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Improved numerical evaluation of the radial groundwater flow equation

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Abstract

The closed-form solution for the hydraulic head, derived for the radial groundwater flow equation subject to the constant-head boundary condition at the wellbore, is in an integral form that covers an integration range from zero to infinity. The integral is difficult to evaluate due to the integrand not only consisting of the product and the square of the Bessel functions but also having a singularity at the origin. A unified numerical method is proposed to evaluate the solution with accuracy to five decimal places and for very wide ranges of dimensionless distances and times. This approach includes a singularity removal scheme, Newton's method, the Gaussian quadrature, and Shanks' method. It gives the dimensionless heads in tabular forms with better accuracy while comparing to those given by other approaches.

A formula describing the flow rate across the wellbore, derived by the authors based on Darcy's law, is proved to equal those presented by Jaeger [Proc Royal Soc Edinburgh 61 (1942) 223] and Jacob and Lohman [Am Geo Union 33 (4) (1952) 559]. The same singularity removal scheme and the Gaussian quadrature are also employed to evaluate the wellbore flow rate. Computed values of dimensionless flow rate versus dimensionless time expressed in tabular forms are also correct to five decimal places. Both the tabular results of dimensionless hydraulic head and dimensionless flow rate may be useful in engineering applications.

Keywords: Radial flow equation; Closed-form solution; Constant head; Numerical approach

1. Introduction

Some physical problems, in the areas of heat conduction, groundwater flow, hydrodynamic problems, and deep mining operations, may be modeled as a diffusion type of partial differential equation in a radial coordinate system. In the mean time, the boundary condition such as the Dirichlet type or the Neuman type is commonly employed at the origin of the coordinate, some distance from the origin, and/or the remote side of the problem domain. The closed-form solution for the radial diffusion equation (e.g., the groundwater flow equation) subject to the Dirichlet-type boundary condition (e.g., constant head or constant temperature) at a cylindrical surface has practical uses in some physical or engineering applications.

Goldstein [10] used Heaviside's operational method, developed by Bromwich, to solve a closed-form solution of the viscous fluid. Using the asymptotic expansion, he found the approximate solutions in small and large dimensionless times. Carlsaw and Jaeger [5] employed the Laplace transformation and the contour integral method to obtain the same closed-form solution to the heat conduction problem. Their solution for temperature distribution in an infinite medium is expressed in an integral form that covers the integration limit from zero to infinity and has an integrand consisting of the product and the square of the Bessel functions. In Harvard's problem report [12], an infinite series expansion was initially adopted to remove the singularity of the integrand at the origin; and then, the numerical integration methods such as the trapezoidal rule, the tangential rule, and the seven-point formulas [7] and cited by Harvard's problem report [12] were applied to evaluate the integral. Results for the temperature distributions in the tabular form are given to five decimal places for dimensionless distances at 2(1)10 and 20 and dimensionless times at 0.1(0.1)1(1)10. Although Harvard's

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problem report [12] had presented the numerical results for the solution to five digits, yet the range of dimensionless times and distances may be too small. Two examples, one is for an idealized pumping test and another one is for a constant-head injection test, given in Batu [2] indicate that the ranges given in Harvard's problem report [12] for dimensionless times from 0.1 to 10 and dimensionless distances from 2 to 20 may not be large enough to meet the need in engineering applications. Moreover, the problems of low accuracy when employing those traditional integration formulas to evaluate the integral may restrict the computations to have the solutions beyond their ranges of dimensionless times and distances and still having the accuracy to five decimal places. Jaeger [18] provided tabular results for the solutions (temperature distributions) for dimensionless times from 10^{-3} to 10^3 and dimensionless distances from 1.1 to 100. However, the numerical values for the solution, obtained by the approximated formulas for the cases of small and large times and by the traditional integration approach for most of times, are only to three decimal places; this accuracy may be not sufficient in engineering applications. Hantush [11] originally introduced Goldstein's solution into the groundwater area for problems having a wedge-shaped aquifer (i.e., non-uniform thickness aquifer) and provided the approximate formulas for relatively small dimensionless time ($\tau < 0.01$) and large dimensionless time ($\tau > 500$).

Smith [25] considered a problem in the deep mining operations, which required finding the heat flux across the wellbore. Using the Bromwich contour integral method, he obtained the solutions for the surface temperature and temperature flux across the wellbore. Tabular values with accuracy to the third decimal place for the heat flux at the wellbore obtained by various approximate formulas can be seen in the literature [15,16,19]. Based on the solution given by Smith [25], Jacob and Lohman [20] presented a formula describing the flow rate across the wellbore and listed tabular values for dimensionless times ranged from 10^{-4} to 10^{12}. Further, Lohman [21] also gave tabular values of dimensionless head, modified from Jacob and Lohman [20], for dimensionless times between 10^{-4} and 10^{12} and at dimensionless times of 10^{13}, 10^{14}, and 10^{15}. Later on, Reed [24] expanded Lohman's table to cover the range of dimensionless times from 10^{12} to 3×10^{15}. Recently, Batu [2] gave exactly the same table for values of dimensionless head versus dimensionless time as Reed's one.

Carslaw and Jaeger [5] briefly described few steps to derive the solution of temperature distribution in a heat conduction problem. The integral in the closed-form solution of Carslaw and Jaeger [5] is difficult to accurately evaluate because of the singularity of the integrand at the origin and the oscillatory nature and slow convergence of the integrand, especially, in small dimensionless times and/or large dimensionless distances. Therefore, the main purpose of this paper is to present a unified numerical approach, including the use of a singularity removal scheme, Newton's method, the Gaussian quadrature, and Shanks' method for efficiently evaluating such a integral with accuracy to five decimal places.

Jaeger [17] and Jacob and Lohman [20] both gave the formulas for describing the flow rate across the wellbore of a well under a constant-head boundary condition. Although these two formulas are in completely different form, yet interestingly both are derived from the same diffusion equation and representing equivalent physical systems. Besides, based on Darcy's law and the closed-form solution for the hydraulic head, we also derive a different formula for wellbore flow rate. Thus, another purpose of this paper is to present the detailed procedures for showing the equivalence of these three formulas mathematically. Finally, we also use the same singularity removal scheme and the Gaussian quadrature to evaluate the wellbore flow-rate formula with the accuracy to five decimal places too.

2. Theoretical background

2.1. Closed-form solution of hydraulic head

Based on the conservation of mass and Darcy's law, the radial flow equation to describe the distribution of the hydraulic head $h(r,t)$ in a homogeneous, isotropic, and confined aquifer may be written as [8]

$$S\frac{\partial h}{\partial t} = T\left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r}\right) \tag{1}$$

where t represents the time from the start test, r represents the radial distance from the centerline of the well, S represents the storage coefficient of the aquifer, and T represents the transmissivity of the aquifer.

The hydraulic head of the aquifer is initially assumed zero, that is

$$h(r,0) = 0 \quad \text{for } r > r_w \tag{2}$$

where r_w represents the well radius. The boundary condition for maintaining a constant head h_0 at $r = r_w$ at any time may be written as

$$h(r_w, t) = h_0 \quad \text{for } t > 0 \tag{3}$$

The hydraulic head is relatively unchanged at any time as r near infinity; such a boundary condition can be expressed as

$$h(\infty, t) = 0 \quad \text{for } t > 0 \tag{4}$$

Carslaw and Jaeger [5] used the Laplace transform and the contour integral method to solve Eq. (1) subject to Eqs. (2)–(4) and gave the time-domain solution as

$$h = h_0 - \frac{2h_0}{\pi}\int_0^\infty e^{(-T/S)u^2 t} \times \frac{[Y_0(ru)J_0(r_w u) - J_0(ru)Y_0(r_w u)]}{[J_0^2(r_w u) + Y_0^2(r_w u)]}\frac{du}{u} \quad (5)$$

where $J_0(\cdot)$ and $Y_0(\cdot)$ are the Bessel functions of the first and second kinds of order zero, respectively.

