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Solutions of semilinear elliptic equations with asymptotic linear nonlinearity[☆]

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1. Introduction

In this paper, we study the existence, uniqueness, and asymptotic behavior of positive solutions of the semilinear elliptic equations with asymptotic linear nonlinearity as follows:

$$\begin{aligned} \Delta u + \lambda \sqrt{(u-b)^2 + \varepsilon} &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded $C^{2,\alpha}$ domain in \mathbf{R}^n , and λ, ε and b are positive real numbers. The solutions structure of (1) is quite different when ε is equal to zero or not. When ε is equal to zero or a nonzero real number but small enough, there are some results of solutions structure of (1) in [1,9]. Here we extend these results to the case when ε is an arbitrary positive real number.

For convenience, we denote $f_\varepsilon = \sqrt{(u-b)^2 + \varepsilon}$ and the main results are as follows:

Theorem 1. *Suppose $\varepsilon > 0$, then there exists exactly one solution $u_\lambda(\cdot, \varepsilon)$ of (1) for all $\lambda \in (0, \lambda_1)$. Here λ_1 is the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary condition.*

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The solutions obtained in Theorem 1 are the minimal solutions of (1) and can be proved by monotone iteration method easily. It is quite natural to ask the existence of solutions of (1) when λ is larger than λ_1 . However, since the nonlinearity is asymptotic linear, it is difficult to apply Mountain–Pass Lemma to show the existence of solutions of (1). Therefore, due to the linear growth of f , we need to carefully estimate the energy levels in order to use the variation method of Nehari-type to prove the existence of solutions, see [8]. The result is as follows:

Theorem 2. *For any $\varepsilon > 0$, there exists $\lambda^*(\varepsilon) > \lambda_1$ such that if $\lambda \in (\lambda_1, \lambda^*(\varepsilon))$, then (1) has at least two solutions.*

The result in Theorem 2 can be obtained by using the blow up method to study the asymptotic behavior of the solutions. However, the global structure of the solutions of (1) is still open.

The paper is organized as follows. In Section 2, we show the existence of non-minimal solution of (1) when ε is small enough. In Section 3, we study the uniqueness and asymptotic behavior of solutions of (1). Applying the results of Sections 2 and 3, we prove the main theorems in Section 4.

2. Existence of non-minimal solutions

In this section, we will firstly use the perturbation method to show the existence of minimal solution of (1). The key lemma is as follows:

Lemma 1 (Perturbation Lemma). *Let u_ε be a minimal solution of (1). Consider*

$$\begin{aligned} \Delta u + \lambda f_\varepsilon(u) + h(u) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $h \in C^1(\mathbb{R}^1, \mathbb{R}^1)$ and $\lambda f_\varepsilon(\cdot) + h(\cdot) > 0$ in $[0, \infty)$. Then there exists $\delta \equiv \delta(u_\varepsilon) > 0$ such that for $\|h\|_\infty < \delta$, (2) has a minimal solution.

Proof. Let w and v be the solutions of

$$\begin{aligned} \Delta w &= -1 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3}$$

and

$$\begin{aligned} \Delta v + \lambda f'_\varepsilon(u_\varepsilon)v &= -\mu v \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

respectively, where $\mu > 0$ is the first eigenvalue and $\|v\|_\infty = 1$.

Define $\tilde{u} = u_\varepsilon + tv + sw$, where t and s are positive real numbers. By using the mean-value theorem, there exists $u_\varepsilon < u_\theta < u_\varepsilon + tv + sw$ and $u_\varepsilon < u_{\theta'} < u_\theta$

such that

$$\begin{aligned}
 &\Delta \bar{u} + \lambda f'_\varepsilon(\bar{u}) + h(\bar{u}) \\
 &= -\lambda f'_\varepsilon(u_\varepsilon) - \lambda t f'_\varepsilon(u_\varepsilon)v - t\mu v - s + \lambda f'_\varepsilon(u_\varepsilon + tv + sw) + h(\bar{u}) \\
 &= \lambda f'_\varepsilon(u_\theta)(tv + sw) - \lambda t f'_\varepsilon(u_\varepsilon)v - t\mu v - s + h(\bar{u}) \\
 &= tv \left(\frac{-1}{2}\mu + \lambda(f'_\varepsilon(u_\theta) - f'_\varepsilon(u_\varepsilon)) \right) + \left(\frac{-1}{2}t\mu v + \lambda sw f'_\varepsilon(u_\theta) \right) - s + h(\bar{u}) \\
 &= tv \left(\frac{-1}{2}\mu + \lambda f''_\varepsilon(u_{\theta'}) (u_\theta - u_\varepsilon) \right) + \left(\frac{-1}{2}t\mu v + \lambda sw f'_\varepsilon(u_\theta) \right) - s + h(\bar{u}). \tag{5}
 \end{aligned}$$

