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The Structural Birnbaum Importance of Consecutive-*k* Systems

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Abstract. This paper is concerned with three types of structural Birnbaum importance of components in a consecutive-k system. We accomplish the following three tasks:

(i) Survey existing results, including pointing out results claimed but proofs are incomplete.
(ii) Give some new results and useful lemmas.
(iii) Observe what are still provable and what are not with the help of extensive computing.

Keywords: consecutive-k systems, system reliabilty, Birnbaum importance, structural importance

1. Introduction

The Birnbaum importance $I(i)$ (Birnbaum, 1969) of a component i measures the improvement of the system reliability $R(S)$ over the improvement of the reliability p_i of that component, i.e.,

$$I(i) = \frac{\partial R(S)}{\partial p_i}.$$

Note that $R(S)$ can be computed only when the reliabilities of all components in the system are well defined. In the *component assignment problem*, n functionally exchangeable components are to be assigned to the n locations in the system to maximize the system reliability. We need to know the relative importance of the locations so that more reliable components can be assigned to the more important locations. Yet it is futile to compute the Birnbaum importance of each component since the reliability of an empty system usually remains zero even after one component is inserted. It is customary to introduce a fictitious set of component reliabilities $(p_1 = p, p_2 = p, \ldots, p_n = p)$ so that $I(i)$ can be computed;

the uniformity of p_i is to assume that $I(i)$ reflects only the merit of the locations. Since p is fictitious, it is reasonable to require $I(i) \geq I(j)$ for all $0 < p < 1$ before saying i is more *uniformly Birnbaum important* than j. In applications, $p \geq 1/2$ is almost always true since no system can tolerate the existence of a component which is more likely than not to be in the "failed" state. We say i is more *half-line Birnbaum important* than j if $I(i) \geq I(j)$ for all $1/2 \leq p < 1$. The special case $I(i) \geq I(j)$ for $p = 1/2$ has been called *structurally more important* in the literature. However, we feel that the uniform and half-line Birnbaum importance also depend essentially on system structure only. Therefore we will rename the $p = 1/2$ case by *combinatorial Birnbaum importance* ($I^c(i)$) since the index merely counts the number of cases for which the system works, and reserve the term "structural" for its generic use.

A consecutive-k system is a linear arrangement of n components such that the system fails if and only if some k consecutive components are all failed. The component assignment problem for a consecutive-2 line was first raised by Derman et al. (1982), and the conjectured optimal assignment depends only on the relative ranks of p_i but is invariant to their actual values. The conjecture was independently proved by Malon (1984), and Du and Hwang (1986). Since a consecutive-k system is symmetric with respect to the middle component(s), throughout this paper the Birnbaum importance $I(i)$ will be discussed only for $i \leq \lceil n/2 \rceil$ (for convenience, we will not state this upper bound each time). Zuo and Kuo (1990) established a linear order of the uniform Birnbaum importance for the consecutive-2 system and this order agrees completely with the optimal assignment mentioned earlier.

For $3 \leq k \leq n - 3$ Malon (1985) proved that an invariant optimal assignment does not exist. This finding has intensified the need to compare structural Birnbaum importance of the components as a certain guide for assignment. However, such comparisons are very difficult to obtain and only a few exist for uniform Birnbaum importance. Recently, Lin et al. (1999) made some breakthrough in comparing combinatorial Birnbaum importance of a consecutive-k system. In this paper, we will summarize our knowledge of the combinatorial case and the uniform case, propose the new half-line case, present some new results and conjectures, and use our extensive computer-generated data to give counterexamples to some plausible conjectures.

2. The literature

In this section we review the literature. Since there exist some confusions as to what has been proved, we will make observations where proofs are incomplete, and fix them or give counterexamples (from computer data) when we can.

First the uniform case. Kuo et al. (1990), extending the results of Tong (1985) and Malon (1985), proved

$$I(1) < I(2) < \cdots < I(k) \tag{2.1}$$

Chang et al. (1999) proved

$$I(1) < I(i) < I(k) \quad \text{for all } i \neq 1, k \tag{2.2}$$

with some special cases first proved by Zakaria et al. (1992), and Zuo (1993).

Chang et al. claimed

$$I(tk+1) < I(tk) \qquad \text{for all } t, \tag{2.3}$$
$$I(tk+1) < I(tk+2) \quad \text{for all } t, \tag{2.4}$$

and

$$I(tk-1) < I(tk) \quad \text{for all } t = (n+1)/2k, \tag{2.5}$$

but latter retracted the claim of (2.3) to $t = 2$, (2.4) to $t = 1$ and (2.5) to $t = 2$. Zuo also claimed (2.3), but his proof is unsatisfactory too (Chang et al., 1999).