Define the dimensionless variables $h_D(\rho,\tau) = h/h_0$, $\tau = Tt/Sr_w^2$, and $\rho = r/r_w$ where the notations of h_D, τ, and ρ respectively represent the dimensionless hydraulic head, time, and distance. Eq. (5) can then be expressed in dimensionless form as

$$h_D(\rho,\tau) = 1 - \frac{2}{\pi}\int_0^\infty F(u)\,du \quad (6)$$

where

$$F(u) = e^{-\tau u^2}\frac{[J_0(u)Y_0(\rho u) - Y_0(u)J_0(\rho u)]}{[J_0^2(u) + Y_0^2(u)]u} \quad (7)$$

2.2. Derivation for formulas of dimensionless flow rate across the wellbore

Employing Darcy's law, the flow rate across the wellbore may be expressed as

$$Q_D = -\left.\frac{\partial h_D(\rho,\tau)}{\partial \rho}\right|_{\rho=1} \quad (8)$$

where $Q_D = Q/2\pi r_w T$ is the dimensionless flow rate. Substituting Eq. (6) into Eq. (8) and with $\rho = 1$, one obtains

$$Q_D = \frac{2}{\pi}\int_0^\infty e^{-\tau u^2}\frac{[J_1(u)Y_0(u) - J_0(u)Y_1(u)]}{J_0^2(u) + Y_0^2(u)}\,du \quad (9)$$

where $J_1(u)$ and $Y_1(u)$ are the Bessel functions of the first and second kinds of order first, respectively.

Jaeger [17] also gave a flow-rate formula in the form

$$Q_D = \frac{4}{\pi^2}\int_0^\infty \frac{e^{-\tau u^2}}{[J_0^2(u) + Y_0^2(u)]u}\,du \quad (10)$$

The bracket term on the right-hand side (RHS) of Eq. (9) is equal to $2/\pi u$ based on the formula given in Abramowitz and Stegun [1, p. 360, Eq. (9.1.16)]; that is

$$J_1(u)Y_0(u) - J_0(u)Y_1(u) = 2/(\pi u) \quad (11)$$

Thus, Eq. (10) can be obtained by simply substituting Eq. (11) into Eq. (9).

Based on the results of Smith [25], Jacob and Lohman [20] also presented another formula to represent the flow rate across the wellbore as follows:

$$Q_D = \frac{4\tau}{\pi}\int_0^\infty u e^{-\tau u^2}\left\{\frac{\pi}{2} + \tan^{-1}\left(\frac{Y_0(u)}{J_0(u)}\right)\right\}du \quad (12)$$

Eq. (12) differs from Eq. (10); but these two formulas can be shown equally. Assume that $U = e^{-\tau u^2}$ and $dV = 1/\{[J_0^2(u) + Y_0^2(u)]u\}\,du$; then, $dU = -2\tau u e^{-\tau u^2}\,du$ and $V = (\pi/2)\tan^{-1}[Y_0(u)/J_0(u)]$ [1, p. 82, Eq. (4.4.54)]. Applying the integration by parts, Eq. (10) may be expressed as

$$Q_D = \frac{4}{\pi^2}\left\{e^{-\tau u^2}\frac{\pi}{2}\tan^{-1}\left[\frac{Y_0(u)}{J_0(u)}\right]\Big|_0^\infty - \int_0^\infty \frac{\pi}{2}\tan^{-1}\left[\frac{Y_0(u)}{J_0(u)}\right](-2\tau u e^{-\tau u^2}\,du)\right\} \quad (13)$$

Notably, $Y_0(0)/J_0(0) = -\infty$ when $u = 0$, thus $\tan^{-1} \times [Y_0(0)/J_0(0)] = -\pi/2$; on the other hand, $Y_0(\infty)/J_0(\infty) = 0$ when $u \to \infty$, then $\tan^{-1}[Y_0(\infty)/J_0(\infty)] = 0$. Therefore, Eq. (13) may be expressed as

$$Q_D = 1 + \frac{4\tau}{\pi}\int_0^\infty u e^{-\tau u^2}\tan^{-1}\left(\frac{Y_0(u)}{J_0(u)}\right)du \quad (14)$$

One can easily prove that

$$\frac{4\tau}{\pi}\int_0^\infty \frac{\pi}{2}u e^{-\tau u^2}\,du = 1 \quad (15)$$

Substituting Eq. (15) into (14) yields Eq. (12). Therefore, we have shown that these three formulas, i.e., Eqs. (9), (10) and (12), are mathematically equal.

2.3. Behavior of integrand

In order to explore the characteristic of Eq. (6), one may separate the integrand, Eq. (7), into three components as $E(\tau,u) = \exp(-\tau u^2)$, $N(\rho,u) = J_0(u)Y_0(\rho u) - Y_0(u)J_0(\rho u)$, and $D(u) = [(J_0^2(u) + Y_0^2(u))u]^{-1}$ [12]. The exponent function $E(\tau,u)$ is a damping factor and has a nature of rapid decay from one to zero while u varies from zero to infinity. The component $N(\rho,u)$ shows a damped oscillation along the u-axis for large u. In addition, $D(u)$ exists a branch point at $u = 0$ and approaches to a constant value for large u.

Fig. 1 demonstrates the plots of the integrand, $F(u)$, versus u for $\rho = 10$ and $\tau = 0.001$, 1, or 10 and shows

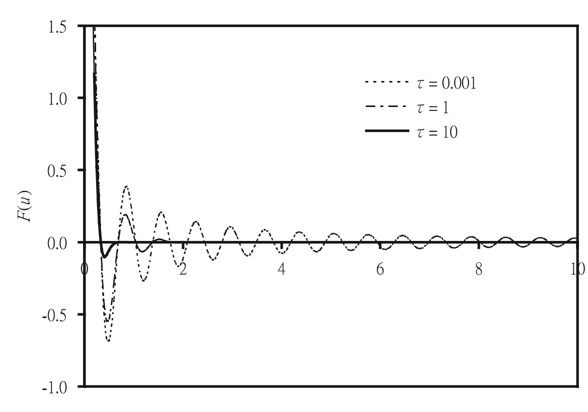

Fig. 1. A plot of the integrand $F(u)$, Eq. (7), versus u for dimensionless distance $\rho = 10$ and dimensionless time $\tau = 0.001$, 1, or 10.