Since f''_ε is bounded and $\|u_\theta - u_\varepsilon\|_\infty \leq \|tv + sw\|_\infty$, the first term in (5) is negative when s and t are small enough. By the Boundary Hopf Lemma, see [4], the second term in (5) is also negative if s is smaller than t and small enough. Therefore, take $\delta = s > 0$ and let $\|h(\bar{u})\|_\infty < \delta$, the right-hand side in (5) is negative. Hence \bar{u} is a supersolution of (2). It is obvious that 0 is a subsolution of (2). By monotone iteration method, we obtain a minimal solution of (2). The proof is complete. \square

In Lemma 1, we had assumed the existence of minimal solution of (1). It is easy to check that (1) has a minimal solution when $\varepsilon = 0$, i.e. $f_0(u) = |u - b|$, since 0 and b are subsolution and supersolution, respectively. In addition, by Lemma 1, we have the following theorem.

Theorem 3. *There exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists $\lambda^*(\varepsilon) > \lambda_1$ such that if $\lambda \in (0, \lambda^*(\varepsilon))$, (1) has a minimal solution.*

Proof. We may rewrite the Eq. (1) as

$$\Delta u + \lambda f_0 + h(u) = 0,$$

where $h(u) = \lambda(f_\varepsilon - f_0)$. Since $\lambda(f_\varepsilon - f_0) \leq \lambda\sqrt{\varepsilon}$, and by Lemma 1, there exists $\varepsilon_0 > 0$ such that when $0 < \varepsilon < \varepsilon_0$, the equations

$$\begin{aligned}
 &\Delta u + \lambda(f_0(u) + \sqrt{\varepsilon}) = 0 \quad \text{in } \Omega, \\
 &u = 0 \quad \text{on } \partial\Omega,
 \end{aligned} \tag{6}$$

have a minimal solution which is also a supersolution of (1). Since 0 is a subsolution of (1), the result follows. \square

From Theorem 3, we show the existence of minimal solution of (1) when λ is smaller than λ^* . It is natural to study the existence of non-minimal solution of (1) when $\lambda < \lambda^*$. The result is as follows:

Theorem 4. *If ε is small enough and if $\lambda_1 < \lambda < \lambda^*(\varepsilon)$, then (1) has a non-minimal solution, where λ_1 is the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary condition.*

Proof. We shall prove the theorem by variation method of Nehari-type. By Theorem 3, let $u_\lambda(\cdot, \varepsilon)$ be a minimal solution of (1). If (1) has a non-minimal solution u , say $u = w + u_\lambda$ and $w > 0$, then w satisfies

$$\begin{aligned} \Delta w + \lambda(f'_\varepsilon(w + u_\lambda) - f'_\varepsilon(u_\lambda)) &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{7}$$

For convenience, let

$$g(x, w) = f'_\varepsilon(w + u_\lambda) - f'_\varepsilon(u_\lambda) \quad \text{and} \quad G(x, w) = \int_0^w g(x, v) \, dv, \tag{8}$$

then w satisfies

$$\begin{aligned} \Delta w + \lambda g(x, w) &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Define

$$\begin{aligned} J_\lambda(w) &= \int_\Omega \frac{1}{2} |\nabla w|^2 - \int_\Omega \lambda G(x, w), \\ I_\lambda(w) &= \int_\Omega |\nabla w|^2 - \lambda \int_\Omega w g(x, w), \end{aligned}$$

and

$$M_\lambda = \{w \in H_0^1 : I_\lambda(w) = 0\}.$$

Since f'_ε is convex and $w > 0$, the function $g(x, w)$ in (8) is convex in w such that

$$g(x, w) = g(x, w) - g(x, 0) \leq g'(x, w)w. \tag{9}$$

By integrating (9) with respect to w , we have $2G(x, w) \leq g(x, w)w$. Therefore, on M_λ ,

$$J_\lambda(w) = \frac{\lambda}{2} \left[\int_\Omega w g(x, w) - 2G(x, w) \right] \geq 0,$$

i.e. $J_\lambda(w)$ is bounded below. Now, if we can show that $M_\lambda \neq \emptyset$ for $\lambda > \lambda_1$, then by the Nehari method, see [6], we can obtain a non-minimal solution of (1).