For the combinatorial case, Lin et al. (1999) proved

$$I^c(k-1) < I^c(k+1), \quad \text{(proof containing a fixable error)} \tag{2.6}$$
$$I^c(k+1) < I^c(k+2) < \cdots < I^c(2k) \tag{2.7}$$
$$I^c(3k) < I^c(2k) < I^c(k) \tag{2.8}$$
$$I^c(1) < I^c(k+1) < I^c(2k+1) \tag{2.9}$$

and

$$I^c(i) < I^c(2k) \quad \text{for all } i > 2k. \tag{2.10}$$

They claimed

$$I^c(2k+1) < I^c(2k+2) < \cdots < I^c(3k-1) \tag{2.11}$$

by saying the proof is similar to the proof of (2.7). However, the proof of (2.7) depends on the validity of (2.6), whose counterpart, i.e.,

$$I^c(2k-1) < I^c(2k+1) \tag{2.12}$$

has not been proved. They also claimed

$$I^c(2k+1) < I^c(i) \quad \text{for all } i > 2k+1 \tag{2.13}$$

whose induction proof depends on the validity of (2.11).

For $k = 3$, Lin et al. claimed

$$I^c(5) < I^c(7) < I^c(10) < I^c(i) < I^c(9), \quad \text{where } i = 8 \text{ or } i \geq 11,$$

without proofs.

3. Main results

For fixed k, let $R(n)$ denote the reliability of a consecutive-k system with n components. Define

$$J_n(i) = R(i-1)R(n-i). \tag{3.1}$$

It is well known that

$$I_n(i) = \frac{J_n(i) - R(n)}{q_i}.$$

Therefore comparing $I_n(i)$ is the same as comparing $J_n(i)$. We will omit the subscript n when no confusion is possible.

We first consider the uniform case. We quote a result from Chang et al. (1999).

Lemma 3.1. $I_n(i+1) \gtreqless I_n(i)$ *if and only if* $I_{n-k}(i+1) \gtreqless I_{n-k}(i-k)$ *for* $i > k$.

Note that the proven part of (2.3) and (2.4) follows from Lemma 3.1 by using (2.2). We also prove a new case of (2.3), but only for the combinatorial importance.

Corollary 3.2. $I^c(3k) > I^c(3k+1)$.

Proof: By using (2.10). □

We expand Lemma 3.1 to allow comparison of $I(j)$ with $I(i)$. We first give a recursive equation of $R(n)$.

Set $R(n) = 0$ for $n \leq -2$, $R(-1) = 1/p$ and $R(n) = 1$ for $0 \leq n \leq k-1$. Then

Lemma 3.3.

(i) $$R(n) = R(i)R(n-i) - pq^k \sum_{s=k+1-i}^{k} R(n-i-s) \quad \textit{for } 1 \leq i < k,$$

(ii) $$R(n) = R(i)R(n-i) - p^2q^k \sum_{s=2}^{k} R(n-i-s) \sum_{r=k+2-s}^{k} q^{r+s-k-2} R(i-r)$$
$$\textit{for } k \leq i \leq \lceil n/2 \rceil.$$

Proof: Let S_n, S_i and S_{n-i} denote the original system and its two subsystems consisting of the first i components and the last $n-i$ components respectively. Then S_n fails while both S_i and S_{n-i} work if and only if there exists a set F of at least k consecutive failed components, including components i and $i+1$, satisfying the following conditions:

Case (i). $s-1$ components in F are in S_{n-i} and are followed by a working component; at least $k-s+1$ components in F are in S_i.

Case (ii). $s-1$ components in F are in S_{n-i} and are followed by a working component; $r-1$ components in F are in S_i and are preceded by a working component.

The negative terms in (i) and (ii) simply sum up all such choices, respectively. The reason to set $R(-1)=1/p$ is to take care of the case that all S_{n-i} are in F in case (i), hence there is no room and no need for the working component following S_{n-i}. □

Lemma 3.4. *Suppose* $i \geq k+1$. *Then*

(i) $$J_n(j)-J_n(i) = pq^k \sum_{s=k+1-j+i}^{k} [J_{n-j+i-s}(i)-J_{n-j+i-s}(i-s)] \quad \text{for } 0<j-i<k,$$