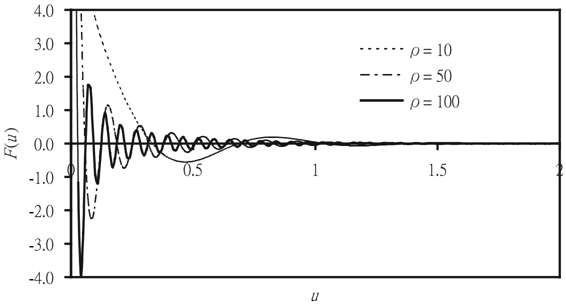

Fig. 2. A plot of the integrand $F(u)$, Eq. (7), versus u for dimensionless time $\tau = 1$ and dimensionless distance $\rho = 10$, 50, or 100.

the oscillatory nature of the integrand. This figure displays that the smaller the τ value, the slower convergence will be the integrand. Besides, the value of $F(u)$ becomes infinity as u close to zero and approaches zero for large u; obviously, the integration of Eq. (6) is difficult to accurately evaluate especially when u is very close to the origin (singular point). Furthermore, for a fixed value of ρ, $F(u)$ not only has the same roots for various values of τ, but also displays decreasing amplitudes of the oscillation with increasing τ. Fig. 2 gives the plots of $F(u)$ versus u for $\tau = 1$ and $\rho = 10$, 50, or 100 and indicates that $F(u)$ increases in the amplitude but decreases in the wavelength of the oscillation while increasing ρ. For small ρ and large τ, the amplitude decreases at a very slow rate. Observed those curves shown in Figs. 1 and 2, one can find the roots of $F(u)$ by a root search scheme and perform the numerical integration for the area under the integrand and between any two consecutive roots. The integrand in Eq. (9) is a monotonically decreasing function, which has a positive large value at the origin (singularity) and rapidly decreases to zero as u goes large.

3. Numerical evaluation of closed-form solution

A unified numerical method is presented in this paper to estimate the values of the closed-form solution, i.e., dimensionless head, at various dimensionless distance and time. This method initially adopts an approach of infinite series expansion given in Harvard's problem report [12] to remove the singularity of the integrand at $u = 0$ so that the numerical integrations for Eqs. (7) and (9), both with the integration limit from zero to infinity, are possible. The Newton's method, which has the merit of quadratic convergence [22], is then employed along with suggested increments to find the consecutive roots of the integrand along the u-axis. For each area under the integrand and between two consecutive roots, the Gaussian quadrature is chosen to perform numerical integrations. Finally, Shanks' method is applied to accelerate the convergence when evaluating the related Bessel functions and the alternating infinite series transformed form the integral.

3.1. Removal of the singularity of integrand at the origin

One efficient way of evaluating the integral in Eq. (6), the closed-form solution, is to transform it to an alternating infinite series. Let $a \ll u_1$ where a is a small value and u_1 be the first root of $F(u)$. The integral over the half-domain may be expressed by piecewise integrations as

$$\int_0^\infty F(u)\,\mathrm{d}u = \int_0^a F(u)\,\mathrm{d}u + \int_a^{u_1} F(u)\,\mathrm{d}u + \sum_{i=1}^{\infty} \int_{u_i}^{u_{i+1}} F(u)\,\mathrm{d}u \tag{16}$$

Harvard's problem report [12] provided an algorithm using an infinite series expansion to remove the singularity of the integrand at the origin. In the interval $0 \leqslant u \leqslant a$, the first term on the RHS of Eq. (16) is subtracted and added a term $(2/\pi)\log\rho$ at the same time for the numerator and expressed as

$$\begin{aligned}\int_0^a F(u)\,\mathrm{d}u &= \int_0^a \frac{\mathrm{e}^{-\tau u^2}[J_0(u)Y_0(\rho u) - Y_0(u)J_0(\rho u)] - \frac{2}{\pi}\log\rho}{[J_0^2(u) + Y_0^2(u)]u}\,\mathrm{d}u \\ &\quad + \int_0^a \frac{\frac{2}{\pi}\log\rho}{[J_0^2(u) + Y_0^2(u)]u}\,\mathrm{d}u\end{aligned} \tag{17}$$

The numerator of the first term on the RHS of Eq. (17) may be approximated by an infinite series. Detailed derivations for expressing the first term of Eq. (17) as an infinite series are given in Appendix A. Accordingly, the first term on the RHS of Eq. (17) reduces to

$$\begin{aligned}&\int_0^a \frac{\mathrm{e}^{-\tau u^2}[J_0(u)Y_0(\rho u) - Y_0(u)J_0(\rho u)] - \frac{2}{\pi}\log\rho}{[J_0^2(u) + Y_0^2(u)]u}\,\mathrm{d}u \\ &\quad = -\frac{2}{\pi}\int_0^a \frac{d_1 u + d_2 u^3 + d_3 u^5 + \Lambda}{J_0^2(u) + Y_0^2(u)}\,\mathrm{d}u\end{aligned} \tag{18}$$

where $d_1, d_2, d_3, \ldots$ are coefficients.

Based on the differentiation formula for arctangent function [1, p. 79, Eq. (4.4.3)] and the relationship of Eq. (11), one can write

$$-\frac{\mathrm{d}}{\mathrm{d}u}\tan^{-1}\left[\frac{J_0(u)}{Y_0(u)}\right] = \frac{2}{\pi u}\frac{1}{J_0^2(u) + Y_0^2(u)} \tag{19}$$

Taking the integration of the RHS of Eq. (19) from zero to a yields

$$\frac{2}{\pi}\int_0^a \frac{\mathrm{d}u}{[J_0^2(u) + Y_0^2(u)]u} = -\tan^{-1}\left[\frac{J_0(a)}{Y_0(a)}\right] \tag{20}$$

Note that $J_0(0)/Y_0(0) = 0$ when $u = 0$, thus $\tan^{-1} \times [J_0(0)/Y_0(0)] = 0$. Based on Eqs. (18) and (20), Eq. (17) can reduce to

$$\int_0^a F(u)\,\mathrm{d}u = -\frac{2}{\pi}\int_0^a \frac{d_1u + d_2u^3 + d_3u^5 + \Lambda}{J_0^2(u) + Y_0^2(u)}\,\mathrm{d}u - \log\rho\tan^{-1}\left[\frac{J_0(a)}{Y_0(a)}\right] \tag{21}$$

The value of the first term on the RHS of Eq. (21) approaches zero as $u \to 0$ because its numerator approaches zero and denominator approaches infinity. The arctangent in the interval $0 \leqslant u \leqslant a$ is a continuous function since $a \ll u_1$; therefore, the second term of Eq. (21) is a finite value when ρ is known. Obviously, the singularity at the origin has been removed and the value of left-hand side (LHS) integral of Eq. (21) can be easily evaluated. In order to obtain good accuracy of the evaluation for Eq. (21), the upper limit a is set as a very small value; say, $a = u_1/10$ for $\tau \leqslant 1$ and $a = u_1/(10\tau)$ for $\tau > 1$.

For the numerical evaluation of the wellbore flow rate, the numerator on the RHS of Eq. (9) can be replaced by $2/\pi u$ based on the relationship of Eq. (11). The integration range of Eq. (9) can be split into [0,a] and (a, ∞) regions. For the first region, one can subtract and add one for the numerator at the same time. Consequently, Eq. (9), the formula for dimensionless flow rate at the wellbore, becomes

$$Q_\mathrm{D} = \frac{4}{\pi^2}\left\{\int_0^a \frac{\mathrm{e}^{-\tau u^2} - 1}{[J_0^2(u) + Y_0^2(u)]u}\,\mathrm{d}u + \int_0^a \frac{1}{[J_0^2(u) + Y_0^2(u)]u}\,\mathrm{d}u + \int_0^\infty \frac{\mathrm{e}^{-\tau u^2}}{[J_0^2(u) + Y_0^2(u)]u}\,\mathrm{d}u\right\} \tag{22}$$

The numerator of the first term on the RHS of Eq. (22) may be approximated by an infinite series as

$$\mathrm{e}^{-\tau u^2} - 1 = -\tau u^2 + \frac{1}{2!}(\tau u^2)^2 - \frac{1}{3!}(\tau u^2)^3 + \frac{1}{4!}(\tau u^2)^4 - \Lambda \tag{23}$$

