega^{\mathrm{T}})})\,. \quad (4.11)$$

Letting $\zeta_{\alpha} = P_{\alpha}^{\varepsilon} - P_{\alpha,b}^{\varepsilon}$, $\eta_{\alpha} = p_{\alpha}^{\varepsilon} - P_{\alpha}^{\varepsilon}$ for $\alpha \in \{\mathrm{w},\mathrm{o}\}$ in (3.22)–(3.25), one obtains

$$\int_{\Omega^{\tau}}(\Psi^{\varepsilon}-\Psi_b^{\varepsilon})\partial_t S^{\varepsilon} + \sum_{\alpha\in\{\mathrm{w},\mathrm{o}\}}\int_{\Omega^{\tau}}\Lambda_{\alpha}(S^{\varepsilon})|\nabla_x P_{\alpha}^{\varepsilon}|^2 + \int_{\mathcal{Q}^{\tau}}(\psi^{\varepsilon}-\Psi_b^{\varepsilon})\partial_t s^{\varepsilon}$$

$$+\sum_{\alpha\in\{\mathrm{w},\mathrm{o}\}}\int_{\mathcal{Q}^{\tau}}\lambda_{\alpha}(s^{\varepsilon})|\nabla_y p_{\alpha}^{\varepsilon}|^2 \le c(\|\nabla_x P^{\varepsilon}, \nabla_x E_{\alpha}, \nabla_x P_{\alpha,b}^{\varepsilon}\|_{L^2(\Omega^{\mathrm{T}})})\,. \quad (4.12)$$

By (3.11), (4.7), (4.11)–(4.12), and A4, A5, we obtain (4.5). Clearly (4.5) implies

$$\int_{\Omega^{\mathrm{T}}}\Lambda_{\mathrm{o}}\Lambda_{\mathrm{w}}(S^{\varepsilon})|\nabla_x\Upsilon(S^{\varepsilon})|^2 + \int_{\mathcal{Q}^{\mathrm{T}}}\lambda_{\mathrm{o}}\lambda_{\mathrm{w}}(s^{\varepsilon})|\nabla_y\upsilon(s^{\varepsilon})|^2 \le c_0\,, \quad (4.13)$$

where c_0 is independent of ε. (4.6) is due to A5, (4.4) and (4.13). □

Lemma 4.2. *Suppose* $2 \le \varpi_0 \in \mathbb{N}$ *and* $\frac{2\mathbf{k}_1}{2^{\varpi_0}} \le \min\{\mathcal{J}\left(\frac{\mathbf{k}_1}{2}\right) - s_l, s_r - \mathcal{J}\left(1-\frac{\mathbf{k}_1}{2}\right)\}$, *where* $\mathbf{k}_1$ *is the one in* A5. *For any* $\tau(\le T)$, $\varpi(\ge 2+\varpi_0)\in\mathbb{N}$, *and* $\varepsilon\left(<\mu \stackrel{\mathrm{def}}{=} \frac{\mathbf{k}_1}{2^{\varpi}}\right)$, *solutions of Theorem* 3.1 *satisfy the following results*: *If* A7(a) *holds, then*

$$\sup_{t\le\tau}(|\{x\in\Omega : S^{\varepsilon}(t)\le\mu\}| + |\{(x,y)\in\mathcal{Q} : \rho^{\varepsilon}(t)\le\mu\}|)$$

$$+\sup_{t\le\tau}(|\{x\in\Omega : 1-\mu\le S^{\varepsilon}(t)\}| + |\{(x,y)\in\mathcal{Q} : 1-\mu\le\rho^{\varepsilon}(t)\}|)$$

$$\le \frac{c_0|c_0\tau|^{\varpi-\varpi_0}}{(\varpi-\varpi_0)^{(\varpi-\varpi_0)\mathbf{f}_{\varpi}}}\,, \quad (4.14)$$

if A7(b) *holds, then*

$$\sup_{t\le\tau}(|\{x\in\Omega:\mathcal{G}^\varepsilon(t)\le\mu+s_l\}|+|\{(x,y)\in\mathcal{Q}:s^\varepsilon(t)\le\mu+s_l\}|)$$
$$+\sup_{t\le\tau}(|\{x\in\Omega:s_r-\mu\le\mathcal{G}^\varepsilon(t)\}|+|\{(x,y)\in\mathcal{Q}:s_r-\mu\le s^\varepsilon(t)\}|)$$
$$\le\frac{c_0|c_0\tau|^{\varpi-\varpi_0}}{(\varpi-\varpi_0)^{(\varpi-\varpi_0)\mathbf{f}_\varpi}}\,, \tag{4.15}$$

if A7(c) *holds, then*

$$\sup_{t\le\tau}(|\{x\in\Omega:S^\varepsilon(t)\le\mu\}|+|\{(x,y)\in\mathcal{Q}:\rho^\varepsilon(t)\le\mu\}|)$$
$$+\sup_{t\le\tau}(|\{x\in\Omega:s_r-\mu\le\mathcal{G}^\varepsilon(t)\}|+|\{(x,y)\in\mathcal{Q}:s_r-\mu\le s^\varepsilon(t)\}|)$$
$$\le\frac{c_0|c_0\tau|^{\varpi-\varpi_0}}{(\varpi-\varpi_0)^{(\varpi-\varpi_0)\mathbf{f}_\varpi}}\,, \tag{4.16}$$

and finally if A7(d) *holds, then*

$$\sup_{t\le\tau}(|\{x\in\Omega:\mathcal{G}^\varepsilon(t)\le\mu+s_l\}|+|\{(x,y)\in\mathcal{Q}:s^\varepsilon(t)\le\mu+s_l\}|)$$
$$+\sup_{t\le\tau}(|\{x\in\Omega:1-\mu\le S^\varepsilon(t)\}|+|\{(x,y)\in\mathcal{Q}:1-\mu\le\rho^\varepsilon(t)\}|)$$
$$\le\frac{c_0|c_0\tau|^{\varpi-\varpi_0}}{(\varpi-\varpi_0)^{(\varpi-\varpi_0)\mathbf{f}_\varpi}}\,, \tag{4.17}$$

where $\lim_{\varpi\to\infty}\mathbf{f}_\varpi=1$ *and* c_0 *is a constant independent of* τ, ϖ, ε, μ.

Proof. Case 1. We claim (4.14). A7(a)$_1$ is assumed here. Define $\mathcal{K}_\mu$, $\mathcal{K}_{\varsigma,\mu}$ as

$$\mathcal{K}_\mu(z)\stackrel{\text{def}}{=}\begin{cases}0\,, & \text{for } 2\mu\le z\,,\\ z-2\mu\,, & \text{for } \mu\le z\le 2\mu\,,\\ -\mu\,, & \text{for } z\le\mu\,,\end{cases}$$

$$\mathcal{K}_{\varsigma,\mu}(z)\stackrel{\text{def}}{=}\begin{cases}0\,, & \text{for } \varsigma(2\mu)\le z\,,\\ z-\varsigma(2\mu)\,, & \text{for } \varsigma(\mu)\le z\le\varsigma(2\mu)\,,\\ \varsigma(\mu)-\varsigma(2\mu)\,, & \text{for } z\le\varsigma(\mu)\,,\end{cases}$$

where

$$\varsigma(z)\stackrel{\text{def}}{=}\int_{0.5}^{z}\frac{\tilde\Lambda^\varepsilon_{\mathrm{w}}}{\tilde\Lambda^\varepsilon}(\xi)d\xi\,,\quad z\in(0,1)\,. \tag{4.18}$$

Define

$$\breve{\mathcal{X}}_\mu(z)\stackrel{\text{def}}{=}\begin{cases}1\,, & \text{for } \mu\le z\le 2\mu\,,\\ 0\,, & \text{otherwise.}\end{cases}$$

Then $\breve{\mathcal{X}}_\mu(z) = \frac{d}{dz}\mathcal{K}_\mu(z) = \frac{d}{dz}\mathcal{K}_{\varsigma,\mu}(\varsigma(z))$, $\frac{d}{dz}\varsigma(z) = \frac{\tilde{\Lambda}^\varepsilon_{\rm w}}{\tilde{\Lambda}^\varepsilon}(z)$. By $2\mu \leq \frac{\mathbf{k}_1}{2}$ and (3.15),

$$(\zeta_1, \zeta_2, \eta) = (\mathcal{K}_\mu(S^\varepsilon), \mathcal{K}_{\varsigma,\mu}(\varsigma(S^\varepsilon)), \mathcal{K}_\mu(\rho^\varepsilon) - \mathcal{K}_\mu(S^\varepsilon)) \in L^2(0,T;\mathcal{W}_1)\,. \tag{4.19}$$

Employ (ζ_1, ζ_2, η) of (4.19) in (3.16)–(3.18) to obtain, by A4,

$$\begin{aligned}\int_{\Omega^\tau} \mathcal{K}_\mu(S^\varepsilon)\partial_t S^\varepsilon + \int_{\Omega^\tau} \Lambda_{\rm w}(S^\varepsilon)\breve{\mathcal{X}}_\mu(S^\varepsilon)\nabla_x\Psi^\varepsilon\nabla_x S^\varepsilon + \int_{\mathcal{Q}^\tau} \mathcal{K}_\mu(\rho^\varepsilon)\partial_t s^\varepsilon \\ \leq c_1 \int_{\Omega^\tau} \tilde{\Lambda}^\varepsilon_{\rm w}\breve{\mathcal{X}}_\mu(S^\varepsilon)|\nabla_x S^\varepsilon|\,,\end{aligned} \tag{4.20}$$

where constant c_1 is independent of ε, μ. Suppose

$$\int_{\Omega^\tau} \mathcal{K}_\mu(S^\varepsilon)\partial_t S^\varepsilon + \int_{\mathcal{Q}^\tau} \mathcal{K}_\mu(\rho^\varepsilon)\partial_t s^\varepsilon \geq 0\,, \tag{4.21}$$

(4.20)–(4.21) imply

$$\begin{aligned}\int_{\Omega^\tau} \tilde{\Lambda}^\varepsilon_{\rm w}\breve{\mathcal{X}}_\mu(S^\varepsilon)|\nabla_x S^\varepsilon| &\leq \sqrt{\int_{\Omega^\tau} \frac{\tilde{\Lambda}^\varepsilon_{\rm w}\breve{\mathcal{X}}_\mu}{|\frac{d\Upsilon}{dS}|}(S^\varepsilon)}\sqrt{\int_{\Omega^\tau} \tilde{\Lambda}^\varepsilon_{\rm w}\breve{\mathcal{X}}_\mu(S^\varepsilon)|\nabla_x\Psi^\varepsilon||\nabla_x S^\varepsilon|} \\ &\leq c_2\sqrt{\int_{\Omega^\tau} \frac{\tilde{\Lambda}^\varepsilon_{\rm w}\breve{\mathcal{X}}_\mu}{|\frac{d\Upsilon}{dS}|}(S^\varepsilon)}\sqrt{\int_{\Omega^\tau} \tilde{\Lambda}^\varepsilon_{\rm w}\breve{\mathcal{X}}_\mu(S^\varepsilon)|\nabla_x S^\varepsilon|}\,,\end{aligned} \tag{4.22}$$

where constant c_2 is independent of ε, μ. (4.20)–(4.22) imply

$$\int_{\Omega^\tau} \mathcal{K}_\mu(S^\varepsilon)\partial_t S^\varepsilon + \int_{\mathcal{Q}^\tau} \mathcal{K}_\mu(\rho^\varepsilon)\partial_t s^\varepsilon \leq c_3 \int_{\Omega^\tau} \frac{\tilde{\Lambda}^\varepsilon_{\rm w}\breve{\mathcal{X}}_\mu}{|\frac{d\Upsilon}{dS}|}(S^\varepsilon)\,. \tag{4.23}$$

Define $\mathcal{Z}^\varepsilon_\mu \stackrel{\rm def}{=} \breve{\mathcal{Z}}_\mu(S^\varepsilon) + \hat{\mathcal{Z}}_\mu(s^\varepsilon)$ where

$$\breve{\mathcal{Z}}_\mu(\xi) \stackrel{\rm def}{=} \int_{2\mu}^{\xi} \mathcal{K}_\mu(z)dz\,, \quad \hat{\mathcal{Z}}_\mu(\xi) \stackrel{\rm def}{=} \int_{\mathcal{J}(2\mu)}^{\xi} \mathcal{K}_\mu(\mathcal{J}^{-1}(z))dz\,.$$

