Method for determining the optical axis and (n_e, n_o) of a birefringent crystal

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There is a phase difference between s and p polarizations when a circularly polarized heterodyne light beam is reflected from a birefringent crystal. It can be measured accurately with a common-path heterodyne interferometric technique. We have derived an equation that describes the relationship between the phase differences and n_e, n_o, and α. Two groups of solutions for (n_e, n_o) can be obtained from this equation by the phase measurements performed at three incident angles under moderate conditions. Each group consists of three pairs of solutions for (n_e, n_o). Finally, by justifying with physical conditions, we obtained the correct solution for (n_e, n_o). Azimuth angle α of the birefringent crystal optical axis can also be determined. And the feasibility of this method is demonstrated.

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1. Introduction

Birefringent crystals have been used to fabricate polarization optical components for a long time. Recently, some birefringent devices such as birefringent laser cavity filters,[1] poled-polymer electro-optic devices,[2] liquid-crystal spatial light modulations,[3] and magneto-optic recording media[4] have been used for many applications. To enhance their quality and performance, it is necessary for one to determine the optical axis and to measure the extraordinary index, n_e, and the ordinary index, n_o, accurately. Several methods have been proposed to measure the (n_e, n_o) of a birefringent crystal. These measurement methods are generally divided into two types: transmission[5–7] and reflection.[8–13] In the transmission type the phase variations of the light beam transmitted through a birefringent crystal are measured, which necessitates the need for accuracy in the thickness, flatness, and parallelism of the two opposite sides of the birefringent crystals. Hence, the measurement processes become tedious. In addition, the estimated data are only for the index difference $|n_e - n_o|$ and not for the individual data of n_e and n_o. Huang *et al.*[7] obtained the data for (n_e, n_o) with a specially designed transmission-type measurement method, but it is suitable only for a wedge-shaped birefringent crystal and is displaced by a linear translator with high resolution. Although the reflection type method, such as the ellipsometric technique, can be used to obtain the data (n_e, n_o), it is related to the light intensity variations. Consequently, it can be easily influenced by the stability of the light source, the scattering light, the internal reflection, etc., and its resolution decreases. Moreover, almost all the above methods cannot be used to determine the optical axis of the birefringent crystal.

We present a simple method for determining the optical axis and (n_e, n_o) of a birefringent crystal. The method uses a common-path heterodyne interferometric technique and Fresnel equations. When a light beam from a circularly polarized heterodyne light source[14] is incident on a birefringent crystal, a phase difference ϕ occurs between the s- and the p-polarization components. From Fresnel equations it is known that ϕ depends on n_e, n_o; incident angle θ; azimuth angle β of the transmission axis of the analyzer, which causes the necessary polarization components to interfere; and azimuth angle α of the optical axis. The phase difference can be measured accurately with a common-path heterodyne interferometric technique under certain conditions. First, let $\beta = 0°$ and condition $\alpha = 0°$ or $90°$ is identified by $\phi = 0°$. Next, β changes to a nonzero

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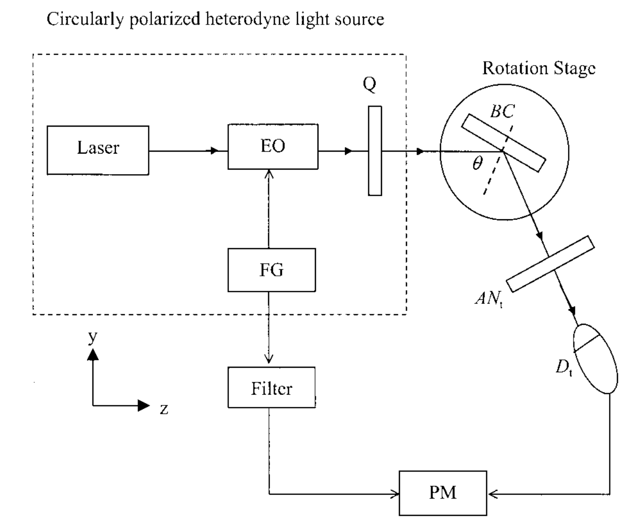

Fig. 1. Schematic structure for the measurement of phase differences owing to reflection at a birefringent crystal: EO, electro-optic modulator; Q, quarter-wave plate; BC, birefringent crystal; AN_t, analyzer; D_t, photodetector; FG, function generator; PM, phasemeter.