Thus, the variable u in the denominator of the first term on the RHS of Eq. (22), which poses the problem of the singularity at $u = 0$, can be cancelled out. Based on the relationships of Eqs. (20) and (23), Eq. (22) can be rewritten as

$$Q_\mathrm{D} = \frac{4}{\pi^2}\left\{\int_0^a \frac{-\tau u + \frac{1}{2}\tau^2u^3 - \frac{1}{3!}\tau^3u^5 + \frac{1}{4!}\tau^4u^7 - \Lambda}{[J_0^2(u) + Y_0^2(u)]}\,\mathrm{d}u - \frac{\pi}{2}\tan^{-1}\left(\frac{J_0(a)}{Y_0(a)}\right) + \int_a^\infty \frac{\mathrm{e}^{-\tau u^2}}{[J_0^2(u) + Y_0^2(u)]u}\,\mathrm{d}u\right\} \tag{24}$$

The value of the first RHS term approaches zero as $u \to 0$ because the numerator of the integrand approaches zero and the denominator approaches infinity. Similarly, the arctangent is a continuous function in the interval $0 \leqslant u \leqslant a$ because $J_0(a)/Y_0(a)$ is a finite value as discussed before. Thus, the formula for dimensionless flow rate, Eq. (24), does not contain the singular point. Because the integrand of the third term on the RHS of Eq. (24) is a monotonically decreasing function, Eq. (24) can be easily evaluated by the Gaussian quadrature. The upper limit a is set as a very small value, say 10^{-5}, for better accuracy.

3.2. Newton's method and suggested increments

The pattern of the oscillatory nature of the integrand $F(u)$ along the u-axis indicates that the result of integration between any two consecutive roots and under the integrand may be considered as a term of an alternating series. Because the oscillation of $F(u)$ is due to the term $N(\rho, u)$, the functions $F(u)$ has the same roots as $N(\rho, u)$; and the roots of $N(\rho, u)$ may be found by a conjunctive use of the following suggested increments for locating the roots and Newton's method for iteratively converging to the root.

In reality, dimensionless distance, ρ, is a critical factor when determining the location of the root. For $\rho > 1$, the asymptotic expansion of the large positive ith root of the nominator in Eq. (7), u_i, is [1, p. 374]

$$u_i = \beta + \frac{\alpha}{\beta} + \frac{\gamma - \alpha^2}{\beta^3} + \frac{\mu - 4\alpha\gamma + 2\alpha^3}{\beta^5} + \Lambda, \quad i = 1, 2, \ldots \tag{25}$$

where $\beta = i\pi/(\rho - 1)$, $\alpha = -1/(8\rho)$, $\gamma = 25(\rho^3 - 1)/\lfloor 6(4\rho)^3(\rho - 1)\rfloor$, and $\mu = -1073(\rho^5 - 1)/\lfloor 5(4\rho)^5 \times (\rho - 1)\rfloor$. When ρ is small, Eq. (25) gives good approximation for first few roots of $F(u)$ in Eq. (6). Thus, the first root u_1 can be approximated by $\pi/(\rho - 1)$ which is obtained by simply neglecting the second and remaining terms of Eq. (25). Therefore, the increment Δ_1 from the origin to the first root approximately equals to $\pi/(\rho - 1)$. When ρ is large, the approximate result of using the increment Δ_1 to estimate the large roots of $F(u)$ will be very poor; on the other word, another increments that give reasonable approximations to the larger roots are needed. Reasonable guess for the second increment is chosen as $\Delta_2 = u_1$; and, therefore, the second root u_2 is approximately equal to $2u_1$. Similarly, the remaining increments Δ_i are chosen as $u_{i-1} - u_{i-2}$ and the remaining roots are approximately equal to $u_i = u_{i-1} + \Delta_i$, where $i = 3, 4, \ldots$.

With the suggested increment Δ_i, Newton's method is capable of finding the sequence of roots in a very efficient manner. The iterative scheme representing Newton's algorithm to find the roots of the integrand is [22]

$$u_i^{j+1} = u_i^j - \frac{F(u_i^j)}{F'(u_i^j)}, \quad j = 1, 2, \ldots \tag{26}$$

where u_i^j represents the ith root at the jth iteration. The derivative of $F(u)$ obtained by direct differentiation is

$$\begin{aligned} F'(u) = {} & 2\tau u F(u) + \mathrm{e}^{-\tau u^2} D(u)\{[Y_1(u)J_0(\rho u) - J_1(u)Y_0(\rho u)] \\ & + \rho[Y_0(u)J_1(\rho u) - J_0(u)Y_1(\rho u)]\} \\ & - F(u)D(u)\{[J_0^2(u) + Y_0^2(u)] \\ & - 2u[J_0(u)J_1(u) + Y_0(u)Y_1(u)]\} \end{aligned} \quad (27)$$

The criteria used to terminate the iteration of Eq. (26) are $|u_i^{j+1} - u_i^j| < \mathrm{UTOL}$ and $|F(u_i^{j+1}) - F(u_i^j)| < \mathrm{FTOL}$, where the values of UTOL and FTOL depend on the desired accuracy of the roots. For example, by taking $\mathrm{UTOL} = 10^{-10}$ and $\mathrm{FTOL} = 10^{-20}$ and starting from $u = 0$ with $\Delta = 0.1$, Newton's method only takes seven iterations to find the first root $u_1 = 0.3313938715$, which has the accuracy to 10 decimal places when $\tau = 1$ and $\rho = 10$.

3.3. Shanks' method

The Shanks transform, also called the ε-algorithm, consists of a family of nonlinear sequence-to-sequence transformations [26]. Shanks [26] proved that these transformations are effective when applied to accelerate the convergence of (some) slowly convergent sequences and when converging (some) divergent sequences. Some examples of the applications of Shanks' method include numerical series, the power series of rational and meromorphic functions, and a wide variety of sequences drawn from integral equations, geometry, fluid mechanics and number theory [26]. Huang et al. [14] had applied the Shanks transform method to evaluate the drawdown solution in a large-diameter well and the solution for groundwater flow under constant-head boundary condition. The results show that this method is very efficient while applying to these two drawdown solutions.

The infinite series transformed from the third term on the RHS of Eq. (16) converges very slowly as demonstrated in Fig. 1 for the case of small τ and large ρ. Therefore, Shanks' method is applied to accelerate the convergence of the running sum for such an infinite series. The RHS of Eq. (16) may be expressed as

$$S_n = S_1 + \sum_{k=2}^{\infty} S_k \quad (28)$$

where S_n represents a sequence of partial sums, S_1 represents the sum of the first and the second terms in Eq. (16), and $\sum_{k=2}^{\infty} S_k$ represents the running sum of the third term in Eq. (16). Such an approach of adding all the terms until the specified tolerance is met for Eq. (28) is called the direct sum. The Shanks transform is a nonlinear iterative algorithm based on the sequence of partial sums and expressed as [28]

$$e_{s+1}(S_n) = e_{s-1}(S_{n+1}) + \frac{1}{e_s(S_{n+1}) - e_s(S_n)}, \quad s = 1, 2, \ldots \quad (29)$$

where $e_0(S_n) = S_n$ and $e_1(S_n) = [e_0(S_{n+1}) - e_0(S_n)]^{-1}$.

The Shanks transform requires using the even-order terms when approximating the sum of S_n; the odd-order terms are purely intermediate quantities in the computations. Applying the Shanks transform to evaluate a given series requires setting a convergence criterion. A convergence factor, ε, is defined as

$$\left| \frac{e_{2r+2}(S_{n-1}) - e_{2r}(S_n)}{e_{2r+1}(S_{n-1})} \right| \leqslant \varepsilon \quad (30)$$

The running sum is terminated when the LHS term of Eq. (30), which indeed includes three successive terms, is less than ε. This procedure can avoid the iteration being stopped prematurely.