(4.23) implies

$$\int_{\mathcal{Q}^\tau} \partial_t\mathcal{Z}^\varepsilon_\mu = \int_{\Omega^\tau} \mathcal{K}_\mu(S^\varepsilon)\partial_t S^\varepsilon + \int_{\mathcal{Q}^\tau} \mathcal{K}_\mu(\rho^\varepsilon)\partial_t s^\varepsilon \leq c_3 \int_{\Omega^\tau} \frac{\tilde{\Lambda}^\varepsilon_{\rm w}\breve{\mathcal{X}}_\mu}{|\frac{d\Upsilon}{dS}|}(S^\varepsilon)\,. \tag{4.24}$$

$(3.9)_2$, (4.24), and A7$(\mathrm{a})_1$ yield that, if $0 \leq t_1 \leq t_2 \leq T$,

$$\int_{t_1}^{t_2}\int_{\mathcal{Q}} \partial_t\mathcal{Z}^\varepsilon_\mu \leq c_4 \int_{t_1}^{t_2}\int_{\mathcal{Q}} \mathcal{Z}^\varepsilon_{2\mu}\,, \tag{4.25}$$

where c_4 is independent of t_1, t_2, μ, ε. Define

$$\mathcal{F}^\varepsilon(\mu,\tau) \stackrel{\rm def}{=} \frac{1}{\mu^2}\sup_{t\leq\tau}\int_{\mathcal{Q}} \mathcal{Z}^\varepsilon_\mu(\cdot,t)\,.$$

(4.25) implies that, for $0 \le t_1 \le t_2 \le T$,

$$\mathcal{F}^\varepsilon(\mu, t_2) - \mathcal{F}^\varepsilon(\mu, t_1) \le c_5(t_2 - t_1)\mathcal{F}^\varepsilon(2\mu, t_2)\,, \tag{4.26}$$

where c_5 is independent of t_1, t_2, μ, ε. By induction and $(3.10)_2$, one obtains, for $j \in \mathbb{N}$, $jh \le T$,

$$\mathcal{F}^\varepsilon\left(\frac{\mathbf{k}_1}{2^\varpi}, jh\right) \le (\varpi - \varpi_0 + 1)^{j-1}|c_5 h|^{\varpi-\varpi_0}\mathcal{F}^\varepsilon\left(\frac{\mathbf{k}_1}{2^{\varpi_0}}, jh\right). \tag{4.27}$$

If $j = \frac{\varpi-\varpi_0}{\log(\varpi-\varpi_0)}$ and $\tau = jh$ in (4.27), then

$$\mathcal{F}^\varepsilon\left(\frac{\mathbf{k}_1}{2^\varpi}, \tau\right) \le \frac{|c_5\tau|^{\varpi-\varpi_0}}{(\varpi-\varpi_0)^{(\varpi-\varpi_0)\mathbf{f}_\varpi}}\mathcal{F}^\varepsilon\left(\frac{\mathbf{k}_1}{2^{\varpi_0}}, \tau\right), \tag{4.28}$$

where $\mathbf{f}_\varpi \to 1$ as $\varpi \to \infty$. Define

$$\begin{cases} \mathcal{B}_1(t) \overset{\text{def}}{=} \left\{x \in \Omega : S^\varepsilon(x,t) \le \mu = \dfrac{\mathbf{k}_1}{2^\varpi}\right\}, \\ \mathcal{B}_2(t) \overset{\text{def}}{=} \left\{(x,y) \in \mathcal{Q} : \rho^\varepsilon(x,y,t) \le \mu = \dfrac{\mathbf{k}_1}{2^\varpi}\right\}. \end{cases}$$

$\mathrm{A7(a)}_1$, (3.11) and (4.28) imply

$$\sup_{t\le\tau}\left(\int \mathcal{X}_{\mathcal{B}_1(t)} + \int \mathcal{X}_{\mathcal{B}_2(t)}\right) \le c_6\mathcal{F}^\varepsilon\left(\frac{\mathbf{k}_1}{2^\varpi}, \tau\right) \le \frac{c_6|c_5\tau|^{\varpi-\varpi_0}}{(\varpi-\varpi_0)^{(\varpi-\varpi_0)\mathbf{f}_\varpi}}\mathcal{F}^\varepsilon\left(\frac{\mathbf{k}_1}{2^{\varpi_0}}, \tau\right),$$

where constant c_6 is independent of τ, ϖ, ε, μ. See (4.3) for $\mathcal{X}_{\mathcal{B}_i}$ $(i = 1, 2)$. So the proof of the first part of (4.14) is complete.

Proof of the second part of (4.14) is similar to that of the first part, so we just sketch the proof. For comparison with proof of the first part, some notations above will be used again. Define $\mathcal{K}_\mu$, $\mathcal{K}_{\varsigma,\mu}$ as

$$\mathcal{K}_\mu(z) \overset{\text{def}}{=} \begin{cases} 0\,, & \text{for } z \le 1 - 2\mu\,, \\ z - 1 + 2\mu\,, & \text{for } 1 - 2\mu \le z \le 1 - \mu\,, \\ \mu\,, & \text{for } 1 - \mu \le z\,, \end{cases}$$

$$\mathcal{K}_{\varsigma,\mu}(z) \overset{\text{def}}{=} \begin{cases} 0\,, & \text{for } z \le \varsigma(1 - 2\mu)\,, \\ z - \varsigma(1 - 2\mu)\,, & \text{for } \varsigma(1 - 2\mu) \le z \le \varsigma(1 - \mu)\,, \\ \varsigma(1-\mu) - \varsigma(1 - 2\mu)\,, & \text{for } \varsigma(1 - \mu) \le z\,, \end{cases}$$

where ς is the one in (4.18). Define

$$\breve{\mathcal{X}}_\mu(z) \overset{\text{def}}{=} \begin{cases} 1\,, & \text{for } 1 - 2\mu \le z \le 1 - \mu\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Then $\breve{\mathcal{X}}_\mu(z) = \frac{d}{dz}\mathcal{K}_\mu(z) = \frac{d}{dz}\mathcal{K}_{\varsigma,\mu}(\varsigma(z))$. By $2\mu \le \frac{\mathbf{k}_1}{2}$ and (3.15),

$$(\zeta_1, \zeta_2, \eta) = (\mathcal{K}_\mu(S^\varepsilon), \mathcal{K}_{\varsigma,\mu}(\varsigma(S^\varepsilon)), \mathcal{K}_\mu(\rho^\varepsilon) - \mathcal{K}_\mu(S^\varepsilon)) \in L^2(0,T;\mathcal{W}_1)\,. \tag{4.29}$$

Employ (ζ_1, ζ_2, η) of (4.29) in (3.16)–(3.18) to get

$$\int_{\Omega^\tau} \mathcal{K}_\mu(S^\varepsilon)\partial_t S^\varepsilon + \int_{\Omega^\tau} \Lambda_{\mathrm{o}}(S^\varepsilon)\breve{\mathcal{X}}_\mu(S^\varepsilon)\nabla_x\Psi^\varepsilon\nabla_x S^\varepsilon + \int_{\mathcal{Q}^\tau} \mathcal{K}_\mu(\rho^\varepsilon)\partial_t s^\varepsilon$$
$$\le c_1 \int_{\Omega^\tau} \tilde{\Lambda}^\varepsilon_{\mathrm{o}}\breve{\mathcal{X}}_\mu(S^\varepsilon)|\nabla_x S^\varepsilon|\,, \tag{4.30}$$

where constant c_1 is independent of ε, μ. Then following the proof of the first part, one can complete the proof of the second part.

Case 2. We assume A7(b)$_1$ and claim (4.15). Proof of this case is similar to that of Case 1. Define $\mathcal{K}_\mu$, $\mathcal{K}_{\varsigma,\mu}$ as

$$\mathcal{K}_\mu(z) \stackrel{\text{def}}{=} \begin{cases} 0\,, & \text{for } 2\mu + s_l \le z\,, \\ z - (2\mu + s_l)\,, & \text{for } \mu + s_l \le z \le 2\mu + s_l\,, \\ -\mu\,, & \text{for } z \le \mu + s_l\,, \end{cases}$$

$$\mathcal{K}_{\varsigma,\mu}(z) \stackrel{\text{def}}{=} \begin{cases} 0\,, & \text{for } \varsigma(2\mu + s_l) \le z\,, \\ z - \varsigma(2\mu + s_l)\,, & \text{for } \varsigma(\mu + s_l) \le z \le \varsigma(2\mu + s_l)\,, \\ \varsigma(\mu + s_l) - \varsigma(2\mu + s_l)\,, & \text{for } z \le \varsigma(\mu + s_l)\,, \end{cases}$$

where

$$\varsigma(z) \stackrel{\text{def}}{=} \int_{\mathcal{J}(0.5)}^{z} \frac{\tilde{\Lambda}^\varepsilon_{\mathrm{w}}}{\tilde{\Lambda}^\varepsilon}(\mathcal{J}^{-1}(\xi))d\xi\,, \quad z \in (s_l,\ s_r)\,.$$

Let

$$\breve{\mathcal{X}}_\mu(z) \stackrel{\text{def}}{=} \begin{cases} 1\,, & \text{for } \mu + s_l \le z \le 2\mu + s_l\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

By $2\mu \le \mathcal{J}(\frac{\mathbf{k}_1}{2}) - s_l$ and (3.15),

$$(\zeta_1, \zeta_2, \eta) = (\mathcal{K}_\mu(\mathcal{G}^\varepsilon), \mathcal{K}_{\varsigma,\mu}(\varsigma(\mathcal{G}^\varepsilon)), \mathcal{K}_\mu(s^\varepsilon) - \mathcal{K}_\mu(\mathcal{G}^\varepsilon)) \in L^2(0,T;\mathcal{W}_1)\,. \tag{4.31}$$

Set (ζ_1, ζ_2, η) of (4.31) in (3.16)–(3.18) to obtain

$$\int_{\Omega^\tau} \mathcal{K}_\mu(\mathcal{G}^\varepsilon)\partial_t S^\varepsilon + \int_{\Omega^\tau} \Lambda_{\mathrm{w}}(S^\varepsilon)\breve{\mathcal{X}}_\mu(\mathcal{G}^\varepsilon)\nabla_x\Psi^\varepsilon\nabla_x\mathcal{G}^\varepsilon + \int_{\mathcal{Q}^\tau} \mathcal{K}_\mu(s^\varepsilon)\partial_t s^\varepsilon$$
$$\le c_1 \int_{\Omega^\tau} \tilde{\Lambda}^\varepsilon_{\mathrm{w}}(S^\varepsilon)\breve{\mathcal{X}}_\mu(\mathcal{G}^\varepsilon)|\nabla_x\mathcal{G}^\varepsilon|\,, \tag{4.32}$$

where constant c_1 is independent of ε, μ. As Case 1, (4.32) implies

$$\int_{\Omega^\tau} \mathcal{K}_\mu(\mathcal{G}^\varepsilon)\partial_t S^\varepsilon + \int_{\mathcal{Q}^\tau} \mathcal{K}_\mu(s^\varepsilon)\partial_t s^\varepsilon \le c_3 \int_{\Omega^\tau} \frac{\tilde{\Lambda}^\varepsilon_{\mathrm{w}}(S^\varepsilon)\breve{\mathcal{X}}_\mu(\mathcal{G}^\varepsilon)}{|\frac{dv}{ds}(\mathcal{G}^\varepsilon)|}\,. \tag{4.33}$$

Define $\mathcal{Z}^\varepsilon_\mu \overset{\text{def}}{=} \breve{\mathcal{Z}}_\mu(S^\varepsilon) + \hat{\mathcal{Z}}_\mu(s^\varepsilon)$ where

$$\breve{\mathcal{Z}}_\mu(\xi) \overset{\text{def}}{=} \int_{\mathcal{J}^{-1}(2\mu+s_l)}^{\xi} \mathcal{K}_\mu(\mathcal{J}(z))dz\,, \quad \hat{\mathcal{Z}}_\mu(\xi) \overset{\text{def}}{=} \int_{2\mu+s_l}^{\xi} \mathcal{K}_\mu(z)dz\,.$$

$(3.9)_2$, (4.33), and A7(b)_1 yield that, if $0 \le t_1 \le t_2 \le T$,

$$\int_{t_1}^{t_2}\int_{\mathcal{Q}} \partial_t \mathcal{Z}^\varepsilon_\mu \le c_4 \int_{t_1}^{t_2}\int_{\mathcal{Q}} \mathcal{Z}^\varepsilon_{2\mu}\,, \tag{4.34}$$

where c_4 is independent of t_1, t_2, μ and ε. Then following the proof of Case 1, one can show the first part of (4.15). The second part of (4.15) can be shown by a similar argument as the first part of (4.15). By tracing proofs of Case 1 and Case 2, one can see that (4.16) and (4.17) also hold. □

Lemma 4.3. *Suppose* $2 \le \varpi_0 \in \mathbb{N}$ *and* $\frac{2\mathbf{k}_1}{2^{\varpi_0}} \le \min\left\{\mathcal{J}\left(\frac{\mathbf{k}_1}{2}\right) - s_l, s_r - \mathcal{J}\left(1-\frac{\mathbf{k}_1}{2}\right)\right\}$ *where* $\mathbf{k}_1$ *is the one in* A5. *If* $1 < r < 2$ *and* $\varepsilon < \frac{\mathbf{k}_1}{2^{2+\varpi_0}}$, *then*

$$\sum_{\alpha\in\{\text{w,o}\}} (\|P^\varepsilon_\alpha\|_{L^r(0,T;W^{1,r}(\Omega))} + \|p^\varepsilon_\alpha\|_{L^r(0,T;\mathcal{W}^{1,r}_y(\mathcal{Q}))}) \le c\,, \tag{4.35}$$

where c *is a constant independent of* ε.