(ii) $$J_n(j)-J_n(i) = p^2q^k \sum_{s=2}^{k} R(j-i-s) \sum_{r=k+2-s}^{k} q^{r+s-k-2} \times [J_{n-j-r+i}(i)-J_{n-j-r+i}(i-r)] \quad \text{for } j-i \geq k.$$

Proof:

(*i*) $$\begin{aligned} J_n(j) &= R(j-1)R(n-j) \\ &= \left[R(i-1)R(j-i) - pq^k \sum_{s=k+1-j+i}^{k} R(i-1-s)\right] R(n-j) \\ &= R(i-1)R(j-i)R(n-j) - pq^k \sum_{s=k+1-j+i}^{k} R(i-1-s)R(n-j) \\ &= R(i-1)\left[R(n-i) + pq^k \sum_{s=k+1-j+i}^{k} R(n-j-s)\right] \\ &\quad - pq^k \sum_{s=k+1-j+i}^{k} R(i-1-s)R(n-j) \\ &= J_n(i) + pq^k \sum_{s=k+1-j+i}^{k} [J_{n-j-s+i}(i) - J_{n-j-s+i}(i-s)]. \end{aligned}$$

(ii) $$\begin{aligned} J_n(j) &= R(j-1)R(n-j) \\ &= \left[R(i-1)R(j-i) - p^2q^k \sum_{s=2}^{k} R(j-i-s) \right. \\ &\quad \left. \times \sum_{r=k+2-s}^{k} q^{r+s-k-2} R(i-1-r)\right] R(n-j) \\ &= R(i-1)R(j-i)R(n-j) \\ &\quad - p^2q^k \sum_{s=2}^{k} R(j-i-s) \sum_{r=k+2-s}^{k} q^{r+s-k-2} R(i-1-r)R(n-j) \end{aligned}$$

$$
\begin{aligned}
&= R(i-1)\left[R(n-i) + p^2q^k \sum_{s=2}^{k} R(j-i-s) \sum_{r=k+2-s}^{k} q^{r+s-k-2}\right.\\
&\quad \left.\times R(n-j-r)\right] - p^2q^k \sum_{s=2}^{k} R(j-i-s)\\
&\quad \times \sum_{r=k+2-s}^{k} q^{r+s-k-2} R(i-1-r)R(n-j)\\
&= J_n(i) + p^2q^k \sum_{s=2}^{k} R(j-i-s) \sum_{r=k+2-s}^{k} q^{r+s-k-2}[J_{n-j-r+i}(i)\\
&\quad -J_{n-j-r+i}(i-r)]. \qquad \square
\end{aligned}
$$

By making use of the well-known (Hwang, 1982) recursive equation:

$$R(n) = pR(n-1) + pqR(n-2) + \cdots + pq^{k-1}R(n-k) \quad \text{for } n \geq k,$$

we obtain

Lemma 3.5.

(i) $$I_n(i) = \sum_{j=0}^{k-1} pq^j I_{n-1-j}(i-1-j) \quad \textit{for } i \geq k+1.$$

(ii) $$I_n(i) = \sum_{j=0}^{k-1} pq^j I_{n-1-j}(i) \quad \textit{for } i \leq n-k.$$

Proof:

(i)
$$
\begin{aligned}
pI_{n-1}(i-1) &= p[R(i-2)R(n-i) - R(n-1)]/q,\\
pqI_{n-2}(i-2) &= pq[R(i-3)R(n-i) - R(n-2)]/q,\\
&\cdots\\
pq^{k-1}I_{n-k}(i-k) &= pq^{k-1}[R(i-k-1)R(n-i) - R(n-k)]/q.
\end{aligned}
$$

Summing up,

$$
\begin{aligned}
&\sum_{j=0}^{k-1} pq^j I_{n-1-j}(i-1-j)\\
&\quad = \left[R(n-i)\sum_{j=0}^{k-1} pq^j R(i-2-j) - \sum_{j=0}^{k-1} pq^j R(n-1-j)\right] \Big/ q\\
&\quad = [R(n-i)R(i-1) - R(n)]/q\\
&\quad = I_n(i).
\end{aligned}
$$

(ii)
$$\begin{aligned} pI_{n-1}(i) &= p[R(i-1)R(n-i-1) - R(n-1)]/q, \\ pqI_{n-2}(i) &= pq[R(i-1)R(n-i-2) - R(n-2)]/q, \\ &\cdots \\ pq^{k-1}I_{n-k}(i) &= pq^{k-1}[R(i-1)R(n-i-k) - R(n-k)]/q. \end{aligned}$$