3.4. Numerical integration

The Gaussian quadrature is a commonly used technique to perform the numerical integration. The integral $\int_a^b f(x)\,\mathrm{d}x$ should be transformed by using the change of variable to the new integration interval of [−1,1] when employing the Gaussian quadrature. The formula of the Gaussian quadrature may be written as [9]

$$\int_{-1}^{1} f(\xi)\,\mathrm{d}\xi = \sum_{i=1}^{n} W_i f(\xi_i) \quad \text{for } n \text{ points} \quad (31)$$

where W_i represents the weighting factor and ξ_i represents the integration point. Values of W_i and ξ_i can be found from the books in the fields of numerical methods (e.g., Refs. [3,9]) and the finite element methods (e.g., Refs. [4,23]).

The first term on the RHS of Eq. (16) can be easily estimated after the removal of the singularity for the integrand at the origin. The second RHS term of Eq. (16) can be directly evaluated from a to u_1 (i.e., the smallest root) and the third RHS term denotes the sum of those integration results representing the areas under $F(u)$ and between u_i and u_{i+1} where $i = 1, 2, \ldots$. Each area is considered as a term of an infinite series and the running sum is terminated by means of Shanks' method, which is applied to accelerate the convergence for the evaluation of the infinite series. In this case, the convergence criterion is set as 10^{-7} for the evaluations of dimensionless head of Eq. (6).

Both the six-point and ten-point formulas of the Gaussian quadrature are used at the same time to carry out the numerical integration for each RHS term of Eq. (16). If the difference of these two results for any interval between two consecutive roots is greater than the prescribed criterion, then, the interval will be divided into two portions. The same integration procedure is repeatedly applied to each portion until the convergence criteria are met to ensure that the result bears the desired accuracy. The result of the integration within the interval, considered as a term of the infinite series, is equal to the sum of the areas obtained from total divided portions. The third term on the RHS of Eq. (16), the running sum of an infinite series, is evaluated and terminated by means of Shanks' method. Adding those three RHS terms of Eq. (16) gives the value of Eq. (6).

The interval for the numerical integration of dimensionless flow rate in Eq. (9) is chosen as 10^{-10}. Then, both the six-point and ten-point formulas of the Gaussian quadrature are also used at the same time to carry out the integration of (9). If the difference of these two integration results is greater than the prescribed criterion, say 10^{-7}, then, the interval will be divided into two portions, and the same integration procedure is again applied to each portion until the integration result for each portion is less than 10^{-7}. Finally, the numerical integration result for dimensionless flow rate can be obtained by simply adding all the results from each interval or portion.

4. Numerical evaluations and discussions

Table 1 lists the suggested increments, the approximate roots, the roots of Eq. (7), and the number of iterations required by Newton's method to find the first five roots for $\rho = 10$ and $\tau = 1$ or 10. It shows that the approximate roots are fairly close to the real roots based on the suggested increments when using Newton's method to determine the roots. Notably, Newton's method takes less than 10 iterations to converge to the roots.

The Shanks transform is employed to accelerate the calculation of dimensionless hydraulic head as well as the Bessel functions of $J_0(u)$, $J_1(u)$, $Y_0(u)$, and $Y_1(u)$. Both Shanks' method and the direct sum are employed to evaluate Eq. (6) for $\rho = 2$, 10, 50, or 100 while $\tau = 0.001$, 0.1, and 10 and the results under the criterion $\varepsilon = 10^{-7}$ are given in Table 2. As indicated in Table 2, Shanks' method needs less than 14 terms to obtain the result while the direct sum generally requires more terms than Shanks' method if ρ is large. Table 2 also indicates that the Shanks transform converges significantly faster than the direct sum when τ is small and ρ is large. Obviously, Shanks' method can efficiently accelerate the evaluation of the integral for those oscillatory functions.

Table 1
Suggested increments and required number of iterations when using Newton's method to find the roots of the integrand, Eq. (7), for $\rho = 10$

i	Increment, Δ_i	Approximate root	Root, U_i	Number of iterations	
				$\tau = 1$	$\tau = 10$
1	0.34907	0.34907	0.33139	6	6
2	0.33139	0.66278	0.68576	7	8
3	0.35436	1.01714	1.03774	8	7
4	0.35198	1.38972	1.38864	9	9
5	0.35090	1.73954	1.73896	9	5

Note: ρ is the dimensionless distance and τ is the dimensionless time.

Table 2
Required terms to achieve the same degree of accuracy when employing the direct sum (DS) and Shanks' method (SM) to evaluate Eq. (6) for various ρ and τ under the criterion $\varepsilon = 10^{-7}$

τ	ρ							
	2		10		50		100	
	DS	SM	DS	SM	DS	SM	DS	SM
0.001	36	13	271	12	>1000	12	>1000	12
0.1	5	8	31	12	147	12	278	12
10	5	8	5	8	17	12	32	12

Notably, all the Bessel functions in the integrand are evaluated with accuracy to 10 decimal places. The results of dimensionless head (h_D) versus dimensionless time (τ), ranging from 10^{-3} to 10^3, and dimensionless distance (ρ), ranging from 1.1 to 100, listed in Tables 3–5 correct to at least five decimal places when applying the proposed unified method to evaluate Eq. (6). Noticed that $h_D(\rho = 1.0) = 1.0$ for any time τ is the constant-head boundary condition. Those results agree with those of Harvard's problem report [12] to five decimal places and with those of Jaeger [18] to three decimal places as indicated from comparisons with those two results.

The integration of Harvard's problem report [12] was, however, solely done by traditional integration formulas such as the trapezoid rule, which may cause the problems of low accuracy and high computing time. Consequently, their integration may also have the problem of slow convergence whenever τ is small and ρ is large and results in poor accurate solutions. Batu [2, p. 199] gives an example of idealized pumping test taken from Wikramaratra [27]. The recorded drawdown starts from 10 s for an aquifer with following data: $T = 86.4$ m^2/day, $S = 0.01$, $r_w = 0.2$ m. Therefore, the minimum value of dimensionless time is $\tau = 25$ which is larger than 10, the upper limit of dimensionless time given in Harvard's problem report [12]. Besides, a constant-head injection test is given in Batu [2, p. 693] to determine the formation horizontal hydraulic conductivity Kr. One of the well radii is 0.1 m and various values of the radius of influence R considered to estimate Kr are 0.5, 1, 2, 4, and 5. For those values $R > 2$, dimensionless distance ρ will be greater than 20 which is beyond the upper limit of dimensionless distance given in Harvard's problem report [12]. These two examples may indicate that the ranges given in Harvard's problem report [12] for dimensionless times from 0.1 to 10 and dimensionless distances from 2 to 20 may not be large enough to meet

Table 3
Values of dimensionless hydraulic head $h_D(\rho,\tau)$, Eq. (6), for ρ from 1.1 to 2.0 and τ from 0.001 to 1000