Proof. We assume A1–A5 and $\text{A7(a)}_{1,2}$ hold. Suppose $\frac{\mathbf{k}_1}{2^{\varpi_*+1}} \le \varepsilon < \frac{\mathbf{k}_1}{2^{\varpi_*}} \le \frac{\mathbf{k}_1}{2^{2+\varpi_0}}$. Due to (3.14), we define

$$\begin{cases} \mathcal{B}_{\varpi_0} \overset{\text{def}}{=} \left\{(x,t)\in\Omega^{\text{T}} : \dfrac{\mathbf{k}_1}{2^{2+\varpi_0}} \le S^\varepsilon\right\}\,, \\ \mathcal{B}_{\varpi} \overset{\text{def}}{=} \left\{(x,t)\in\Omega^{\text{T}} : \dfrac{\mathbf{k}_1}{2^{\varpi+1}} \le S^\varepsilon < \dfrac{\mathbf{k}_1}{2^{\varpi}}\right\}, \text{ for } 2+\varpi_0 \le \varpi \le \varpi_* - 1\,, \\ \mathcal{B}_{\varpi_*} \overset{\text{def}}{=} \left\{(x,t)\in\Omega^{\text{T}} : \dfrac{\mathbf{k}_1}{2^{\varpi_*+1}} \le \varepsilon \le S^\varepsilon < \dfrac{\mathbf{k}_1}{2^{\varpi_*}}\right\}. \end{cases}$$

Lemmas 4.1, 4.2, (3.7) and Hölder inequality imply

$$\begin{aligned} \int_{\Omega^{\text{T}}} |\nabla_x P^\varepsilon_{\text{w}}|^r &\le \left(\int_{\Omega^{\text{T}}} \Lambda_{\text{w}}(S^\varepsilon)|\nabla_x P^\varepsilon_{\text{w}}|^2\right)^{r/2} \left(\int_{\Omega^{\text{T}}} |\Lambda_{\text{w}}(S^\varepsilon)|^{\frac{-r}{2-r}}\right)^{(2-r)/2} \\ &\le c_1\left(\int_{\Omega^{\text{T}}} |\Lambda_{\text{w}}(S^\varepsilon)|^{\frac{-r}{2-r}}\right)^{(2-r)/2} \\ &= c_1\left(\int_{\Omega^{\text{T}}} |\Lambda_{\text{w}}(S^\varepsilon)|^{\frac{-r}{2-r}} \sum_{\varpi=\varpi_0}^{\varpi_*} \mathcal{X}_{\mathcal{B}_\varpi}\right)^{(2-r)/2} \\ &\le c_2 \text{ (independent of } \varepsilon)\,. \end{aligned} \tag{4.36}$$

See (4.3) for $\mathcal{X}_{\mathcal{B}}$. By (3.11), $\|P^\varepsilon_{\text{w}}\|_{L^r(0,T;W^{1,r}(\Omega))}$ is bounded independently of ε. By a similar argument, one can show the rest of (4.35). Furthermore, a similar argument will show (4.35) if one of the conditions A7(b), A7(c), and A7(d) holds. □

Lemma 4.4. *For $f \in C_0^\infty(\Omega)$ and sufficiently small ϖ, solutions of Theorem* 3.1 *satisfy*

$$\int_\varpi^{\mathrm{T}} \int_\Omega f(x)(S^\varepsilon(t) - S^\varepsilon(t-\varpi))(\Phi^\varepsilon(t) - \Phi^\varepsilon(t-\varpi)) \le c\varpi \|f\|_{W^{1,\infty}(\Omega)}\,,$$

where c is independent of ε, ϖ.

Proof. Let $f \in C_0^\infty(\Omega)$. One can see

$$\zeta_1(x,t) \overset{\text{def}}{=} f(x) \int_{\max(t,\varpi)}^{\min(t+\varpi,T)} \varpi\ \partial^{-\varpi}\Phi^\varepsilon(x,\tau)d\tau \in L^2(0,T;\mathcal{V})\,,$$

$$\eta(x,y,t) \overset{\text{def}}{=} f(x) \int_{\max(t,\varpi)}^{\min(t+\varpi,T)} \varpi\ \partial^{-\varpi}(\phi^\varepsilon - \Phi^\varepsilon)(x,y,\tau)d\tau \in L^2(0,T;\mathcal{U}_0)\,.$$

See $(2.1)_1$ for time differentiation. Employ ζ_1 and η above in (3.16) and (3.18) respectively to obtain, by Fubini's theorem and Lemma 4.1,

$$\begin{aligned} &\int_\varpi^{\mathrm{T}} \int_\Omega f(x)\varpi^2 \partial^{-\varpi} S^\varepsilon\ \partial^{-\varpi}\Phi^\varepsilon(x,\tau) + \int_\varpi^{\mathrm{T}} \int_{\mathcal{Q}} f(x)\varpi^2\partial^{-\varpi} s^\varepsilon\ \partial^{-\varpi}\phi^\varepsilon(x,y,\tau) \\ &\quad = \int_{\Omega^{\mathrm{T}}} \partial_t S^\varepsilon(x,t)\zeta_1 + \int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s^\varepsilon(x,y,t)(\eta+\zeta_1) \\ &\quad = -\int_{\Omega^{\mathrm{T}}} \left(\tilde{\Lambda}_{\mathrm{w}}^\varepsilon \nabla_x(P^\varepsilon - E_{\mathrm{w}}) - \frac{\Lambda_{\mathrm{w}}\Lambda_{\mathrm{o}}}{\Lambda}\nabla_x \Upsilon(S^\varepsilon)\right)\nabla_x\zeta_1 + \int_{\mathcal{Q}^{\mathrm{T}}} \frac{\lambda_{\mathrm{w}}\lambda_{\mathrm{o}}}{\lambda}\nabla_y \upsilon(s^\varepsilon)\nabla_y\eta \\ &\quad \le c\varpi\|f\|_{W^{1,\infty}(\Omega)}\,, \end{aligned} \tag{4.37}$$

where c is independent of ε, ϖ. So the proof is complete. □

Let $\mathbf{m} \in \mathbb{N}$, $\delta = \frac{T}{\mathbf{m}}$, $\mathcal{I}_{i,\delta} = [(i-1)\delta, i\delta)$. We define $\mathcal{A}^\delta : L^1([0,T]) \to L^1([0,T])$ by

$$\mathcal{A}^\delta(\zeta)(t) \overset{\text{def}}{=} \frac{1}{\delta}\int_{\mathcal{I}_{i,\delta}} \zeta(\tau)d\tau\,, \quad \text{for } t \in \mathcal{I}_{i,\delta}\,. \tag{4.38}$$

Lemma 4.5. *As $\delta \to 0$, $\|\Phi^\varepsilon - \mathcal{A}^\delta(\Phi^\varepsilon)\|_{L^2(\Omega^{\mathrm{T}})}$ converges to* 0 *uniformly in ε.*

Proof. Let $0 \ne f \in C_0^\infty(\Omega)$. Define

$$\begin{aligned} \mathcal{B}(\varepsilon,\varpi,\mathbf{n}) \overset{\text{def}}{=} \Bigg\{ & t \in (\varpi,T) : \|\Phi^\varepsilon\|_{H^1(\Omega)}(t) + \|\Phi^\varepsilon\|_{H^1(\Omega)}(t-\varpi) \\ & + \frac{1}{\varpi}\int_\Omega \frac{\varpi^2 f(x)}{\|f\|_{W^{1,\infty}(\Omega)}}\partial^{-\varpi}S^\varepsilon(x,t)\partial^{-\varpi}\Phi^\varepsilon(x,t)dx > \mathbf{n}\Bigg\}\,. \end{aligned} \tag{4.39}$$

By Lemmas 4.1, 4.4 and (4.39), $\int_{\mathcal{B}(\varepsilon,\varpi,\mathbf{n})} \mathbf{n}\ dt \le c$, where c is independent of ε, ϖ. So

$$|\mathcal{B}(\varepsilon,\varpi,\mathbf{n})| \le c/\mathbf{n}\,, \quad \text{for all } \varepsilon,\varpi\,. \tag{4.40}$$

Next we claim: *If* $\mathbf{n}$ *is fixed, then as* $\varpi \to 0$,

$$\|\Phi^\varepsilon(\cdot,t) - \Phi^\varepsilon(\cdot,t-\varpi)\|_{L^2(\Omega)} \to 0\,, \quad \textit{uniformly in } \varepsilon \textit{ and } t\,, \tag{4.41}$$

where $t \in (\varpi, T) \setminus \mathcal{B}(\varepsilon, \varpi, \mathbf{n})$.

Proof of claim: If not, there is a constant $c_1 > 0$ and a sequence $\{t_\varpi, \varepsilon_\varpi\}$ such that, as $\varpi \to 0$,

$$\begin{cases} t_\varpi \in (\varpi, T) \setminus \mathcal{B}(\varepsilon_\varpi, \varpi, \mathbf{n})\,, \\ \|\Phi^{\varepsilon_\varpi}\|_{H^1(\Omega)}(t_\varpi) + \|\Phi^{\varepsilon_\varpi}\|_{H^1(\Omega)}(t_\varpi - \varpi) \le \mathbf{n}\,, \\ \displaystyle\int_\Omega \frac{\varpi^2 f(x)}{\|f\|_{W^{1,\infty}(\Omega)}} \partial^{-\varpi} S^{\varepsilon_\varpi}(x,t_\varpi) \partial^{-\varpi} \Phi^{\varepsilon_\varpi}(x,t_\varpi) dx \le \mathbf{n}\varpi\,, \\ \|\Phi^{\varepsilon_\varpi}(t_\varpi) - \Phi^{\varepsilon_\varpi}(t_\varpi - \varpi)\|_{L^2(\Omega)} \ge c_1\,. \end{cases} \tag{4.42}$$

By $(4.42)_2$ and compactness principle, there is a subsequence (not be relabelled) of $\{\Phi^{\varepsilon_\varpi}(t_\varpi), \Phi^{\varepsilon_\varpi}(t_\varpi - \varpi)\}$ converging to $\{g_1, g_2\}$ strongly in $L^2(\Omega)$ and pointwise almost everywhere. By $(4.42)_4$,

$$\|g_1 - g_2\|_{L^2(\Omega)} \ge c_1\,. \tag{4.43}$$

Since $\mathcal{E}^{-1}$ is bounded on $\mathbb{R}$, by $(4.42)_3$,

$$\begin{aligned} &\int_\Omega (\mathcal{E}^{-1}(g_1) - \mathcal{E}^{-1}(g_2))(g_1 - g_2) \frac{f(x)}{\|f\|_{W^{1,\infty}(\Omega)}} dx \\ &\qquad = \lim_{\varpi \to 0} \int_\Omega \frac{\varpi^2 f(x)}{\|f\|_{W^{1,\infty}(\Omega)}} \partial^{-\varpi} S^{\varepsilon_\varpi}(x,t_\varpi) \partial^{-\varpi} \Phi^{\varepsilon_\varpi}(x,t_\varpi) dx = 0\,. \end{aligned} \tag{4.44}$$

Since $\mathcal{E}^{-1}$ is strictly increasing on $\mathbb{R}$ and because f can be any non-negative smooth function, (4.44) implies $g_1 = g_2$ almost everywhere, which contradicts (4.43). So the claim is true.