angle, and three phase differences, ϕ_1, ϕ_2, and ϕ_3, are obtained under three incident angles θ_1, θ_2, and θ_3. Substituting the data of (θ_1, ϕ_1), (θ_2, ϕ_2), and (θ_3, ϕ_3) into the special equation derived from Fresnel equations results in three equations. Any two of the equations with $\alpha = 0°$ or $90°$ yield two groups of solutions for (n_e, n_o). Each group has three pairs of solutions for (n_e, n_o). After justification, only one group of solutions is correct, with average values for indices (n_e, n_o) of the birefringent crystal. Its corresponding α value is the azimuth angle of the optical axis.

2. Principle

The schematic diagram of this method is shown in Fig. 1. A linearly polarized laser light passes through an electro-optic (EO) modulator and quarter-wave plate. The EO modulator is driven by a function generator. A sawtooth signal with angular frequency ω and the half-voltage amplitude $V_{\lambda/2}$ is applied to the EO modulator. The light beam is incident at θ on a birefringent crystal, of which the optical axis is at α with the incidence plane, as shown in Fig. 2. The light reflected from the air–crystal interface passes through an analyzer and enters a photodetector. If the amplitude of the light is E_t, then D_t measures the intensity $I_t = |E_t|^2$. Here, I_t acts as a test signal.

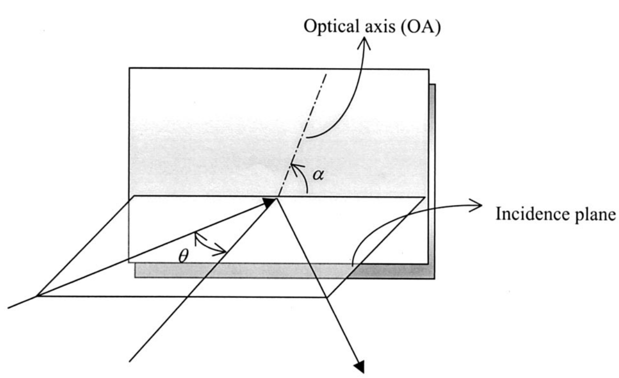

Fig. 2. Reflection at the surface of a birefringent crystal.

For convenience, the $+z$ axis is chosen along the propagation direction and the y axis is along the vertical direction. Let the laser light be horizontally linearly polarized, the fast axis of the EO modulator and the transmission axis of Q be 45° and 0° with respect to the x axis, respectively, then the Jones vector of the light that is incident on the birefringent crystal can be written as

$$E_i = Q(0°)EO(\omega t)E_0 = \begin{bmatrix} \cos\left(\dfrac{\omega t}{2}\right) \\ -\sin\left(\dfrac{\omega t}{2}\right) \end{bmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}\exp\left(i\,\frac{\omega t}{2}\right) + \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}\exp\left(-i\,\frac{\omega t}{2}\right). \quad (1)$$

From Eq. (1), an angular frequency difference ω can be seen between the left and the right-circular polarizations. And a linearly polarized laser, an EO modulator driven by a function generator and a quarter-wave plate form a circularly polarized heterodyne light source. If the transmission axis of AN_t is located at β with respect to the x axis, we then have

$$E_t = AN(\beta)SE_i = AN(\beta)\begin{pmatrix} r_{pp} & r_{ps} \\ r_{sp} & r_{ss} \end{pmatrix}E_i = \left[(r_{pp}\cos\beta + r_{sp}\sin\beta)\cos\frac{\omega t}{2} - (r_{ps}\cos\beta + r_{ss}\sin\beta)\sin\frac{\omega t}{2}\right]\begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}, \quad (2)$$

where S is the Jones matrix for the birefringent crystal, r_{pp} and r_{ss} are the direct-reflection coefficients, and r_{ps} and r_{sp} are the cross-reflection coefficients.[8,15,16] And they can be expressed as