τ	Dimensionless distance ρ									
	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0.001	0.02417	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.002	0.10858	0.00143	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.003	0.18763	0.00897	0.00009	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.004	0.25141	0.02315	0.00070	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.005	0.30272	0.04157	0.00237	0.00005	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.006	0.34472	0.06203	0.00542	0.00022	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.007	0.37977	0.08312	0.00986	0.00061	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000
0.008	0.40954	0.10404	0.01555	0.00132	0.00006	0.00000	0.00000	0.00000	0.00000	0.00000
0.009	0.43520	0.12433	0.02226	0.00243	0.00016	0.00001	0.00000	0.00000	0.00000	0.00000
0.01	0.45759	0.14378	0.02977	0.00396	0.00033	0.00002	0.00000	0.00000	0.00000	0.00000
0.02	0.58914	0.29025	0.11747	0.03855	0.01017	0.00214	0.00036	0.00005	0.00000	0.00000
0.03	0.65238	0.37911	0.19415	0.08690	0.03378	0.01135	0.00328	0.00082	0.00017	0.00003
0.04	0.69132	0.43908	0.25429	0.13350	0.06323	0.02692	0.01027	0.00350	0.00107	0.00029
0.05	0.71838	0.48285	0.30195	0.17487	0.09343	0.04592	0.02071	0.00855	0.00323	0.00111
0.06	0.73859	0.51659	0.34061	0.21093	0.12230	0.06622	0.03342	0.01569	0.00684	0.00277
0.07	0.75444	0.54360	0.37269	0.24237	0.14911	0.08660	0.04739	0.02440	0.01180	0.00536
0.08	0.76730	0.56586	0.39982	0.26994	0.17374	0.10641	0.06191	0.03417	0.01787	0.00885
0.09	0.77800	0.58462	0.42313	0.29429	0.19630	0.12535	0.07652	0.04460	0.02479	0.01313
0.1	0.78710	0.60071	0.44345	0.31598	0.21694	0.14329	0.09093	0.05537	0.03233	0.01808
0.2	0.83704	0.69116	0.56234	0.45037	0.35472	0.27454	0.20866	0.15564	0.11387	0.08167
0.3	0.85958	0.73299	0.61955	0.51869	0.42984	0.35241	0.28570	0.22893	0.18124	0.14172
0.4	0.87319	0.75847	0.65493	0.56185	0.47862	0.40468	0.33948	0.28246	0.23302	0.19054
0.5	0.88256	0.77611	0.67963	0.59233	0.51358	0.44283	0.37958	0.32335	0.27367	0.23008
0.6	0.88954	0.78928	0.69817	0.61537	0.54025	0.47226	0.41093	0.35582	0.30653	0.26266
0.7	0.89501	0.79962	0.71276	0.63361	0.56149	0.49588	0.43632	0.38240	0.33375	0.29002
0.8	0.89944	0.80802	0.72466	0.64852	0.57894	0.51540	0.45744	0.40468	0.35676	0.31337
0.9	0.90314	0.81504	0.73461	0.66102	0.59363	0.53190	0.47538	0.42371	0.37656	0.33361
1	0.90629	0.82101	0.74309	0.67171	0.60622	0.54608	0.49088	0.44023	0.39382	0.35137
2	0.92371	0.85417	0.79040	0.73164	0.67729	0.62689	0.58003	0.53641	0.49574	0.45781
3	0.93175	0.86950	0.81235	0.75959	0.71068	0.66517	0.62270	0.58298	0.54575	0.51081
4	0.93668	0.87892	0.82585	0.77683	0.73133	0.68892	0.64928	0.61212	0.57720	0.54433
5	0.94013	0.88551	0.83531	0.78891	0.74582	0.70563	0.66801	0.63270	0.59948	0.56814
6	0.94273	0.89048	0.84244	0.79804	0.75677	0.71826	0.68220	0.64831	0.61640	0.58627
7	0.94479	0.89441	0.84809	0.80526	0.76545	0.72829	0.69346	0.66073	0.62987	0.60072
8	0.94648	0.89763	0.85273	0.81119	0.77258	0.73651	0.70271	0.67093	0.64095	0.61261
9	0.94790	0.90034	0.85662	0.81618	0.77857	0.74344	0.71051	0.67953	0.65030	0.62265
10	0.94911	0.90267	0.85997	0.82046	0.78372	0.74939	0.71720	0.68692	0.65833	0.63129
20	0.95605	0.91593	0.87903	0.84488	0.81310	0.78339	0.75551	0.72924	0.70442	0.68090
30	0.95939	0.92232	0.88822	0.85665	0.82728	0.79981	0.77402	0.74971	0.72674	0.70496
40	0.96150	0.92635	0.89402	0.86409	0.83623	0.81018	0.78572	0.76266	0.74087	0.72020
50	0.96300	0.92923	0.89816	0.86940	0.84263	0.81759	0.79408	0.77192	0.75096	0.73109
60	0.96416	0.93143	0.90133	0.87347	0.84753	0.82327	0.80049	0.77901	0.75871	0.73945
70	0.96508	0.93320	0.90388	0.87674	0.85147	0.82783	0.80563	0.78471	0.76492	0.74616
80	0.96585	0.93467	0.90599	0.87944	0.85473	0.83161	0.80990	0.78943	0.77008	0.75172
90	0.96650	0.93592	0.90779	0.88174	0.85750	0.83482	0.81352	0.79345	0.77446	0.75645
100	0.96706	0.93700	0.90934	0.88373	0.85990	0.83760	0.81666	0.79692	0.77825	0.76054
200	0.97038	0.94334	0.91846	0.89543	0.87399	0.85393	0.83509	0.81733	0.80053	0.78460
300	0.97204	0.94652	0.92304	0.90130	0.88106	0.86213	0.84435	0.82759	0.81173	0.79669
400	0.97312	0.94858	0.92601	0.90510	0.88565	0.86745	0.85035	0.83423	0.81899	0.80452
500	0.97390	0.95008	0.92816	0.90787	0.88898	0.87131	0.85471	0.83906	0.82426	0.81022
600	0.97451	0.95124	0.92984	0.91002	0.89157	0.87431	0.85810	0.84281	0.82835	0.81464
700	0.97501	0.95219	0.93120	0.91176	0.89367	0.87674	0.86085	0.84586	0.83168	0.81823
800	0.97542	0.95298	0.93233	0.91322	0.89543	0.87878	0.86315	0.84841	0.83446	0.82123
900	0.97577	0.95365	0.93331	0.91447	0.89693	0.88053	0.86512	0.85059	0.83684	0.82381
1000	0.97608	0.95424	0.93416	0.91556	0.89824	0.88205	0.86683	0.85249	0.83892	0.82605

Note: $h_D(\rho = 1, \tau) = 1.0$.

the need in engineering applications. Jaeger [18] also presented tabular values of dimensionless temperatures for the same ranges of τ and ρ as we give, yet, his solutions only correct to three decimal places. He calculated Eq. (6) by dividing τ into three regions, $\tau < 0.3$, $0.3 < \tau < 10$, and $\tau > 10$. For the region of the early time, $\tau < 0.3$, a simple formula expressed in the form of the repeated integral of the error function was used to

Table 4
Values of dimensionless hydraulic head $h_D(\rho, \tau)$, Eq. (6), for ρ from 2 to 10 and τ from 0.1 to 1000