(4.40) and (4.41) imply, as $\varpi \to 0$,

$$\int_\varpi^{\mathrm{T}} \|\Phi^\varepsilon(\cdot,t) - \Phi^\varepsilon(\cdot,t-\varpi)\|^2_{L^2(\Omega)} dt \to 0\,, \quad \text{uniformly in } \varepsilon\,. \tag{4.45}$$

By (4.38) and (4.45), if $\delta = T/\mathbf{m}$, then

$$\begin{aligned} \int_0^{\mathrm{T}} \|\Phi^\varepsilon - \mathcal{A}^\delta(\Phi^\varepsilon)\|^2_{L^2(\Omega)} dt &= \sum_{i=1}^{\mathbf{m}} \int_{\mathcal{I}_{i,\delta}} \left\| \frac{1}{\delta} \int_{\mathcal{I}_{i,\delta}} (\Phi^\varepsilon(x,t) - \Phi^\varepsilon(x,\tau)) d\tau \right\|^2_{L^2(\Omega)} dt \\ &\le \sum_{i=1}^{\mathbf{m}} \int_{\mathcal{I}_{i,\delta}} \frac{1}{\delta} \int_{t-i\delta}^{t-(i-1)\delta} \|\Phi^\varepsilon(\cdot,t) - \Phi^\varepsilon(\cdot,t-\varpi)\|^2_{L^2(\Omega)} d\varpi dt \\ &\le \frac{2}{\delta} \int_0^\delta \int_\varpi^{\mathrm{T}} \|\Phi^\varepsilon(\cdot,t) - \Phi^\varepsilon(\cdot,t-\varpi)\|^2_{L^2(\Omega)} dt d\varpi\,. \end{aligned} \tag{4.46}$$

Right-hand side of (4.46) converges to 0 uniformly in ε as $\delta \to 0$. So the lemma follows. □

Lemma 4.6. *There is a convergent subsequence of* $\{S^\varepsilon, \mathcal{G}^\varepsilon\}$ *in* $L^2(\Omega^{\rm T})$.

Proof. By Lemma 4.1, $\|\Phi^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le c_1$, which is independent of ε. So for all δ,

$$\|\mathcal{A}^\delta(\Phi^\varepsilon)\|_{L^2(0,T;H^1(\Omega))} \le c_2 \text{ (independent of } \varepsilon\text{)}\,. \tag{4.47}$$

By Lemma 4.5, (4.47), and diagonal process, one can find a subsequence of $\{\Phi^\varepsilon\}$ converging to Φ in $L^2(\Omega^{\rm T})$ strongly and pointwise almost everywhere. By boundedness and continuity of $\mathcal{E}^{-1}$ as well as convergence of $\{\Phi^\varepsilon\}$ in $L^2(\Omega^{\rm T})$, it is not difficult to find a convergent subsequence for $\{S^\varepsilon\}$. Convergence of $\{\mathcal{G}^\varepsilon\}$ is due to the convergence of $\{S^\varepsilon\}$ and boundedness of $\mathcal{J}$. □

For convenience, it is assumed that S^ε converges to S in $L^2(\Omega^{\rm T})$ and pointwise almost everywhere.

Lemma 4.7. $0 < S < 1$.

Proof. Suppose A7(a)$_1$ holds. By Theorem 3.1 and Lemma 4.6, $0 \le S \le 1$. We claim $\mathbf{b} \stackrel{\rm def}{=} |\{(x,t) \in \Omega^{\rm T} : S = 0\}| = 0$. If not, by Egoroff's theorem[15] and Lemma 4.6, there is a set $\mathcal{B} \subset \Omega^{\rm T}$ such that (i) $|\mathcal{B}| < \mathbf{b}/3 \ne 0$ and (ii) S^ε converges uniformly to S in $\Omega^{\rm T} \setminus \mathcal{B}$.

Take ϖ_0, ϖ_1 large enough so that

$$\begin{cases} 2 < \varpi_0 < \varpi_1 - 2\,, \\ \dfrac{2\mathbf{k}_1}{2^{\varpi_0}} \le \min\left\{\mathcal{J}\left(\dfrac{\mathbf{k}_1}{2}\right) - s_l, s_r - \mathcal{J}\left(1 - \dfrac{\mathbf{k}_1}{2}\right)\right\}\,, \\ \dfrac{|c_0 T|^{\varpi_1 - \varpi_0 + 1}}{(\varpi_1 - \varpi_0)^{(\varpi_1 - \varpi_0)\mathbf{f}_{\varpi_1}}} \le \dfrac{\mathbf{b}}{3}\,, \end{cases} \tag{4.48}$$

where $\mathbf{k}_1$ is the one in A5 and c_0, $\mathbf{f}_\varpi$ are those in Lemma 4.2. By Lemma 4.2 and (4.48), for all $\varepsilon < \mu \stackrel{\rm def}{=} \frac{\mathbf{k}_1}{2^{\varpi_1}}$,

$$|\{(x,t) \in \Omega^{\rm T} : S^\varepsilon \le \mu\}| \le \frac{|c_0 T|^{\varpi_1 - \varpi_0 + 1}}{(\varpi_1 - \varpi_0)^{(\varpi_1 - \varpi_0)\mathbf{f}_{\varpi_1}}} \le \frac{\mathbf{b}}{3}\,. \tag{4.49}$$

Since S^ε converges uniformly to S in $\Omega^{\rm T} \setminus \mathcal{B}$, there is a $\varepsilon_0 \le \mu \left(= \frac{\mathbf{k}_1}{2^{\varpi_1}}\right)$ such that, for any $\varepsilon < \varepsilon_0$,

$$|S^\varepsilon - S|(x,t) \le \mu\,, \quad \text{for } (x,t) \in \Omega^{\rm T} \setminus \mathcal{B}\,. \tag{4.50}$$

However, (4.49) and (4.50) imply, for any $\varepsilon < \varepsilon_0$,

$$\frac{2\mathbf{b}}{3} \le |\{(x,t) \in \Omega^{\rm T} \setminus \mathcal{B} : S = 0\}| \le |\{(x,t) \in \Omega^{\rm T} \setminus \mathcal{B} : S^\varepsilon \le \mu\}| \le \frac{\mathbf{b}}{3}\,, \tag{4.51}$$

that is in contradiction to $\mathbf{b} \ne 0$. So $0 < S$. By a similar argument, one can prove $S < 1$. Moreover, a similar argument will show the lemma if one of the conditions A7(b), A7(c), and A7(d) holds. So proof of this lemma is complete. □

Lemma 4.8. $\{\mathcal{D}(\mathcal{G}^\varepsilon)\}$ *is a Cauchy sequence in* $L^2(\Omega^{\mathrm{T}})$. *See* (2.1) *for* $\mathcal{D}$.

Proof. Case 1. Suppose A7(a) or A7(c) holds. If $\mathcal{D}$ is a bounded function on $(s_l, s_r]$, the lemma is obvious by Lemmas 4.6 and 4.7. If not, for any $\delta > 0$, one can find $\varpi_0, \varpi_1 \in \mathbb{N}$ and a positive number $\mathbf{b}$ such that, by A7(a)$_3$ or A7(c)$_5$,

$$\begin{cases} 2 < \varpi_0 < \varpi_1 - 2\,, \\ \dfrac{2\mathbf{k}_1}{2^{\varpi_0}} \le \min\left\{\mathcal{J}\left(\dfrac{\mathbf{k}_1}{2}\right) - s_l, s_r - \mathcal{J}\left(1 - \dfrac{\mathbf{k}_1}{2}\right)\right\}, \\ \mathcal{D}\left(\mathcal{J}\left(\dfrac{\mathbf{k}_1}{2^{\varpi_1}}\right)\right) < 0\,, \\ \displaystyle\sum_{\varpi=\varpi_1}^{\infty} \left|\mathcal{D}\left(\mathcal{J}\left(\frac{\mathbf{k}_1}{2^{\varpi+1}}\right)\right)\right|^2 \frac{|c_0 T|^{\varpi-\varpi_0+1}}{(\varpi-\varpi_0)^{(\varpi-\varpi_0)\mathbf{f}_\varpi}} < \delta\,, \\ \left(\left|\mathcal{D}\left(\mathcal{J}\left(\dfrac{\mathbf{k}_1}{2^{\varpi_1}}\right)\right)\right|^2 + |\mathcal{D}(\mathcal{J}(1))|^2\right) \max\left\{\mathbf{b},\ \dfrac{c_0|c_0T|^{\varpi_1-\varpi_0}}{(\varpi_1-\varpi_0)^{(\varpi_1-\varpi_0)\mathbf{f}_{\varpi_1}}}\right\} < \delta\,, \end{cases} \tag{4.52}$$

where $\mathbf{k}_1$ is the one in A5 and c_0, $\mathbf{f}_\varpi$ are those in Lemma 4.2. Suppose $\frac{\mathbf{k}_1}{2^{\varpi_*+1}} \le \varepsilon < \frac{\mathbf{k}_1}{2^{\varpi_*}} \le \frac{\mathbf{k}_1}{2^{\varpi_1}}$. Because of (3.14), we define

$$\begin{cases} \mathcal{B}^{\varepsilon,\varpi_1} \overset{\text{def}}{=} \left\{(x,t) \in \Omega^{\mathrm{T}}\colon S^\varepsilon < \dfrac{\mathbf{k}_1}{2^{\varpi_1}}\right\}, \\ \mathcal{B}_\varpi \overset{\text{def}}{=} \left\{(x,t) \in \Omega^{\mathrm{T}}\colon \dfrac{\mathbf{k}_1}{2^{\varpi+1}} \le S^\varepsilon < \dfrac{\mathbf{k}_1}{2^{\varpi}}\right\},\ \text{for } \varpi_1 \le \varpi \le \varpi_* - 1\,, \\ \mathcal{B}_{\varpi_*} \overset{\text{def}}{=} \left\{(x,t) \in \Omega^{\mathrm{T}}\colon \dfrac{\mathbf{k}_1}{2^{\varpi_*+1}} \le \varepsilon \le S^\varepsilon < \dfrac{\mathbf{k}_1}{2^{\varpi_*}}\right\}. \end{cases}$$

Then $\mathcal{B}^{\varepsilon,\varpi_1} = \bigcup_{\varpi=\varpi_1}^{\varpi_*} \mathcal{B}_\varpi$. Lemma 4.2 and (4.52)$_4$ imply

$$\begin{aligned} \int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^\varepsilon))|^2 \mathcal{X}_{\mathcal{B}^{\varepsilon,\varpi_1}} &= \int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^\varepsilon))|^2 \sum_{\varpi=\varpi_1}^{\varpi_*} \mathcal{X}_{\mathcal{B}_\varpi} \\ &\le \sum_{\varpi=\varpi_1}^{\varpi_*} \left|\mathcal{D}\left(\mathcal{J}\left(\frac{\mathbf{k}_1}{2^{\varpi+1}}\right)\right)\right|^2 \frac{|c_0T|^{\varpi-\varpi_0+1}}{(\varpi-\varpi_0)^{(\varpi-\varpi_0)\mathbf{f}_\varpi}} < \delta\,. \end{aligned} \tag{4.53}$$

See (4.3) for characteristic function $\mathcal{X}_{\mathcal{B}}$.

Let both ε_i, $\varepsilon_j < \mathbf{k}_1/2^{\varpi_1}$. Define

$$\begin{cases} \mathcal{K}^{i,j} \overset{\text{def}}{=} \left\{(x,t) \in \Omega^{\mathrm{T}}\colon \dfrac{\mathbf{k}_1}{2^{\varpi_1}} \le \min\{S^{\varepsilon_i}, S^{\varepsilon_j}\}\right\}, \\ \tilde{\mathcal{K}}_{i,j} \overset{\text{def}}{=} \left\{(x,t) \in \Omega^{\mathrm{T}}\colon S^{\varepsilon_i} \le \dfrac{\mathbf{k}_1}{2^{\varpi_1}} \le S^{\varepsilon_j}\right\}. \end{cases}$$

Consider the following

$$\begin{aligned}\int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i})) - \mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 &\le \int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i})) - \mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\mathcal{K}^{i,j}} \\ &+ \int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i}))|^2 \mathcal{X}_{\tilde{\mathcal{K}}_{j,i}} + \int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\tilde{\mathcal{K}}_{i,j}} \\ &+ \int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i}))|^2 \mathcal{X}_{\mathcal{B}^{\varepsilon_i,\varpi_1}} + \int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\mathcal{B}^{\varepsilon_j,\varpi_1}} . \end{aligned} \tag{4.54}$$

By (4.53),

$$\int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i}))|^2 \mathcal{X}_{\mathcal{B}^{\varepsilon_i,\varpi_1}} + \int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\mathcal{B}^{\varepsilon_j,\varpi_1}} \le 2\delta . \tag{4.55}$$

By Lemma 4.2 and $(4.52)_5$,

$$\int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i}))|^2 \mathcal{X}_{\tilde{\mathcal{K}}_{j,i}} + \int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\tilde{\mathcal{K}}_{i,j}} \le c_1\delta . \tag{4.56}$$

Lemmas 4.6 and 4.7 imply that $\mathcal{D}(\mathcal{J}(S^\varepsilon))$ converges to $\mathcal{D}(\mathcal{J}(S))$ pointwise almost everywhere. By Egoroff's theorem,[15] one can select a set $\mathcal{B}$ such that (i) $|\mathcal{B}| < \mathbf{b}$ ($\mathbf{b}$ is the one in $(4.52)_5$) and (ii) $\mathcal{D}(\mathcal{J}(S^\varepsilon))$ converges to $\mathcal{D}(\mathcal{J}(S))$ uniformly in $\Omega^{\mathrm{T}} \setminus \mathcal{B}$. By $(4.52)_5$,

$$\int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i})) - \mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\mathcal{K}^{i,j}\cap\mathcal{B}} < c_2\delta , \tag{4.57}$$

and there is a $\varepsilon_0 < \frac{\mathbf{k}_1}{2^{\varpi_1}}$ such that, for both ε_i, $\varepsilon_j < \varepsilon_0$,

$$\int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i})) - \mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \mathcal{X}_{\mathcal{K}^{i,j}\setminus\mathcal{B}} \le \delta . \tag{4.58}$$

Therefore, by (4.54)–(4.58), for any $\delta > 0$, there is a ε_0 such that, as ε_i, $\varepsilon_j \le \varepsilon_0$,

$$\int_{\Omega^{\mathrm{T}}} |\mathcal{D}(\mathcal{J}(S^{\varepsilon_i})) - \mathcal{D}(\mathcal{J}(S^{\varepsilon_j}))|^2 \le c_3\delta . \tag{4.59}$$

So convergence of $\{\mathcal{D}(\mathcal{J}(S^\varepsilon))\}$ (i.e. $\{\mathcal{D}(\mathcal{G}^\varepsilon)\}$) is proved.