$$r_{pp} = \frac{A_1A_6 + A_2A_5}{A_1 + A_2}, \quad (3a)$$

$$r_{ps} = \frac{A_1A_2(A_4 - A_3)}{A_1 + A_2}, \quad (3b)$$

$$r_{sp} = \frac{A_6 - A_5}{A_1 + A_2}, \quad (3c)$$

$$r_{ss} = \frac{A_1A_3 + A_2A_4}{A_1 + A_2}, \quad (3d)$$

where

$$A_1 = \frac{C}{(\sin^2\theta + C\cos\theta)\tan\alpha}, \tag{4a}$$

$$A_2 = \frac{n_o \tan\alpha(B + n_o\cos\theta)}{Bn_o\cos\theta + C^2}, \tag{4b}$$

$$A_3 = \frac{\cos\theta - C}{\cos\theta + C}, \tag{4c}$$

$$A_4 = \frac{n_o\cos\theta - B}{n_o\cos\theta + B}, \tag{4d}$$

$$A_5 = \frac{n_o^2\cos\theta - C}{n_o^2\cos\theta + C}, \tag{4e}$$

$$A_6 = \frac{Bn_o\cos\theta - C^2}{Bn_o\cos\theta + C^2}, \tag{4f}$$

$$B^2 = n_o^2 n_e^2 - \sin^2\theta(n_o^2\sin^2\alpha + n_o^2\cos^2\alpha), \tag{4g}$$

$$C^2 = n_o^2 - \sin^2\theta. \tag{4h}$$

Hence, we have

$$I_t = |E_r|^2 = I_0[1 + \cos(\omega t + \phi)], \tag{5}$$

where

$$I_0 = \frac{(r_{pp}\cos\beta + r_{sp}\sin\beta)^2 + (r_{ps}\cos\beta + r_{ss}\sin\beta)^2}{2}, \tag{6}$$

$$\phi = \tan^{-1}\left[\frac{2(r_{pp}\cos\beta + r_{sp}\sin\beta)(r_{ps}\cos\beta + r_{ss}\sin\beta)}{(r_{pp}\cos\beta + r_{sp}\sin\beta)^2 - (r_{ps}\cos\beta + r_{ss}\sin\beta)^2}\right]. \tag{7}$$

On the other hand, the electric signal generated by the function generator is filtered and acts as the reference signal. Both the test signal and the reference signal are sinusoidal signals, which are sent to a phasemeter, where ϕ can be measured accurately.

From Eqs. (3)–(5) and (7), it can be seen that ϕ depends on n_e, n_o, α, θ, and β. In practical measurement processes, θ and β can be obtained from the direct angle readouts of the division mark of the rotatory stage. Consequently, only three factors, n_e, n_o, and α, should be solved. That is, we have

$$\phi = \phi(n_e, n_o, \alpha). \tag{8}$$

Theoretically, the data of ϕ, which corresponds to three different conditions, should be measured. If we substitute the data of ϕ into Eq. (8), n_e, n_o and α could be obtained. But these equations are complicated, and it is difficult to solve them directly. For easier operations and estimations, θ and β could be chosen to simplify Eq. (7). When we choose $\beta = 0°$, Eq. (7) can be rewritten as

$$\phi = \tan^{-1}\left(\frac{2r_{pp}r_{ps}}{r_{pp}^2 - r_{ps}^2}\right). \tag{9}$$

It is obvious from Eqs. (3) and (4) that either r_{ps} or r_{sp} equals zero when α equals either 0° or 90°, respectively. Hence, when $\beta = 0°$, the optical axis of the birefringent crystal can be rotated until $\phi = 0°$ is satisfied. Then the optical axis is located at either 0° or 90° with respect to the incidence plane.

Next, when AN_t is rotated so that β is nonzero, Eq. (7) can be rewritten as

$$\phi = \tan^{-1}\left(\frac{\sin 2\beta\, r_{pp}r_{ss}}{r_{pp}^2\cos^2\beta - r_{ss}^2\sin^2\beta}\right). \tag{10}$$

We now consider two particular conditions:

(i) If $\alpha = 0°$, then

$$r_{pp} = \frac{n_o n_e\cos\theta - (n_o^2 - \sin^2\theta)^{1/2}}{\mathrm{n_o n_e}\cos\theta + (n_o^2 - \sin^2\theta)^{1/2}}, \tag{11a}$$

$$r_{ss} = \frac{\cos\theta - (n_o^2 - \sin^2\theta)^{1/2}}{\cos\theta + (n_o^2 - \sin^2\theta)^{1/2}}. \tag{11b}$$