To prove that $M_\lambda \neq \emptyset$, let ϕ_1 be the first eigenfunction of $-\Delta$ in Ω with Dirichlet boundary condition and $\int_\Omega \phi_1^2 = 1$. If $\lambda_1 < \lambda$, then

$$\begin{aligned} I_\lambda(t\phi_1) &= \lambda_1 t^2 - \lambda \int_\Omega t\phi_1 g(x, t\phi_1) \\ &= \lambda_1 t^2 - \lambda t^2 \int_\Omega \phi_1^2 \frac{t\phi_1 + 2u_\lambda - 2b}{f'_\varepsilon(t\phi_1 + u_\lambda) + f'_\varepsilon(u_\lambda)} \\ &< 0, \end{aligned}$$

when t tend to infinity. On the other hand, let w_1 be the eigenfunction with $\int_\Omega w_1^2 = 1$ of the first eigenvalue μ_1 of the following equation:

$$\begin{aligned} \Delta w + \lambda f'_\varepsilon(u_\lambda)w &= -\mu_1 w \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{10}$$

By the result of [2], we know $\mu_1 > 0$. Therefore,

$$\begin{aligned} I_\lambda(sw_1) &= s^2 \int |\nabla w_1|^2 - \lambda s \int w_1 g(x, sw_1) \\ &= s^2 \int |\nabla w_1|^2 - \lambda s \int [f'(u_\lambda)sw_1^2 + O(s^2)] \\ &= s^2(\mu_1 + O(s)), \end{aligned}$$

when s is near 0.

Hence, $I_\lambda(sw_1) > 0$ when s is small enough. Therefore, M_λ is non-empty and we obtain a positive solution of (1) by using the Nehari method. The proof is complete. \square

3. Uniqueness and asymptotic behavior of solutions

In this section, we prove the uniqueness of solutions of (1) when λ is smaller than λ_1 . In addition, we also use the blow up method to study the asymptotic behavior of solutions as λ varies and which gives us the multiplicity of solutions.

Theorem 5. *For any $\varepsilon > 0$, if $0 < \lambda \leq \lambda_1$, then the solution of (1) is unique.*

Proof. Suppose that u and v are solutions of (1). Let $w = u - v$, then w satisfies

$$\begin{aligned} \Delta w + \lambda(f_\varepsilon(u) - f_\varepsilon(v)) &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{11}$$

By mean-value theorem, w satisfies

$$\Delta w + \lambda f'_\varepsilon(\bar{w})w = 0,$$

where \bar{w} lies between u and v and

$$-\int_\Omega |\nabla w|^2 + \lambda \int_\Omega f'_\varepsilon(\bar{w})w^2 = 0.$$

Since $|f'_\varepsilon(\bar{w})| < 1$, we have $\lambda \int_\Omega f'_\varepsilon(\bar{w})w^2 \geq \lambda_1 \int_\Omega w^2$. Hence, if $0 \leq \lambda \leq \lambda_1$, then $w = 0$ and the result follows. The proof is complete. \square

Corollary 1. *The solution of (1) is unique in the set $\mathbf{B}_b = \{u \in C^0(\bar{\Omega}) : \|u\|_\infty \leq b\}$.*

Proof. In the proof of Theorem 5, we find $f'_\varepsilon(\bar{w})$ is negative under the assumption $|w| \leq b$. Hence, the uniqueness of solution follows in \mathbf{B}_b . \square

From Theorem 5, we know that (1) can have non-minimal solutions only if $\lambda > \lambda_1$. To study the existence of non-minimal solutions, we need the following lemma:

Lemma 2. *Fix $\varepsilon > 0$, if $\{u_\lambda\}$ is a family of solutions of (1) such that $\|u_\lambda\|_\infty$ tends to infinity, then λ converges to λ_1 .*

Proof. By the uniqueness result of Theorem 5, we know that λ is bounded away from zero, and it is easy to check that $\lambda^*(\varepsilon) < \infty$. Hence $\{\lambda\}$ is bounded and has a convergent subsequence. We still use the same λ and assume that $\lambda \rightarrow \bar{\lambda}$, where $\bar{\lambda} \in (0, \lambda^*(\varepsilon))$ and u_λ satisfies

$$\begin{aligned} \Delta u_\lambda + \lambda f_\varepsilon(u_\lambda) &= 0 \quad \text{in } \Omega, \\ u_\lambda &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{12}$$