Case 2. If A7(b) or A7(d) holds, the convergence of $\{\mathcal{D}(\mathcal{G}^\varepsilon)\}$ can be shown by a similar argument as Case 1. □

Lemma 4.9. *There is a convergent subsequence of* $\{s^\varepsilon\}$ *in* $L^2(\mathcal{Q}^{\mathrm{T}})$.

Proof. Step 1. By (3.11), Theorem 3.1 and Lemmas 4.1, 4.6, 4.8, there is a subsequence (not be relabelled) of $\{S^\varepsilon, s^\varepsilon\}$ such that, as $\varepsilon \to 0$,

$$\begin{cases} S^\varepsilon, \mathcal{G}^\varepsilon \to S, \mathcal{G} , & \text{in } L^2(\Omega^{\mathrm{T}}) \text{ strongly} , \\ \mathcal{D}(s^\varepsilon) \to \hat{\mathcal{D}} , & \text{in } L^2(0,T;\mathcal{U}) \text{ weakly} , \\ s^\varepsilon \to s , & \text{in } L^2(\mathcal{Q}^{\mathrm{T}}) \text{ weakly} , \\ \partial_t s^\varepsilon \to \partial_t s , & \text{in dual } L^2(0,T;\mathcal{U}_0) \text{ weakly} , \\ s^\varepsilon(T) \to \hat{s} , & \text{in } L^2(\mathcal{Q}) \text{ weakly} , \\ s^\varepsilon(0) \to s_0 , & \text{in } L^2(\mathcal{Q}) \text{ strongly} . \end{cases} \tag{4.60}$$

Suppose $\mathbf{v}_i$ ($i \in \mathbb{N}$) is a smooth function in $\mathcal{Q}$ and $\{\mathbf{v}_i\}_{i=1}^{\infty}$ forms a basis of $\mathcal{U}_0$. For each i and $f \in C^1[0,T]$, one obtains, by (2.1) and (3.18),

$$-\int_{\mathcal{Q}^{\mathrm{T}}} s^{\varepsilon}\ \partial_t f(t)\mathbf{v}_i + \int_{\mathcal{Q}^{\mathrm{T}}} \nabla_y \mathcal{D}(s^{\varepsilon})\ f(t)\nabla_y \mathbf{v}_i = \int_{\mathcal{Q}} (s^{\varepsilon}(0)f(0) - s^{\varepsilon}(T)f(T))\mathbf{v}_i\,. \tag{4.61}$$

As $\varepsilon \to 0$, (4.60) implies

$$-\int_{\mathcal{Q}^{\mathrm{T}}} s\ \partial_t f(t)\mathbf{v}_i + \int_{\mathcal{Q}^{\mathrm{T}}} \nabla_y \hat{\mathcal{D}}\ f(t)\nabla_y \mathbf{v}_i = -\int_{\mathcal{Q}} \hat{s}\ f(T)\mathbf{v}_i + \int_{\mathcal{Q}} s_0\ f(0)\mathbf{v}_i\,. \tag{4.62}$$

Applying Green's theorem for (4.62) in the t variable yields

$$\begin{aligned}\int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s\ f(t)\mathbf{v}_i + \int_{\mathcal{Q}^{\mathrm{T}}} \nabla_y \hat{\mathcal{D}}\ f(t)\nabla_y \mathbf{v}_i \\ = -\int_{\mathcal{Q}} (\hat{s} - s(T))f(T)\mathbf{v}_i + \int_{\mathcal{Q}} (s_0 - s(0))f(0)\mathbf{v}_i\,.\end{aligned} \tag{4.63}$$

Since $\{\mathbf{v}_i\}_{i=1}^{\infty}$ is a basis of $\mathcal{U}_0$, (4.63) implies

$$\hat{s} = s(T)\,, \quad s(0) = s_0\,, \tag{4.64}$$

and

$$\int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s\ \eta + \int_{\mathcal{Q}^{\mathrm{T}}} \nabla_y \hat{\mathcal{D}}\ \nabla_y \eta = 0\,, \qquad \text{for } \eta \in L^2(0,T;\mathcal{U}_0)\,. \tag{4.65}$$

Step 2. We claim $\mathcal{D}^{-1}(\hat{\mathcal{D}}) = s$. See Remark 2.1 for $\mathcal{D}^{-1}$. Let us find φ^{ε}, $\varphi \in L^2(0,T;\mathcal{U}_0)$ by solving, for all $(x,t) \in \Omega^{\mathrm{T}}$,

$$\begin{cases} -\Delta_y \varphi^{\varepsilon} = s^{\varepsilon}\,, & y \in \mathcal{M}\,, \\ \varphi^{\varepsilon}|_{\partial\mathcal{M}} = 0\,, \end{cases} \qquad \begin{cases} -\Delta_y \varphi = s\,, & y \in \mathcal{M}\,, \\ \varphi|_{\partial\mathcal{M}} = 0\,. \end{cases} \tag{4.66}$$

(3.18), (4.66), and Green's theorem imply

$$\begin{aligned}\int_{\mathcal{Q}^{\mathrm{T}}} \mathcal{D}(s^{\varepsilon})s^{\varepsilon} &= \int_{\mathcal{Q}^{\mathrm{T}}} \mathcal{D}(\mathcal{G}^{\varepsilon})s^{\varepsilon} - \int_{\mathcal{Q}^{\mathrm{T}}} (\mathcal{D}(s^{\varepsilon}) - \mathcal{D}(\mathcal{G}^{\varepsilon}))\Delta_y \varphi^{\varepsilon} \\ &= \int_{\mathcal{Q}^{\mathrm{T}}} \mathcal{D}(\mathcal{G}^{\varepsilon})s^{\varepsilon} - \int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s^{\varepsilon}\varphi^{\varepsilon}\,.\end{aligned} \tag{4.67}$$

Note that

$$-\int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s^{\varepsilon}\varphi^{\varepsilon} = -\int_{\mathcal{Q}} \frac{|\nabla_y \varphi^{\varepsilon}|^2}{2}(T) + \int_{\mathcal{Q}} \frac{|\nabla_y \varphi^{\varepsilon}|^2}{2}(0)\,. \tag{4.68}$$

By $(4.60)_5$ and (4.64), $s^{\varepsilon}(T)$ converges weakly to $s(T)$ in $L^2(\mathcal{Q})$. (4.66), Hölder inequality, and Green's theorem imply

$$\int_{\mathcal{Q}} |\nabla_y \varphi|^2(T) \leq \varliminf_{\varepsilon \to 0} \int_{\mathcal{Q}} |\nabla_y \varphi^{\varepsilon}|^2(T)\,. \tag{4.69}$$

Take limit supremum on both sides of (4.67) to obtain, by (4.60) and Lemma 4.8,

$$\overline{\lim_{\varepsilon\to 0}}\int_{\mathcal{Q}^{\mathrm{T}}}\mathcal{D}(s^\varepsilon)s^\varepsilon \le \int_{\mathcal{Q}^{\mathrm{T}}}\mathcal{D}(\mathcal{G})s - \int_{\mathcal{Q}}\frac{|\nabla_y\varphi|^2}{2}(T) + \int_{\mathcal{Q}}\frac{|\nabla_y\varphi|^2}{2}(0)\,. \tag{4.70}$$

Set $\eta = \varphi$ in (4.65) to obtain

$$0 = \int_{\mathcal{Q}}\frac{|\nabla_y\varphi|^2}{2}(T) - \int_{\mathcal{Q}}\frac{|\nabla_y\varphi|^2}{2}(0) + \int_{\mathcal{Q}^{\mathrm{T}}}(\hat{\mathcal{D}} - \mathcal{D}(\mathcal{G}))s\,. \tag{4.71}$$

By (4.70) and (4.71),

$$\overline{\lim_{\varepsilon\to 0}}\int_{\mathcal{Q}^{\mathrm{T}}}\mathcal{D}(s^\varepsilon)s^\varepsilon \le \int_{\mathcal{Q}^{\mathrm{T}}}\hat{\mathcal{D}}s\,. \tag{4.72}$$

Since $\mathcal{D}^{-1}$ is strictly increasing on $\mathbb{R}$, for any $f \in L^2(\mathcal{Q}^{\mathrm{T}})$,

$$0 \le \int_{\mathcal{Q}^{\mathrm{T}}}(\mathcal{D}^{-1}(\mathcal{D}(s^\varepsilon)) - \mathcal{D}^{-1}(f))(\mathcal{D}(s^\varepsilon) - f)\,. \tag{4.73}$$

By (4.60), (4.72) and (4.73), and monotonicity argument,[16] one can easily obtain

$$\mathcal{D}^{-1}(\hat{\mathcal{D}}) = s\,. \tag{4.74}$$

Step 3. We claim that $\{s^\varepsilon\}$ is a convergent sequence in $L^2(\mathcal{Q}^{\mathrm{T}})$. By (4.60), (4.72) and (4.74),

$$\lim_{\varepsilon\to 0}\int_{\mathcal{Q}^{\mathrm{T}}}(\mathcal{D}(s^\varepsilon) - \hat{\mathcal{D}})(s^\varepsilon - s) = 0\,. \tag{4.75}$$

Define $\mathcal{F}_{1,\varepsilon} \stackrel{\text{def}}{=} (\mathcal{D}(s^\varepsilon) - \hat{\mathcal{D}})(s^\varepsilon - s)$. By (4.74) and (4.75), $\mathcal{F}_{1,\varepsilon}$ converges to 0 in $L^1(\mathcal{Q}^{\mathrm{T}})$. So there is a subsequence (not be relabelled) of $\{\mathcal{F}_{1,\varepsilon}\}$ converging to 0 pointwise almost everywhere.

Let us consider a point $(x_0, y_0, t_0) \in \mathcal{Q}^{\mathrm{T}}$ which satisfies $\lim_{\varepsilon\to 0}\mathcal{F}_{1,\varepsilon}(x_0, y_0, t_0) = 0$. It is not difficult to see that $\{\mathcal{D}(s^\varepsilon(x_0, y_0, t_0))\}$ is a bounded set. For any accumulation point $\mathcal{D}_{x_0,y_0,t_0}$ of $\{\mathcal{D}(s^\varepsilon(x_0, y_0, t_0))\}$, one may find a subsequence (not be relabelled) of $\{\mathcal{D}(s^\varepsilon(x_0, y_0, t_0))\}$ such that $\lim_{\varepsilon\to 0}\mathcal{D}(s^\varepsilon(x_0, y_0, t_0)) = \mathcal{D}_{x_0,y_0,t_0}$. Since $\mathcal{D}^{-1}$ is continuous,

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0}(\mathcal{D}(s^\varepsilon(x_0, y_0, t_0)) - \hat{\mathcal{D}}(x_0, y_0, t_0))(s^\varepsilon(x_0, y_0, t_0) - s(x_0, y_0, t_0)) \\ &= (\mathcal{D}_{x_0,y_0,t_0} - \hat{\mathcal{D}}(x_0, y_0, t_0))(\mathcal{D}^{-1}(\mathcal{D}_{x_0,y_0,t_0}) - s(x_0, y_0, t_0))\,. \end{aligned} \tag{4.76}$$

(4.74) and (4.76) imply $\mathcal{D}_{x_0,y_0,t_0} = \hat{\mathcal{D}}(x_0, y_0, t_0)$. So $\{\mathcal{D}(s^\varepsilon(x_0, y_0, t_0))\}$ has only one accumulation point $\hat{\mathcal{D}}(x_0, y_0, t_0)$. Since $\mathcal{F}_{1,\varepsilon}$ converges to 0 pointwise almost everywhere, $\mathcal{D}(s^\varepsilon)$ converges to $\hat{\mathcal{D}}$ pointwise almost everywhere. By continuity of $\mathcal{D}^{-1}$ and boundedness of $\{s^\varepsilon\}$ in $\mathcal{Q}^{\mathrm{T}}$, s^ε converges to s in $L^2(\mathcal{Q}^{\mathrm{T}})$. □

By Lemma 4.9 and a similar argument as Lemma 4.7, one can obtain the following result:

Lemma 4.10. $s_l < s < s_r$.