(ii) If $\alpha = 90°$, then

$$r_{pp} = \frac{n_o^2\cos\theta - (n_o^2 - \sin^2\theta)^{1/2}}{n_o^2\cos\theta + (n_o^2 - \sin^2\theta)^{1/2}}, \tag{12a}$$

$$r_{ss} = \frac{\cos\theta - (n_e^2 - \sin^2\theta)^{1/2}}{\cos\theta + (n_e^2 - \sin^2\theta)^{1/2}}. \tag{12b}$$

Since three unknowns (n_e, n_o, and α) are to be solved, we need three equations, which we obtained by measuring ϕ at three incident angles: θ_1, θ_2, and θ_3. We obtained three corresponding phase differences, ϕ_1, ϕ_2, and ϕ_3, that can be represented as

$$\phi_1 = \phi_1(n_e, n_o, \alpha), \tag{13a}$$

$$\phi_2 = \phi_2(n_e, n_o, \alpha), \tag{13b}$$

$$\phi_3 = \phi_3(n_e, n_o, \alpha). \tag{13c}$$

Any two of Eqs. (13a)–(13c) can be combined to form a set of simultaneous equations, and we obtained three sets. Any set of the simultaneous equations can be solved under either condition (i) or (ii), resulting in two corresponding pairs of solutions for (n_e, n_o). Therefore, there are six pairs of solutions for (n_e, n_o). Among them, three pairs are derived under condition (i) and form a group of solutions. The other three are derived under condition (ii) and form another-

Table 1. Experimental Conditions and Measurement Results

Material	Incident Angles (deg) θ_1	θ_2	θ_3	Phase Differences (deg) ϕ_1	ϕ_2	ϕ_3
Calcite	55	60	65	24.52	−6.85	−25.85
Quartz	55	60	65	17.46	−24.40	−62.56

group of solutions. Then the justification of correct solutions can be achieved by the following approaches:

(1) Rationale of the solution. In general, both n_e and n_o are within the range of 1 and 5. If any estimated data of n_e and n_o is not within this range, it is obvious that the estimated data could be incorrect.

(2) Comparison of n_e and n_o. Either a positive or a negative crystal is tested, and all three pairs of solutions of either group should meet with only $n_e > n_o$ or $n_e < n_o$. If not, that group is incorrect.

Hence, only one group of solutions is correct, and the corresponding data of α are the azimuth angle of its optical axis.

3. Experiments and Results

To demonstrate the feasibility of this method, we used a 632.8-nm wavelength He–Ne laser to measure the refractive indices of calcite and quartz. The frequency of the sawtooth signal that is applied to the EO modulator is 800 Hz. We used a high-resolution rotation stage (PS-θ-90) with an angular resolution of 0.005° (Japan Chuo Precision Industrial Company, Ltd.) to mount and rotate the test material and a high-resolution phasemeter with an angular resolution of 0.01° to measure the phase difference. In addition, we used a personal computer to record and analyze the data. The data of the three incident angles and their corresponding phase differences are listed in Table 1. These simultaneous equations are solved with the two dimensional Newton method[17] and the Mathematica software. And two groups of solutions are calculated and summarized in Table 2. The rightmost column lists the results according to the above approaches; the O and x, respectively, represent correct and incorrect solutions. The measured data of (n_e, n_o) and their averages for calcite and quartz are listed in the first two rows in Tables 3 and 4, respectively. And $\alpha = 90°$ where we tested these two crystals.

Table 2. Calculated Solutions and Results

Material	α	(n_e, n_o) (ϕ_1, ϕ_2)	(ϕ_2, ϕ_3)	(ϕ_3, ϕ_1)	Justification
Calcite	0°	(1.6695, 1.5453)	(0.5041, 1.0007)	(580.71, −22.545)	x
	90°	(1.4333, 1.6233)	(1.4267, 1.6144)	(1.4333, 1.6233)	o
Quartz	0°	(1.5522, 1.5627)	(1.5128, 1.4638)	(1.5293, 1.5132)	x
	90°	(1.5552, 1.5449)	(1.5560, 1.5243)	(1.5647, 1.5195)	o

Table 3. Estimated Results and Their Average for Calcite[a]

Factors	Phase Differences (ϕ_1, ϕ_2)	(ϕ_2, ϕ_3)	(ϕ_3, ϕ_1)	Average
n_e	1.4333	1.4267	1.4333	1.4311
n_o	1.6233	1.6144	1.6233	1.6203
$\lvert\Delta n_e\rvert$	9.977×10^{-4}	1.196×10^{-3}	6.178×10^{-4}	9.371×10^{-4}
$\lvert\Delta n_o\rvert$	1.947×10^{-4}	3.69×10^{-4}	2.248×10^{-4}	2.628×10^{-4}
$\lvert\Delta\alpha\rvert$	0.0043°	0.0076°	0.0043°	0.0162°

[a]Values from Ref. 18: (n_e, n_o) are (1.4852, 1.6559) at 627.8 nm.