τ	Dimensionless distance ρ								
	2	3	4	5	6	7	8	9	10
0.1	0.01808	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	0.08167	0.00092	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.3	0.14172	0.00577	0.00005	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.4	0.19054	0.01496	0.00041	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.5	0.23008	0.02696	0.00138	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000
0.6	0.26266	0.04035	0.00317	0.00012	0.00000	0.00000	0.00000	0.00000	0.00000
0.7	0.29002	0.05424	0.00578	0.00033	0.00001	0.00000	0.00000	0.00000	0.00000
0.8	0.31337	0.06807	0.00915	0.00072	0.00003	0.00000	0.00000	0.00000	0.00000
0.9	0.33361	0.08155	0.01313	0.00132	0.00008	0.00000	0.00000	0.00000	0.00000
1	0.35137	0.09452	0.01760	0.00217	0.00017	0.00001	0.00000	0.00000	0.00000
2	0.45781	0.19414	0.07081	0.02151	0.00534	0.00107	0.00017	0.00002	0.00000
3	0.51081	0.25650	0.11862	0.04918	0.01800	0.00576	0.00160	0.00038	0.00008
4	0.54433	0.29964	0.15698	0.07640	0.03409	0.01382	0.00507	0.00167	0.00049
5	0.56814	0.33179	0.18797	0.10101	0.05087	0.02382	0.01032	0.00412	0.00151
6	0.58627	0.35702	0.21354	0.12281	0.06714	0.03465	0.01680	0.00762	0.00323
7	0.60072	0.37754	0.23508	0.14208	0.08246	0.04566	0.02401	0.01194	0.00561
8	0.61261	0.39470	0.25354	0.15920	0.09672	0.05649	0.03158	0.01685	0.00856
9	0.62265	0.40935	0.26961	0.17451	0.10991	0.06695	0.03928	0.02213	0.01195
10	0.63129	0.42206	0.28377	0.18829	0.12212	0.07696	0.04695	0.02764	0.01567
20	0.68090	0.49680	0.37034	0.27739	0.20705	0.15328	0.11219	0.08101	0.05760
30	0.70496	0.53387	0.41494	0.32578	0.25641	0.20142	0.15747	0.12228	0.09418
40	0.72020	0.55755	0.44384	0.35779	0.28993	0.23519	0.19046	0.15371	0.12345
50	0.73109	0.57455	0.46476	0.38121	0.31481	0.26068	0.21590	0.17852	0.14719
60	0.73945	0.58763	0.48092	0.39944	0.33434	0.28092	0.23634	0.19877	0.16690
70	0.74616	0.59816	0.49397	0.41422	0.35027	0.29755	0.25330	0.21573	0.18361
80	0.75172	0.60690	0.50484	0.42657	0.36365	0.31159	0.26770	0.23025	0.19802
90	0.75645	0.61433	0.51410	0.43711	0.37511	0.32366	0.28016	0.24287	0.21064
100	0.76054	0.62078	0.52213	0.44628	0.38510	0.33423	0.29109	0.25400	0.22183
200	0.78460	0.65872	0.56963	0.50080	0.44489	0.39798	0.35775	0.32268	0.29175
300	0.79669	0.67784	0.59364	0.52849	0.47547	0.43085	0.39245	0.35883	0.32902
400	0.80452	0.69023	0.60922	0.54651	0.49540	0.45235	0.41523	0.38266	0.35372
500	0.81022	0.69924	0.62057	0.55963	0.50995	0.46806	0.43190	0.40015	0.37189
600	0.81464	0.70624	0.62939	0.56984	0.52128	0.48031	0.44492	0.41382	0.38611
700	0.81823	0.71193	0.63655	0.57814	0.53048	0.49027	0.45552	0.42496	0.39771
800	0.82123	0.71668	0.64254	0.58508	0.53820	0.49862	0.46440	0.43430	0.40746
900	0.82381	0.72076	0.64768	0.59104	0.54480	0.50577	0.47203	0.44233	0.41583
1000	0.82605	0.72431	0.65215	0.59622	0.55057	0.51202	0.47868	0.44933	0.42314

approximate Eq. (6). For the region of moderate time, $0.3 < \tau < 10$, the integration limits were spitted into two parts, (0, 0.2) and (0.2, ∞). In the first part, a convenient series expansion was used, while in the second part the integral of Eq. (6), was evaluated directly. For the region of large time ($\tau > 10$) and ρ is not too large, the integral of Eq. (6) was directly evaluated for moderate τ and was approximated by the asymptotic expansions for large τ. Finally, for $\tau > 10$ and ρ is large, Eq. (6) was written in a slightly different form, while u was replaced by u/ρ, and evaluated by the same methods described above. The integral of Eq. (6) was evaluated in such a cumbersome manner; consequently, the accuracy of his final results is generally unknown or even poor. Contrarily, our approach can be applied not only to directly evaluate the integral and obtain the results to five decimal places, but also significantly reduce the computing time due to the use of Shanks' method to accelerate the convergence.

Batu [2, p. 699] also gave another example for the constant-head injection test, which was taken from Lohman [21]. After the well was shut in for a period of several days, the static head just prior to the test was 92.33 ft (28.14 m), where four significant figures were used, above discharge point. This may reflect that Jaeger's tabular values with only three significant figures for the solutions may be not sufficient for engineering applications. The values of Eq. (9) estimated by the Gaussian quadrature after the removal of singularity are listed in column 2 of Table 6, in which dimensionless flow rate is correct to at least five decimal places for dimensionless times from 0.01 to 1000. Those flow rates given by Jacob and Lohman [20] and Jaeger and Clarke [19] and shown in Table 6 only have, at most, to third decimal places, which may also have the problem of the insufficient accuracy. It is noteworthy that Eq. (9) is a monotonically decreasing function, the values of Eq. (9)

Table 5
Values of dimensionless hydraulic head $h_D(\rho, \tau)$, Eq. (6), for ρ from 10 to 100 and τ from 10 to 1000

τ	Dimensionless distance ρ									
	10	20	30	40	50	60	70	80	90	100
10	0.01567	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
20	0.05760	0.00068	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
30	0.09418	0.00373	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
40	0.12345	0.00910	0.00025	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
50	0.14719	0.01588	0.00082	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
60	0.16690	0.02333	0.00181	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
70	0.18361	0.03099	0.00324	0.00019	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000
80	0.19802	0.03860	0.00504	0.00040	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000
90	0.21064	0.04603	0.00717	0.00072	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000
100	0.22183	0.05320	0.00954	0.00116	0.00009	0.00000	0.00000	0.00000	0.00000	0.00000
200	0.29175	0.10951	0.03783	0.01117	0.00273	0.00054	0.00009	0.00001	0.00000	0.00000
300	0.32902	0.14646	0.06397	0.02565	0.00919	0.00290	0.00080	0.00019	0.00004	0.00001
400	0.35372	0.17302	0.08559	0.04022	0.01753	0.00699	0.00253	0.00083	0.00024	0.00006
500	0.37189	0.19344	0.10352	0.05369	0.02638	0.01213	0.00519	0.00205	0.00075	0.00025
600	0.38611	0.20986	0.11866	0.06586	0.03512	0.01779	0.00850	0.00381	0.00160	0.00063
700	0.39771	0.22352	0.13166	0.07682	0.04348	0.02362	0.01224	0.00602	0.00280	0.00123
800	0.40746	0.23515	0.14300	0.08671	0.05137	0.02944	0.01621	0.00854	0.00429	0.00205
900	0.41583	0.24524	0.15302	0.09568	0.05878	0.03512	0.02029	0.01129	0.00603	0.00309
1000	0.42314	0.25412	0.16198	0.10386	0.06571	0.04063	0.02441	0.01419	0.00796	0.00430

for dimenssionless times beyond the ranges given in Table 6 can also be easily evaluated by the Gaussian quadrature after removing the singularity at the origion.