Proof of Theorem 2.1. By Theorem 3.1 and integration by parts,

$$\int_{\Omega^{\mathrm{T}}} \partial_t S^\varepsilon \zeta + \int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s^\varepsilon (\zeta+\eta) = \int_{\Omega^{\mathrm{T}}} (S_0^\varepsilon - S^\varepsilon)\partial_t \zeta + \int_{\mathcal{Q}^{\mathrm{T}}} (s_0^\varepsilon - s^\varepsilon)\partial_t(\zeta+\eta)\,,$$

for $\zeta \in L^2(0,T;\mathcal{V}) \cap H^1(\Omega^{\mathrm{T}}), \eta \in L^2(0,T;\mathcal{U}_0) \cap H^1(0,T;L^2(\mathcal{Q}))$, and $\zeta(T) = \eta(T) = 0$. By (3.11) and Lemmas 4.6, 4.9, we obtain (2.6). Indeed, Theorem 2.1 is a direct consequence of Theorem 3.1, Lemmas 4.1, 4.3 and 4.6–4.10.

5. Existence of the Auxiliary Problem

Now we prove Theorem 3.1, which is done by Galerkin's method. Let $I = (0,T]$, $\ell \in \mathbb{N}$, $h = T/\ell$, $t_m = mh$, and $I_m^* = (t_{m-1}, t_m]$. For a Banach space X,

$$I_h(X) \overset{\mathrm{def}}{=} \{f \in L^\infty(0,T;X) : f \text{ is constant in time on each } I_m^* \subset I\}\,. \tag{5.1}$$

If $f \in I_h(X)$, $f|_{I_m^*} = f(t_m)$ for $m \le \ell = T/h$. For $f \in L^\infty(\mathcal{Q}^{\mathrm{T}})$,

$$\wp(f)(x,y,t) \overset{\mathrm{def}}{=} \frac{1}{h}\int_{I_m^*} f(x,y,\tau)d\tau\,, \quad \text{for } t \in I_m^*\,. \tag{5.2}$$

One approximates $\mathcal{G}_b^\varepsilon$, P_b^ε, E_α for $\alpha \in \{\mathrm{w},\mathrm{o}\}$ by

$$\mathcal{G}_b^{\varepsilon,h} \overset{\mathrm{def}}{=} \wp(\mathcal{G}_b^\varepsilon)\,, \qquad P_b^{\varepsilon,h} \overset{\mathrm{def}}{=} \wp(P_b^\varepsilon)\,, \qquad E_\alpha^h \overset{\mathrm{def}}{=} \wp(E_\alpha)\,. \tag{5.3}$$

By A4 and (3.11)–(3.12), it is not difficult to see, for $\alpha \in \{\mathrm{w},\mathrm{o}\}$,

$$\begin{cases} \mathcal{G}_b^{\varepsilon,h} \to \mathcal{G}_b^\varepsilon\,, \\ P_b^{\varepsilon,h} \to P_b^\varepsilon\,, \qquad \text{in } L^2(0,T;H^1(\Omega)) \text{ as } h \to 0\,. \\ E_\alpha^h \to E_\alpha\,, \end{cases} \tag{5.4}$$

Suppose that $\{\mathbf{e}_i\}_{i=1}^\infty$ (resp. $\{\mathbf{v}_i\}_{i=1}^\infty$) is a basis of $\mathcal{V}$ (resp. $\mathcal{U}_0$), and $\mathbf{v}_i$ satisfies

$$\begin{cases} -\Delta_y \mathbf{v}_i = c_i \mathbf{v}_i\,, \quad \text{in } \mathcal{Q}\,, \\ \mathbf{v}_i|_{\Omega\times\partial\mathcal{M}} = 0\,, \end{cases} \tag{5.5}$$

for some constant c_i. Let $\mathcal{V}^h$ (resp. $\mathcal{U}_0^h$) denote the linear span of $\{\mathbf{e}_i\}_{i=1}^\ell$ (resp. $\{\mathbf{v}_i\}_{i=1}^\ell$) where $\ell = T/h$. $\mathcal{W}_1^h \overset{\mathrm{def}}{=} \mathcal{V}^h \times \mathcal{V}^h \times \mathcal{U}_0^h$. By $(3.10)_4$, one may find $\mathcal{G}_0^{\varepsilon,h}$ such that $\mathcal{G}_0^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h}(0)$ is the L^2 projection of $\mathcal{G}_0^\varepsilon - \mathcal{G}_b^\varepsilon(0)$ on $\mathcal{V}^h$. Let $s_0^{\varepsilon,h} \overset{\mathrm{def}}{=} \mathcal{G}_0^{\varepsilon,h}$.

A discretized scheme for (3.13)–(3.19) is to find $\{S^{\varepsilon,h}, \mathcal{G}^{\varepsilon,h}, P^{\varepsilon,h}, s^{\varepsilon,h}\}$ such that

$$(\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h}, P^{\varepsilon,h} - P_b^{\varepsilon,h}, s^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h}) \in I_h(\mathcal{W}_1^h)\,, \tag{5.6}$$

$$S^{\varepsilon,h}(0) = \mathcal{J}^{\varepsilon,-1}(\mathcal{G}^{\varepsilon,h}(0))\,, \qquad \mathcal{G}^{\varepsilon,h}(0) = \mathcal{G}_0^{\varepsilon,h}\,, \qquad s^{\varepsilon,h}(0) = s_0^{\varepsilon,h}\,, \tag{5.7}$$

and if $\{S^{\varepsilon,h}, \mathcal{G}^{\varepsilon,h}, s^{\varepsilon,h}\}(t_{m-1})$ is given, then $(\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h}, P^{\varepsilon,h} - P_b^{\varepsilon,h}, s^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h})(t_m)$ is a zero of the mapping $\mathcal{H}^{\varepsilon,h} : \mathbb{R}^{3\ell} \to \mathbb{R}^{3\ell}$, $\ell = T/h$ defined by

$$\mathcal{H}^{\varepsilon,h}(\xi_{1,1}, \cdot, \xi_{1,\ell}, \xi_{2,1}, \cdot, \xi_{2,\ell}, \xi_{3,1}, \cdot, \xi_{3,\ell}) = (\bar{\xi}_{1,1}, \cdot, \bar{\xi}_{1,\ell}, \bar{\xi}_{2,1}, \cdot, \bar{\xi}_{2,\ell}, \bar{\xi}_{3,1}, \cdot, \bar{\xi}_{3,\ell})\,, \tag{5.8}$$

where

$$(\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h}, P^{\varepsilon,h} - P_b^{\varepsilon,h}, s^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h})(t_m) = \sum_{i=1}^{\ell} (\xi_{1,i}\mathbf{e}_i,\ \xi_{2,i}\mathbf{e}_i,\ \xi_{3,i}\mathbf{v}_i) \in \mathcal{W}_1^h\,, \tag{5.9}$$

$$S^{\varepsilon,h}(t_m) = \mathcal{J}^{\varepsilon,-1}(\mathcal{G}^{\varepsilon,h}(t_m))\,, \tag{5.10}$$

$$\begin{aligned}\bar{\xi}_{1,i} = {} & \int_\Omega \partial^{-h} S^{\varepsilon,h}(t_m)\mathbf{e}_i + \int_\Omega \tilde{\Lambda}_{\mathrm{w}}^\varepsilon(S^{\varepsilon,h})\nabla_x(P^{\varepsilon,h} - E_{\mathrm{w}}^h)(t_m)\nabla_x \mathbf{e}_i \\ & - \int_\Omega \frac{\Lambda_{\mathrm{w}}^\varepsilon \Lambda_{\mathrm{o}}^\varepsilon(S^{\varepsilon,h})\frac{d\upsilon^\varepsilon}{ds}(\mathcal{G}^{\varepsilon,h})}{\Lambda^\varepsilon(S^{\varepsilon,h})}\nabla_x \mathcal{G}^{\varepsilon,h}(t_m)\nabla_x\mathbf{e}_i + \int_{\mathcal{Q}} \partial^{-h} s^{\varepsilon,h}(t_m)\mathbf{e}_i\,,\end{aligned} \tag{5.11}$$

$$\bar{\xi}_{2,i} = \beta_\varepsilon \int_\Omega \left(\tilde{\Lambda}^\varepsilon(S^{\varepsilon,h})\nabla_x P^{\varepsilon,h}(t_m) - \sum_{\alpha\in\{\mathrm{w},\mathrm{o}\}} \tilde{\Lambda}_\alpha^\varepsilon(S^{\varepsilon,h}(t_m))\nabla_x E_\alpha^h \right) \nabla_x \mathbf{e}_i\,, \tag{5.12}$$

$$\bar{\xi}_{3,i} = \int_{\mathcal{Q}} \partial^{-h} s^{\varepsilon,h}(t_m)\mathbf{v}_i - \int_{\mathcal{Q}} \frac{\lambda_{\mathrm{w}}^\varepsilon \lambda_{\mathrm{o}}^\varepsilon \frac{d\upsilon^\varepsilon}{ds}(s^{\varepsilon,h})}{\lambda^\varepsilon(s^{\varepsilon,h})}\nabla_y s^{\varepsilon,h}(t_m)\nabla_y \mathbf{v}_i\,. \tag{5.13}$$

See (3.9) for $\mathcal{J}^{\varepsilon,-1}$ and $(2.1)_1$ for time differentiation. β_ε in (5.12) is a constant satisfying $\beta_\varepsilon > \sup_{z\in\mathbb{R}} \frac{2\Lambda_{\mathrm{w}}^\varepsilon(z)}{\Lambda_{\mathrm{o}}^\varepsilon(z)\left|\frac{d\upsilon^\varepsilon}{ds}(\mathcal{J}^\varepsilon(z))\right|}$.

Theorem 3.1 is proved by the following steps: First we show that zeros of (5.8)–(5.13) exist and are bounded independently of h (see Lemma 5.1), next prove a subset of these zeros forms a convergent sequence (see Lemmas 5.2 and 5.3), and finally conclude the existence of a weak solution of (3.13)–(3.25) (see Lemmas 5.4 and 5.5). Let us define a non-negative function $\Gamma : \mathbb{R} \to \mathbb{R}_0^+$ by

$$\Gamma(z) \stackrel{\text{def}}{=} \int_0^z (\mathcal{J}^{\varepsilon,-1}(z) - \mathcal{J}^{\varepsilon,-1}(\xi))d\xi\,.$$

By (3.9), $\mathcal{J}^{\varepsilon,-1}$ is a strictly increasing function. As Remark 1.2 of Ref. 2,

$$\begin{cases} \Gamma(z_1) - \Gamma(z_2) \le (\mathcal{J}^{\varepsilon,-1}(z_1) - \mathcal{J}^{\varepsilon,-1}(z_2))z_1\,, & \text{for } z_1, z_2 \in \mathbb{R}\,, \\ |\mathcal{J}^{\varepsilon,-1}(z)| \le \varpi\Gamma(z) + \sup\limits_{|\xi|\le 1/\varpi} |\mathcal{J}^{\varepsilon,-1}(\xi)|\,, & \text{for } z\in\mathbb{R}, \varpi > 0\,. \end{cases} \tag{5.14}$$