Let

$$M_\lambda = \|u_\lambda\|_\infty \quad \text{and} \quad v_\lambda = \frac{u_\lambda}{M_\lambda},$$

then

$$\begin{aligned} \Delta v_\lambda + \lambda \frac{f_\varepsilon(v_\lambda M_\lambda)}{M_\lambda v_\lambda} v_\lambda &= 0 \quad \text{in } \Omega, \\ v_\lambda &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{13}$$

Since $f_\varepsilon(v_\lambda M_\lambda)/M_\lambda$ are bounded, we have $f_\varepsilon(v_\lambda M_\lambda)/(M_\lambda v_\lambda) \rightarrow 1$ a.e. in Ω and uniformly in any compact subset D of Ω . By standard elliptic estimates, we obtain $v_\lambda \rightarrow v$ in $C^{2,\alpha}$ as $\lambda \rightarrow \bar{\lambda}$. Hence,

$$\begin{aligned} \Delta v + \bar{\lambda} v &= 0 \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{14}$$

Since $\|v\|_\infty = 1$ and $v > 0$, we have $\bar{\lambda} = \lambda_1$, the first eigenvalue of $-\Delta$ and the result follows. \square

If we denote the solutions set of (1) by S_ε , then by the global bifurcation theorem of Crandall and Rabinowitz in [2], S_ε contains a branch C_ε which is unbounded. Therefore, by Lemma 2, the following result is immediate.

Theorem 6. *The solutions set S_ε of (1) contains an unbounded component C_ε . If $u_\lambda \in C_\varepsilon$ and $\|u_\lambda\|_\infty \rightarrow \infty$, then $\lambda \rightarrow \lambda_1$.*

Remark 1. If $\Omega = \mathbf{B}_1$, the unit ball in \mathbf{R}^2 . The uniqueness problem of non-minimal solution is still open.

4. Proof of main theorems

In this section, we will use the results of previous sections to prove the main theorems.

Proof of Theorem 1.

If $\varepsilon = 0$, then (1) has a minimal solution for all $\lambda > 0$. Now rewrite Eq. (1) as

$$\Delta u + \lambda f_0 + \lambda(f_\varepsilon - f_0) = 0.$$

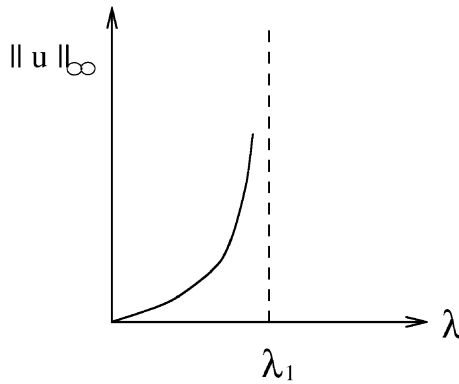


Fig. 1.

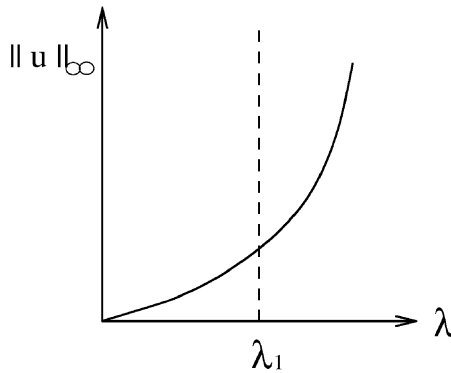


Fig. 2.

Since $\lambda(f_\varepsilon - f_0) \leq \lambda\sqrt{\varepsilon}$, and by Lemma 1, (1) has a minimal solution for any fixed $\varepsilon > 0$ if λ is small enough. Let S_ε be the solution set of (1), then by the global bifurcation theorem in [2,7] and Theorem 5, we have the bifurcation diagrams of solutions of (1) as Fig. 1 or 2. In either case, there exists exactly one solution for $\lambda \in (0, \lambda_1)$. The proof is complete. \square

Proof of Theorem 2.

First, by Theorems 4 and 6, we know that Theorem 2 is true if ε is small enough. Hence, the bifurcation diagram of solutions of (1) is roughly as in Fig. 3.