5. Conclusions

The closed-form solution for the radial diffusion equation subject to the Dirichlet-type boundary condition at a cylindrical surface is expressed in an integral form that covers a range from zero to infinity and has an integrand consisting of the product and the square of the Bessel functions. A unified numerical method, including the use of a singularity removal scheme, Newton's method, the Gaussian quadrature, and Shanks' method, is proposed for efficiently evaluating the integral of the closed-form solution for very wide range of dimensionless times and distances with accuracy to five decimal places. This method initially adopts an approach of infinite series expansion to remove the singularity of the integrand at $u = 0$ before performing the numerical integrations. Newton's method is then employed along with suggested increments to find the consecutive roots of the integrand along the horizontal axis. Our suggested increments have been shown to give good estimates to the roots and Newton's method usually takes less than 10 iterations to converge to the roots. For each area under the integrand and between two consecutive roots, the Gaussian quadrature is chosen to perform the numerical integrations. Finally, Shanks' method is applied to accelerate the convergence when evaluating the related Bessel functions and the alternating infinite series transformed form the integral. Tabular results for the evaluations of dimensionless head correct to five decimal places, which should be sufficient in terms of the accuracy for engineering applications to the proposed approach. It has clearly been demonstrated that our approach can not only directly evaluate the values of dimensionless head to five decimal places for very wide ranges of dimensionless times and distances, but also significantly reduce the computing time due to the use of Shanks' method to accelerate the convergence.

Based on Darcy's law and the solution for dimensionless head, we also derive a formula for dimensionless flow rate across the wellbore. This formula differs from the earlier ones given by Jaeger [17] and Jacob and Lohman [20]; yet, we have proven that these three formulas are essentially equivalent. This flow-rate formula is also difficult to accurately evaluate because of a singularity point exiting at the origin of the integrand. The same singularity removal scheme is also used when evaluating dimensionless flow rate across the wellbore for a wide range of dimensionless time and distance. The results estimated by the proposed numerical approach also have accuracy to five decimal places.

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Table 6
Dimensionless flow rates estimated by the proposed method, Jacob and Lohman [20], and Jaeger and Clarke [19] for dimensionless times r from 0.01 to 1000

τ	Proposed method	Jacob and Lohman	Jaeger and Clarke
0.01	6.12891	6.13	6.129
0.02	4.47163	4.47	4.472
0.03	3.73605	3.74	3.736
0.04	3.29681	3.30	3.297
0.05	2.99658	3.00	2.997
0.06	2.77462	2.78	2.775
0.07	2.60186	2.60	2.602
0.08	2.46240	2.46	2.463
0.09	2.34673	2.35	2.347
0.1	2.24875	2.249	2.249
0.2	1.71522	1.716	1.715
0.3	1.47625	1.477	1.476
0.4	1.33248	1.333	1.333
0.5	1.23357	1.234	1.234
0.6	1.16001	1.160	1.160
0.7	1.10246	1.103	1.102
0.8	1.05577	1.057	1.056
0.9	1.01686	1.018	1.017
1	0.98377	0.985	0.984
2	0.80058	0.803	0.800
3	0.71620	0.719	0.716
4	0.66440	0.667	0.664
5	0.62818	0.630	0.628
6	0.60088	0.602	0.601
7	0.57928	0.580	0.579
8	0.56157	0.562	0.562
9	0.54668	0.547	0.547
10	0.53392	0.534	0.534
20	0.46114	0.461	0.461
30	0.42610	0.427	0.426
40	0.40398	0.405	0.404
50	0.38818	0.389	0.388
60	0.37608	0.377	0.376
70	0.36637	0.367	0.366
80	0.35832	0.359	0.358
90	0.35148	0.352	0.352
100	0.34556	0.346	0.346
200	0.31080	0.311	0.311
300	0.29334	0.294	0.294
400	0.28203	0.283	0.282
500	0.27381	0.274	0.274
600	0.26743	0.268	0.268
700	0.26225	0.263	0.263
800	0.25791	0.258	0.258
900	0.25420	0.254	0.255
1000	0.25096	0.251	0.251

Appendix A. Singularity removal for the first term of Eq. (17)

The numerator of the integrand of the first term on the RHS of Eq. (17) consists of two components, $e^{-\tau u^2}[J_0(u)Y_0(\rho u) - Y_0(u)J_0(\rho u)]$ and $(2/\pi)\log\rho$. The exponential function in the first components may be expanded to a series as

$$e^{-\tau u^2} = 1 - \tau u^2 + \frac{\tau^2}{2}u^4 - \frac{\tau^3}{6}u^6 + \frac{\tau^4}{24}u^8 - \Lambda \tag{A.1}$$

Besides, the two cross products $J_0(u)Y_0(\rho u) - Y_0(u)J_0(\rho u)$ may be expressed in terms of a series in even powers of u as [12]

$$J_0(u)Y_0(\rho u) - Y_0(u)J_0(\rho u) = -\frac{2}{\pi}\sum_{m=0}^{\infty}(-1)^m A_m\left(\frac{u}{2}\right)^{2m} \tag{A.2}$$

where

$$A_m = \sum_{n=0}^{m} \frac{\rho^{2n}}{n!n!(m-n)!(m-n)!}\left[-\log\rho + \left(1+\frac{1}{2}+\frac{1}{3}+\Lambda+\frac{1}{n}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\Lambda+\frac{1}{m-n}\right)\right] \quad (A.3)$$

Furthermore, expanding the LHS of Eq. (A.2) into power series has

$$J_0(u)Y_0(\rho u) - Y_0(u)J_0(\rho u) = -\frac{2}{\pi}\left[C_0 - C_1\left(\frac{u}{2}\right)^2 + C_2\left(\frac{u}{2}\right)^4 - C_3\left(\frac{u}{2}\right)^6 + \Lambda\right] \quad (A.4)$$

where

$$C_0 = -\log\rho \quad (A.5)$$

$$C_1 = -(\rho^2+1)\log\rho + (\rho^2-1) \quad (A.6)$$

$$C_2 = -\frac{1}{4}(\rho^4+4\rho^2+1)\log\rho + \frac{3}{8}(\rho^4-1) \quad (A.7)$$

and

$$C_3 = -\frac{1}{36}(\rho^6+9\rho^4+9\rho^2+1)\log\rho + \frac{1}{216}(11\rho^6+27\rho^4-27\rho^4-11) \quad (A.8)$$

Multiplication of Eqs. (A.1) and (A.4) gives

$$\mathrm{e}^{-\tau u^2}[J_0(u)Y_0(\rho u) - Y_0(u)J_0(\rho u)] = -\frac{2}{\pi}[-\log\rho + d_1u^2 + d_2u^4 + d_3u^6 + \Lambda] \quad (A.9)$$

where

$$d_1 = -\left(C_0\tau + \frac{C_1}{4}\right) \quad (A.10)$$

$$d_2 = \frac{C_0\tau^2}{2} + \frac{C_1\tau}{4} + \frac{C_2}{16} \quad (A.11)$$

and

$$d_3 = -\left(\frac{C_0\tau^3}{6} + \frac{C_1\tau^2}{8} + \frac{C_2\tau}{16} + \frac{C_3}{64}\right) \quad (A.12)$$

Substituting Eqs. (A.9)–(A.12) into the first term on the RHS of Eq. (17) yields

$$\frac{\mathrm{e}^{-\tau u^2}[J_0(u)Y_0(\rho u) - Y_0(u)J_0(\rho u)] - \frac{2}{\pi}\log\rho}{[J_0^2(u)+Y_0^2(u)]u} = \frac{-\frac{2}{\pi}[d_1u + d_2u^3 + d_3u^5 + \Lambda]}{J_0^2(u) + Y_0(u)^2} \quad (A.13)$$

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