Lemma 5.1. *Under* (3.7)–(3.11), (5.6)–(5.13) *are solvable for all* $h(= T/\ell)$, *and solutions satisfy, for* $(\zeta_1, \zeta_2, \eta) \in \mathcal{W}_1^h$, *in* I_m^*,

$$S^{\varepsilon,h}(t_m) = \mathcal{J}^{\varepsilon,-1}(\mathcal{G}^{\varepsilon,h}(t_m))\,, \tag{5.15}$$

$$\begin{aligned} 0 = {} & \int_\Omega \partial^{-h} S^{\varepsilon,h}(t_m)\zeta_1 + \int_\Omega \tilde{\Lambda}_{\mathrm{w}}^\varepsilon(S^{\varepsilon,h})\nabla_x(P^{\varepsilon,h} - E_{\mathrm{w}}^h)(t_m)\nabla_x\zeta_1 \\ & - \int_\Omega \frac{\Lambda_{\mathrm{w}}^\varepsilon\Lambda_{\mathrm{o}}^\varepsilon}{\Lambda^\varepsilon}(S^{\varepsilon,h})\nabla_x \Upsilon^\varepsilon(S^{\varepsilon,h})(t_m)\nabla_x\zeta_1 + \int_{\mathcal{Q}} \partial^{-h} s^{\varepsilon,h}(t_m)\zeta_1\,,\end{aligned} \tag{5.16}$$

$$0 = \int_\Omega \tilde{\Lambda}^\varepsilon(S^{\varepsilon,h})\nabla_x P^{\varepsilon,h}(t_m)\nabla_x\zeta_2 - \sum_{\alpha\in\{\mathrm{w,o}\}}\int_\Omega \tilde{\Lambda}^\varepsilon_\alpha(S^{\varepsilon,h})\nabla_x E^h_\alpha(t_m)\nabla_x\zeta_2\,, \quad (5.17)$$

$$0 = \int_{\mathcal{Q}} \partial^{-h}s^{\varepsilon,h}(t_m)\eta - \int_{\mathcal{Q}} \frac{\lambda^\varepsilon_{\mathrm{w}}\lambda^\varepsilon_{\mathrm{o}}}{\lambda^\varepsilon}(s^{\varepsilon,h})\nabla_y\upsilon^\varepsilon(s^{\varepsilon,h})(t_m)\nabla_y\eta\,. \quad (5.18)$$

Moreover,

$$\sup_{t\le T}\|s^{\varepsilon,h}(t)\|_{L^2(\mathcal{Q})} + \|\mathcal{G}^{\varepsilon,h}\|_{L^2(0,T;H^1(\Omega))} + \|P^{\varepsilon,h}\|_{L^2(0,T;H^1(\Omega))} + \|s^{\varepsilon,h}\|_{L^2(0,T;\mathcal{U})} \le c_0\,, \quad (5.19)$$

where c_0 is a constant independent of h.

Proof. The solvability of (5.6)–(5.13) is done by induction. $\{S^{\varepsilon,h},\mathcal{G}^{\varepsilon,h},s^{\varepsilon,h}\}(0)$ is given in (5.7). Suppose $\{S^{\varepsilon,h},\mathcal{G}^{\varepsilon,h},s^{\varepsilon,h}\}(t_{m-1})$ is solved. Since $\mathcal{H}^{\varepsilon,h}$ of (5.8) is continuous, (5.4) and (5.9)–(5.13) imply

$$\mathcal{H}^{\varepsilon,h}(\xi_{1,1},\ldots,\xi_{3,\ell})(\xi_{1,1},\ldots,\xi_{3,\ell}) \ge \int_\Omega(\mathcal{G}^{\varepsilon,h}-\mathcal{G}^{\varepsilon,h}_b)\partial^{-h}S^{\varepsilon,h}(t_m) + c_1\left(\int_{\mathcal{Q}}\frac{|s^{\varepsilon,h}|^2}{h} + \int_\Omega|\nabla_x\mathcal{G}^{\varepsilon,h}|^2 + \int_\Omega|\nabla_x P^{\varepsilon,h}|^2 + \int_{\mathcal{Q}}|\nabla_y s^{\varepsilon,h}|^2\right)(t_m) - c_2\,, \quad (5.20)$$

where c_1, c_2 are positive constants. By $(5.14)_1$,

$$\partial^{-h}\Gamma(\mathcal{G}^{\varepsilon,h})(t_m) \le (\mathcal{G}^{\varepsilon,h}-\mathcal{G}^{\varepsilon,h}_b)\partial^{-h}S^{\varepsilon,h}(t_m) + \mathcal{G}^{\varepsilon,h}_b\partial^{-h}S^{\varepsilon,h}(t_m)\,. \quad (5.21)$$

(5.20), (5.21) and $(5.14)_2$ imply

$$\mathcal{H}^{\varepsilon,h}(\xi_{1,1},\ldots,\xi_{3,\ell})(\xi_{1,1},\ldots,\xi_{3,\ell}) \ge c_3\left(\int_\Omega\frac{\Gamma(\mathcal{G}^{\varepsilon,h})}{h} + |\nabla_x\mathcal{G}^{\varepsilon,h}|^2 + |\nabla_x P^{\varepsilon,h}|^2 + \int_{\mathcal{Q}}\frac{|s^{\varepsilon,h}|^2}{h} + |\nabla_y s^{\varepsilon,h}|^2\right) - c_4\,. \quad (5.22)$$

If norm of $(\xi_{1,1},\ldots,\xi_{3,\ell})$ is large enough, right-hand side of (5.22) is strictly positive. So $\mathcal{H}^{\varepsilon,h}$ has a zero for $t=t_m$. By induction, it is easy to see that (5.6)–(5.13) are solvable. Clearly the zero of (5.6)–(5.13) satisfies (5.16)–(5.18).

If $(\mathcal{G}^{\varepsilon,h}-\mathcal{G}^{\varepsilon,h}_b, P^{\varepsilon,h}-P^h_b, s^{\varepsilon,h}-\mathcal{G}^{\varepsilon,h}) = \sum_{i=1}^{\ell}(\xi_{1,i}\mathbf{e}_i,\ \xi_{2,i}\mathbf{e}_i,\ \xi_{3,i}\mathbf{v}_i)$ is a zero of (5.8), then

$$\mathcal{H}^{\varepsilon,h}(\xi_{1,1},\ldots,\xi_{3,\ell})(\xi_{1,1},\ldots,\xi_{3,\ell}) = 0\,. \quad (5.23)$$

Integrating (5.23) over $[0, t_m]$, one obtains, by (5.4),

$$\int_0^{t_m}\int_\Omega (\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h})\partial^{-h} S^{\varepsilon,h} + \int_0^{t_m}\int_{\mathcal{Q}} (s^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h})\partial^{-h} s^{\varepsilon,h}$$
$$+ c_5\left(\int_0^{t_m}\int_\Omega |\nabla_x \mathcal{G}^{\varepsilon,h}|^2 + \int_0^{t_m}\int_\Omega |\nabla_x P^{\varepsilon,h}|^2 + \int_0^{t_m}\int_{\mathcal{Q}} |\nabla_y s^{\varepsilon,h}|^2\right) \le c_6\,, \tag{5.24}$$

where c_5, c_6 are constants independent of h. By $(5.14)_1$,

$$\partial^{-h}\Gamma(\mathcal{G}^{\varepsilon,h})(t) \le (\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h})\partial^{-h} S^{\varepsilon,h}(t) + \mathcal{G}_b^{\varepsilon,h}\partial^{-h} S^{\varepsilon,h}(t)\,. \tag{5.25}$$

Integrate (5.25) over $\Omega \times [0, t_m]$ to get

$$\frac{1}{h}\int_{t_m-h}^{t_m}\int_\Omega \Gamma(\mathcal{G}^{\varepsilon,h}) \le \int_0^{t_m}\int_\Omega (\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h})\partial^{-h} S^{\varepsilon,h} + \int_\Omega \Gamma(\mathcal{G}^{\varepsilon,h}(0))$$
$$- \int_0^{t_m-h}\int_\Omega (S^{\varepsilon,h} - S^{\varepsilon,h}(0))\partial^h \mathcal{G}_b^{\varepsilon,h} + \frac{1}{h}\int_{t_m-h}^{t_m}\int_\Omega (S^{\varepsilon,h} - S^{\varepsilon,h}(0))\mathcal{G}_b^{\varepsilon,h}\,, \tag{5.26}$$

where $S^{\varepsilon,h}(t) = S^{\varepsilon,h}(0)$ for $-h < t < 0$. Similar to (5.26), we have

$$\frac{1}{h}\int_{t_m-h}^{t_m}\int_{\mathcal{Q}} \frac{|s^{\varepsilon,h}|^2}{2} \le \int_0^{t_m}\int_{\mathcal{Q}} (s^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h})\partial^{-h} s^{\varepsilon,h} + \int_{\mathcal{Q}} \frac{|s^{\varepsilon,h}(0)|^2}{2}$$
$$- \int_0^{t_m-h}\int_{\mathcal{Q}} (s^{\varepsilon,h} - s^{\varepsilon,h}(0))\partial^h \mathcal{G}_b^{\varepsilon,h} + \frac{1}{h}\int_{t_m-h}^{t_m}\int_{\mathcal{Q}} (s^{\varepsilon,h} - s^{\varepsilon,h}(0))\mathcal{G}_b^{\varepsilon,h}\,, \tag{5.27}$$

where $s^{\varepsilon,h}(t) = s^{\varepsilon,h}(0)$ for $-h < t < 0$. Note $\|\partial^h \mathcal{G}_b^{\varepsilon,h}\|_{L^1(0,T;L^\infty(\Omega))\cap L^2(\Omega^{\mathrm{T}})}$ and $\|\mathcal{G}_b^{\varepsilon,h}\|_{L^\infty(\Omega^{\mathrm{T}})}$ are bounded by a constant independent of h. (5.24), (5.26)–(5.27), $(5.14)_2$, and discrete Gronwall's inequality imply (5.19). □

Lemma 5.2. *For any small* $\varpi(> 0)$, *solutions of* (5.15)–(5.18) *satisfy*

$$\int_\varpi^{\mathrm{T}}\int_\Omega |\mathcal{G}^{\varepsilon,h}(t) - \mathcal{G}^{\varepsilon,h}(t-\varpi)|^2 \le c_0\varpi\,,$$

where c_0 *is a constant independent of* ϖ, $h(= T/\ell)$.

Proof. For fixed μ, we add (5.16) (resp. (5.18)) for $m = j+1, \dots, j+\mu$, and test the resulting equation by $\zeta_j = h^2\mu\partial^{-h\mu}(\mathcal{G}^{\varepsilon,h} - \mathcal{G}_b^{\varepsilon,h})(t_{j+\mu})$ (resp. $\eta_j = h^2\mu\partial^{-h\mu}(s^{\varepsilon,h} - \mathcal{G}^{\varepsilon,h})(t_{j+\mu})$), where $t_{j+\mu} = (j+\mu)h$. Then we sum above two equations for $j = 1, \ldots, \ell-\mu$ to obtain, by Lemma 5.1,

$$\sum_{j=1}^{\ell-\mu}\left\{\int_\Omega |h\mu|^2\partial^{-h\mu}S^{\varepsilon,h}(t_{j+\mu})\partial^{-h\mu}\mathcal{G}^{\varepsilon,h}(t_{j+\mu}) + \int_{\mathcal{Q}} |h\mu\partial^{-h\mu}s^{\varepsilon,h}(t_{j+\mu})|^2\right\}$$

$$= \sum_{j=1}^{\ell-\mu}\left\{\int_\Omega |h\mu|^2\partial^{-h\mu}S^{\varepsilon,h}\,\partial^{-h\mu}\mathcal{G}_b^{\varepsilon,h}(t_{j+\mu}) + \int_{\mathcal{Q}} |h\mu|^2\partial^{-h\mu}s^{\varepsilon,h}\,\partial^{-h\mu}\mathcal{G}_b^{\varepsilon,h}(t_{j+\mu})\right\}$$

$$- \sum_{j=1}^{\ell-\mu}\sum_{m=j+1}^{j+\mu}\left\{\int_\Omega\left(\tilde{\Lambda}_{\mathrm{w}}^\varepsilon(S^{\varepsilon,h})\nabla_x(P^{\varepsilon,h}-E_{\mathrm{w}}^h) - \frac{\Lambda_{\mathrm{w}}^\varepsilon\Lambda_{\mathrm{o}}^\varepsilon}{\Lambda^\varepsilon}(S^{\varepsilon,h})\nabla_x\upsilon^\varepsilon(\mathcal{G}^{\varepsilon,h})\right)(t_m)\nabla_x\zeta_j\right.$$

$$\left. - \int_{\mathcal{Q}}\frac{\lambda_{\mathrm{w}}^\varepsilon\lambda_{\mathrm{o}}^\varepsilon}{\lambda^\varepsilon}\nabla_y\upsilon^\varepsilon(s^{\varepsilon,h})(t_m)\nabla_y\eta_j\right\}. \tag{5.28}$$

By Lemma 5.1 and rearranging the indices j and m, right-hand side of (5.28) is bounded by $c\mu$. So

$$\int_{h\mu}^T\int_\Omega |h\mu|^2\partial^{-h\mu}S^{\varepsilon,h}(t)\partial^{-h\mu}\mathcal{G}^{\varepsilon,h}(t) \le ch\mu\,. \tag{5.29}$$

Since $\mathcal{G}^{\varepsilon,h}$ is a step function in time, inequality (5.29) is also satisfied if one replaces $h\mu$ by any positive constant ϖ. So the lemma is complete. □

Arguing as Lemmas 4.5 and 4.6, one obtains:

Lemma 5.3. *There is a subsequence of* $\{\mathcal{G}^{\varepsilon,h}, S^{\varepsilon,h}\}$ *converging to* $\{\mathcal{G}^\varepsilon, S^\varepsilon\}$ *pointwise almost everywhere and in* $L^2(\Omega^{\mathrm{T}})$ *strongly.*

Remark 5.1. Let us define $\mathcal{D}^\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{D}^\varepsilon(z) \stackrel{\text{def}}{=} \int_{\mathcal{J}(0.5)}^z \frac{\lambda_{\mathrm{w}}^\varepsilon\lambda_{\mathrm{o}}^\varepsilon}{\lambda^\varepsilon}\left|\frac{d\upsilon^\varepsilon}{ds}\right|(\xi)d\xi\,.$$

By $(3.9)_1$ and Lemma 5.3, $\mathcal{D}^\varepsilon(\mathcal{G}^{\varepsilon,h})$ converges to $\mathcal{D}^\varepsilon(\mathcal{G}^\varepsilon)$ in $L^2(\Omega^{\mathrm{T}})$, and $\mathcal{D}^\varepsilon(s^{\varepsilon,h})$ is bounded in $L^2(0,T;\mathcal{U})$.