Table 4. Estimated Results and Their Average for Quartz[a]

Factors	Phase Differences (ϕ_1, ϕ_2)	(ϕ_2, ϕ_3)	(ϕ_3, ϕ_1)	Average
n_e	1.5552	1.5560	1.5647	1.5586
n_o	1.5449	1.5243	1.5195	1.5295
$\lvert\Delta n_e\rvert$	1.626×10^{-3}	1.9763×10^{-3}	1.046×10^{-3}	1.549×10^{-3}
$\lvert\Delta n_o\rvert$	2.14×10^{-4}	5.90×10^{-4}	2.18×10^{-4}	3.406×10^{-4}
$\lvert\Delta\alpha\rvert$	0.1454°	0.0243°	0.0373°	0.069°

[a]Values from Ref. 19: (n_e, n_o) are (1.5518, 1.5428) at 627.8 nm.

4. Discussions

From Eq (7) we get

$$|\Delta\alpha| = \frac{1}{|d\phi/d\alpha|}\,|\Delta\phi|, \tag{14}$$

$$|\Delta\phi_1| = \frac{\partial\phi_1}{\partial n_e}\,|\Delta n_e| + \frac{\partial\phi_1}{\partial n_o}\,|\Delta n_o|, \tag{15}$$

$$|\Delta\phi_2| = \frac{\partial\phi_2}{\partial n_e}\,|\Delta n_e| + \frac{\partial\phi_2}{\partial n_o}\,|\Delta n_o|. \tag{16}$$

Equations (15) and (16) can be rewritten as

$$|\Delta n_e| = \frac{\left|\frac{\partial\phi_2}{\partial n_o}\right||\Delta\phi_1| + \left|\frac{\partial\phi_1}{\partial n_o}\right||\Delta\phi_2|}{\left|\frac{\partial\phi_1}{\partial n_e}\frac{\partial\phi_2}{\partial n_o} - \frac{\partial\phi_2}{\partial n_e}\frac{\partial\phi_1}{\partial n_o}\right|}, \tag{17}$$

$$|\Delta n_o| = \frac{\left|\frac{\partial\phi_1}{\partial n_e}\right||\Delta\phi_1| + \left|\frac{\partial\phi_2}{\partial n_e}\right||\Delta\phi_2|}{\left|\frac{\partial\phi_1}{\partial n_e}\frac{\partial\phi_2}{\partial n_o} - \frac{\partial\phi_2}{\partial n_e}\frac{\partial\phi_1}{\partial n_o}\right|}, \tag{18}$$

where $\Delta\alpha$, Δn_e, and Δn_o are the errors in α, n_e, and n_o, and $\Delta\phi_i$ and $\Delta\phi_j$ are the errors in the phase differences at incident angles θ_i and θ_j. Either i or j is an integer between 1 and 3, and $i \neq j$. If we take into consideration the angular resolution of the phasemeter, the second-harmonic error, and the polarization mixing error, $\Delta\phi = \Delta\phi_i = \Delta\phi_j \cong 0.03°$ can be estimated with our experiments.[20] Substituting these data and the experimental conditions into Eqs (14), (17), and (18), we were able to calculate the corresponding data of $\Delta\alpha$, Δn_e, and Δn_o for three sets of simultaneous equations. We list their averages in the last three rows in Tables 3 and 4.

5. Conclusion

A novel method for determining the optical axis and (n_e, n_o) of a birefringent crystal has been presented with a common-path heterodyne interferometric technique and Fresnel equations. Our method does not have the drawbacks of conventional methods. It does, however, have the advantages of a common-path interferometer and a heterodyne interferometer, which include simple optical setup, high stability, easier operation, and better resolution.

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