Let

$$\Gamma = \{\bar{\varepsilon} \in R^+ : \text{Theorem 2 is false, when } \varepsilon = \bar{\varepsilon}\}.$$

If the result in Theorem 2 is false, then the bifurcation diagram of solutions must be Fig. 1. We will prove that the set Γ is an empty set. Suppose that Γ is a non-empty

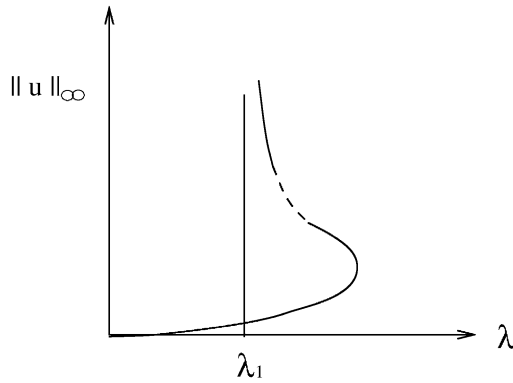


Fig. 3.

set and denote ε by $\varepsilon_0 = \inf \Gamma$, then it is clear that $\varepsilon_0 \neq 0$. Now, we can consider the following two cases:

Case 1: $\varepsilon_0 \notin \Gamma$. In this case, there exists $\lambda_{\varepsilon_0} > \lambda_1$ such that Eq. (1) has a minimal solution for $\lambda \in [\lambda_1, \lambda_{\varepsilon_0}]$ when $\varepsilon = \varepsilon_0$. Hence, by Lemma 1, (1) must have a minimal solution for some $\lambda \geq \lambda_1$, when ε is sufficiently close to ε_0 , but this contradicts $\varepsilon_0 = \inf \Gamma$.

Case 2: $\varepsilon_0 \in \Gamma$. In this case, let η be a positive fixed real number. We will claim that the choice of δ as in the proof of Lemma 1 is independent of ε for $\varepsilon \in [\varepsilon_0 - \eta, \varepsilon_0)$. The proof is similar as in proving Lemma 1. Let \bar{u} be as the \bar{u} in Lemma 1 with $v = v_\varepsilon$, $\mu = \mu_\varepsilon$ and M, K be positive real numbers such that

$$|f'_\varepsilon(x)| \leq M \quad \text{and} \quad |f'_\varepsilon(x)| \leq K, \quad \text{for } \varepsilon \in [\varepsilon_0 - \eta, \varepsilon_0) \text{ and } x \geq 0.$$

By (5), we have

$$\begin{aligned} & \Delta \bar{u} + \lambda f'_\varepsilon(\bar{u}) + h(\bar{u}) \\ & \leq tv_\varepsilon \left(\frac{-1}{2} \mu_\varepsilon + \lambda M \|tv_\varepsilon + sw\|_\infty \right) + \left(\frac{-1}{2} t\mu_\varepsilon v_\varepsilon + \lambda Ksw \right) - s + h(\bar{u}). \end{aligned}$$

Now, we claim that there exist $\bar{v} \leq v_\varepsilon$ for all $\varepsilon \in [\varepsilon_0 - \eta, \varepsilon_0)$ with $\bar{v} > 0$ in Ω and $\partial \bar{v} / \partial n < 0$ on $\partial \Omega$, where n is the unit outer normal of Ω . Suppose that there exists no such \bar{v} . Let $\{v_n\}$ be a convergence subsequence of $\{v_\varepsilon\}$ which converge to $\bar{v} \geq 0$ and $\bar{v} = 0$ at some $x_0 \in \Omega$ or $\partial \bar{v} / \partial n = 0$ on some $x_0 \in \partial \Omega$. If we consider the following equations:

$$\begin{aligned} & \Delta v_n + \lambda f'_\varepsilon(u_n)v_n = -\mu_n v_n \quad \text{in } \Omega, \\ & v_n = 0 \quad \text{on } \partial \Omega \end{aligned} \tag{15}$$

and using the same method as in the proof of Lemma 2, we have

$$\begin{aligned} & \Delta \bar{v} + \lambda \tilde{f} \bar{v} = -\tilde{\mu} \bar{v} \quad \text{in } \Omega, \\ & \bar{v} = 0 \quad \text{on } \partial \Omega \end{aligned} \tag{16}$$

for some $\tilde{\mu} > 0$. But (16) is a linear eigenvalue problem which gives us a contradiction. Hence, if $s < t$ and small enough, we have

$$\Delta \bar{u} + \lambda f_\varepsilon(\bar{u}) + h(\bar{u}) < -s + \|h(\bar{u})\|_\infty \leq 0,$$

by choosing $\delta = s$. Since the choice of δ is independent of ε when ε is sufficiently close to ε_0 which contradicts $\varepsilon_0 = \inf \Gamma$, we have $\Gamma = \emptyset$ and complete the proof. \square

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