Lemma 5.4. *There is* $\{S^\varepsilon, \mathcal{G}^\varepsilon, P^\varepsilon, s^\varepsilon\}$ *such that, for* $(\zeta_1,\zeta_2,\eta) \in L^2(0,T;\mathcal{W}_1)$,

$$\partial_t S^\varepsilon + \int_{\mathcal{M}}\partial_t s^\varepsilon \in \text{dual } L^2(0,T;\mathcal{V})\,, \quad \partial_t s^\varepsilon \in \text{dual } L^2(0,T;\mathcal{U}_0)\,, \tag{5.30}$$

$$\mathcal{G}^\varepsilon = \mathcal{J}^\varepsilon(S^\varepsilon)\,, \quad (\mathcal{G}^\varepsilon - \mathcal{G}_b^\varepsilon, P^\varepsilon - P_b^\varepsilon, s^\varepsilon - \mathcal{G}^\varepsilon) \in L^2(0,T;\mathcal{W}_1)\,, \tag{5.31}$$

$$\int_{\Omega^{\mathrm{T}}}\partial_t S^\varepsilon\zeta_1 + \int_{\Omega^{\mathrm{T}}}\left(\tilde{\Lambda}_{\mathrm{w}}^\varepsilon(S^\varepsilon)\nabla_x(P^\varepsilon - E_{\mathrm{w}}) - \frac{\Lambda_{\mathrm{w}}^\varepsilon\Lambda_{\mathrm{o}}^\varepsilon}{\Lambda^\varepsilon}(S^\varepsilon)\nabla_x\Upsilon^\varepsilon(S^\varepsilon)\right)\nabla_x\zeta_1$$

$$+ \int_{\mathcal{Q}^{\mathrm{T}}}\partial_t s^\varepsilon\zeta_1 = 0\,, \tag{5.32}$$

$$\int_{\Omega^{\mathrm{T}}} \tilde{\Lambda}^{\varepsilon}(S^{\varepsilon})\nabla_x P^{\varepsilon}\nabla_x \zeta_2 - \sum_{\alpha\in\{\mathrm{w},\mathrm{o}\}} \int_{\Omega^{\mathrm{T}}} \tilde{\Lambda}^{\varepsilon}_{\alpha}(S^{\varepsilon})\nabla_x E_{\alpha}\nabla_x \zeta_2 = 0\,, \tag{5.33}$$

$$\int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s^{\varepsilon}\eta - \int_{\mathcal{Q}^{\mathrm{T}}} \frac{\lambda^{\varepsilon}_{\mathrm{w}}\lambda^{\varepsilon}_{\mathrm{o}}}{\lambda^{\varepsilon}}\nabla_y \upsilon^{\varepsilon}(s^{\varepsilon})\nabla_y \eta = 0\,, \tag{5.34}$$

$$\mathcal{G}^{\varepsilon}(x,0) = \mathcal{G}^{\varepsilon}_0\,, \quad s^{\varepsilon}(x,y,0) = s^{\varepsilon}_0\,. \tag{5.35}$$

Proof. By (5.7) and Lemmas 5.1, 5.3, there is $\{S^{\varepsilon}, \mathcal{G}^{\varepsilon}, P^{\varepsilon}, s^{\varepsilon}, \hat{\mathcal{D}}^{\varepsilon}, \hat{s}^{\varepsilon}\}$ such that, as $h \to 0$,

$$\begin{cases} S^{\varepsilon,h}, \mathcal{G}^{\varepsilon,h} \to S^{\varepsilon}, \mathcal{G}^{\varepsilon}\,, & \text{in } L^2(\Omega^{\mathrm{T}}) \text{ strongly}\,,\\ S^{\varepsilon,h}, \mathcal{G}^{\varepsilon,h}, P^{\varepsilon,h} \to S^{\varepsilon}, \mathcal{G}^{\varepsilon}, P^{\varepsilon}\,, & \text{in } L^2(0,T;H^1(\Omega)) \text{ weakly}\,,\\ s^{\varepsilon,h}, \mathcal{D}^{\varepsilon}(s^{\varepsilon,h}) \to s^{\varepsilon}, \hat{\mathcal{D}}^{\varepsilon}\,, & \text{in } L^2(0,T;\mathcal{U}) \text{ weakly}\,,\\ s^{\varepsilon,h}(T) \to \hat{s}^{\varepsilon}\,, & \text{in } L^2(\mathcal{Q}) \text{ weakly}\,,\\ s^{\varepsilon,h}(0) \to s^{\varepsilon}_0\,, & \text{in } L^2(\mathcal{Q}) \text{ strongly}\,,\\ \partial^{-h}S^{\varepsilon,h} + \displaystyle\int_{\mathcal{M}} \partial^{-h}s^{\varepsilon,h} \to \partial_t S^{\varepsilon} + \int_{\mathcal{M}} \partial_t s^{\varepsilon}\,, & \text{in dual } L^2(0,T;\mathcal{V}) \text{ weakly}\,,\\ \partial^{-h}s^{\varepsilon,h} \to \partial_t s^{\varepsilon}\,, & \text{in dual } L^2(0,T;\mathcal{U}_0) \text{ weakly}\,. \end{cases} \tag{5.36}$$

If one can show

$$\begin{cases} \hat{s}^{\varepsilon} = s^{\varepsilon}(T)\,,\\ \hat{\mathcal{D}}^{\varepsilon} = D^{\varepsilon}(s^{\varepsilon})\,, \end{cases} \tag{5.37}$$

then (5.15)–(5.18) imply the lemma as $h \to 0$.

For each $i \geq 1$ and $f \in C^1[0,T]$, (5.18) implies

$$\begin{aligned} &-\int_0^{T-h}\int_{\mathcal{Q}} s^{\varepsilon,h}\partial^h \wp(f)(t)\mathbf{v}_i + \int_{\mathcal{Q}^{\mathrm{T}}} \nabla_y \mathcal{D}^{\varepsilon}(s^{\varepsilon,h})\wp(f)(t)\nabla_y \mathbf{v}_i\\ &\qquad = -\frac{1}{h}\int_{T-h}^{T}\int_{\mathcal{Q}} s^{\varepsilon,h}(T)\wp(f)(t)\mathbf{v}_i + \int_{\mathcal{Q}} s^{\varepsilon,h}(0)f(0)\mathbf{v}_i\,. \end{aligned} \tag{5.38}$$

See (5.2) for $\wp(f)$. Letting $h \to 0$ and following the argument in Step 1 of Lemma 4.9, one obtains (i) $\hat{s}^{\varepsilon} = s^{\varepsilon}(T)$ (i.e. $(5.37)_1$), and (ii) for $\eta \in L^2(0,T;\mathcal{U}_0)$,

$$\int_{\mathcal{Q}^{\mathrm{T}}} \partial_t s^{\varepsilon}\eta + \int_{\mathcal{Q}^{\mathrm{T}}} \nabla_y \hat{\mathcal{D}}^{\varepsilon}\nabla_y \eta = 0\,. \tag{5.39}$$

To show $\hat{\mathcal{D}}^{\varepsilon} = \mathcal{D}^{\varepsilon}(s^{\varepsilon})$, one follows the argument in Step 2 of Lemma 4.9 and employs (5.5). □

Lemma 5.5. *$\varepsilon \leq S^{\varepsilon} \leq 1-\varepsilon$ and $\mathcal{J}(\varepsilon) \leq s^{\varepsilon} \leq \mathcal{J}(1-\varepsilon)$.*

Proof. By (3.6), (3.10) and (5.31), $\zeta \overset{\text{def}}{=} \max\{\mathcal{G}^{\varepsilon} - \mathcal{J}^{\varepsilon}(1-\varepsilon), 0\} \in L^2(0,T;\mathcal{V})$. Let $\zeta_1 = \zeta_2 = \zeta$ in (5.32) and (5.33) and $\eta = \max\{s^{\varepsilon} - \mathcal{J}^{\varepsilon}(1-\varepsilon), 0\} - \zeta_1$ in

(5.34). By $(3.9)_4$ and (5.35), we see $S^\varepsilon \le 1-\varepsilon$, $s^\varepsilon \le \mathcal{J}(1-\varepsilon)$. Similarly, letting $\zeta_1 = \max\{-\mathcal{G}^\varepsilon + \mathcal{J}^\varepsilon(\varepsilon), 0\}$ in (5.32) and $\eta = \max\{-s^\varepsilon + \mathcal{J}^\varepsilon(\varepsilon), 0\} - \zeta_1$ in (5.34), one gets $S^\varepsilon \ge \varepsilon$, $s^\varepsilon \ge \mathcal{J}(\varepsilon)$. □

Based on Lemmas 5.4, 5.5 and Theorem 3.1 is proved below.

Proof of Theorem 3.1. (3.13)–(3.19) is a direct result of Lemmas 5.4 and 5.5. Define

$$\begin{cases} P_{\rm w}^\varepsilon \overset{\rm def}{=} P^\varepsilon - \dfrac{1}{2}\left(\Upsilon^\varepsilon(S^\varepsilon) + \displaystyle\int_0^{\Upsilon^\varepsilon(S^\varepsilon)} \left(\frac{\Lambda_{\rm o}^\varepsilon}{\Lambda^\varepsilon}(\Upsilon^{\varepsilon,-1}) - \frac{\Lambda_{\rm w}^\varepsilon}{\Lambda^\varepsilon}(\Upsilon^{\varepsilon,-1})\right) d\xi\right), \\ P_{\rm o}^\varepsilon \overset{\rm def}{=} \Upsilon^\varepsilon(S^\varepsilon) + P_{\rm w}^\varepsilon, \\ p^\varepsilon \overset{\rm def}{=} P^\varepsilon - \dfrac{1}{2}\left(\displaystyle\int_0^{\Upsilon^\varepsilon(S^\varepsilon)} \left(\frac{\Lambda_{\rm o}^\varepsilon}{\Lambda^\varepsilon}(\Upsilon^{\varepsilon,-1}) - \frac{\Lambda_{\rm w}^\varepsilon}{\Lambda^\varepsilon}(\Upsilon^{\varepsilon,-1}) - \frac{\lambda_{\rm o}^\varepsilon}{\lambda^\varepsilon}(\upsilon^{\varepsilon,-1}) + \frac{\lambda_{\rm w}^\varepsilon}{\lambda^\varepsilon}(\upsilon^{\varepsilon,-1})\right) d\xi\right), \\ p_{\rm w}^\varepsilon \overset{\rm def}{=} p^\varepsilon - \dfrac{1}{2}\left(\upsilon^\varepsilon(s^\varepsilon) + \displaystyle\int_0^{\upsilon^\varepsilon(s^\varepsilon)} \left(\frac{\lambda_{\rm o}^\varepsilon}{\lambda^\varepsilon}(\upsilon^{\varepsilon,-1}) - \frac{\lambda_{\rm w}^\varepsilon}{\lambda^\varepsilon}(\upsilon^{\varepsilon,-1})\right) d\xi\right), \\ p_{\rm o}^\varepsilon \overset{\rm def}{=} \upsilon^\varepsilon(s^\varepsilon) + p_{\rm w}^\varepsilon. \end{cases}$$

Clearly $\{S^\varepsilon, \mathcal{G}^\varepsilon, P^\varepsilon, s^\varepsilon, P_\alpha^\varepsilon, p_\alpha^\varepsilon\ (\alpha = {\rm w}, {\rm o})\}$ satisfies (3.20)–(3.25).

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