Summing up,

$$\begin{aligned} &\sum_{j=0}^{k-1} pq^j I_{n-1-j}(i) \\ &= \left[R(i-1) \sum_{j=0}^{k-1} pq^j R(n-i-j-1) - \sum_{j=0}^{k-1} pq^j R(n-1-j) \right] \Big/ q \\ &= [R(i-1)R(n-i) - R(n)]/q \\ &= I_n(i). \end{aligned}$$

□

Corollary 3.6.

$$\begin{aligned} I_n(i+s) - I_n(i) &= \sum_{j=0}^{k-1} pq^j [I_{n-1-j}(i+s-1-j) - I_{n-1-j}(i)] \quad \textit{for } i+s \geq k+1, \\ &= \sum_{j=0}^{k-1} pq^j [I_{n-1-j}(i+s) - I_{n-1-j}(i-1-j)] \quad \textit{for } i \geq k+1, \\ &= \sum_{j=0}^{k-1} pq^j [I_{n-1-j}(i+s-1-j) - I_{n-1-j}(i-1-j)] \\ &\qquad \textit{for } i \geq k+1, \\ &= \sum_{j=0}^{k-1} pq^j [I_{n-1-j}(i+s) - I_{n-1-j}(i)]. \end{aligned}$$

Our first result on uniform importance extends the proven part of (2.5), i.e.,

$$I_{4k-1}(2k-1) < I_{4k-1}(2k).$$

Theorem 3.7. $I_{tk-1}((t-2)k-1) < I_{tk-1}((t-2)k)$ *for* $t \geq 3$.

Proof: By Lemma 3.1

$$I_{tk-1}((t-2)k-1) < I_{tk-1}((t-2)k)$$

if

$$I_{(t-1)k-1}((t-3)k-1) < I_{(t-1)k-1}((t-2)k) = I_{(t-1)k-1}(k),$$

where the last equality holds due to the symmetry between k and $(t-2)k$ when $n = (t-1)k-1$. Theorem 3.7 now follows from (2.2). □

While we will show (2.3) does not hold in general in Section 4, it might hold for $n = 2tk+1$. While $t=2$ is a special case of the proven part of (2.3), the next result extends it to $t=3$.

Theorem 3.8. $I_{2tk+1}(tk+1) < I_{2tk+1}(tk)$ *for* $t=3$.

Proof: By Lemma 3.1

$$I_{6k+1}(3k) > I_{6k+1}(3k+1)$$

if

$$I_{5k+1}(2k) > I_{5k+1}(3k+1) = I_{5k+1}(2k+1).$$

Theorem 3.8 now follows immediately from the proven part of (2.3). □

We now fix the proof of (2.6) by actually proving its truth for the more general half-line case. Let $I^h(i)$ denote the importance of component i in the half-line case.

Theorem 3.9. $I^h(k-1) < I^h(k+1)$.

Proof:

$$\begin{aligned} J^h(k+1) - J^h(k-1) &= (1-q^k)R(n-k-1) \\ &\quad - [R(n-k-1) - pq^k R(n-2k) - pq^k R(n-2k-1)] \\ &= q^k[pR(n-2k) + pR(n-2k-1) - R(n-k-1)] \\ &= q^k[\{pR(n-2k) - (1/2)R(n-k-1)\} \\ &\quad + \{pR(n-2k-1) - (1/2)R(n-k-1)\}] > 0, \end{aligned}$$

since $p \geq 1/2$ and $R(u) > R(v)$ for $u < v$. □

Remark. In Lin et al. (1999), the factor p is missing from both terms $R(n-2k)$ and $R(n-2k-1)$.

Corollary 3.10. $I^h(1) < I^h(2) < \cdots < I^h(k-1) < I^h(k+1)$.

Proof: The last inequality is from Theorem 3.9. The other inequalities are from (2.1). □

Theorem 3.11. $I^h(k+1) < I^h(i)$ *for all* $k+1 < i$.

Proof: The case $k + 1 < i \leq 2k$ follows from Lemma 3.4 (i) and Corollary 3.10. The case $2k < i$ follows from Lemma 3.4 (ii) and Corollary 3.10. □

Corollary 3.12. $I^h(2k + 2) > I^h(2k + 1)$.

Proof: By Lemma 3.1,

$$I^h(2k + 2) > I^h(2k + 1)$$

if

$$I^h(2k + 2) > I^h(k + 1).$$

Corollary 3.12 follows immediately from Theorem 3.11. □

Theorem 3.13. $I^h(i) < I^h(i + 1)$ *for* $k + 1 \leq i \leq 2k - 1$.

Proof: By Theorem 3.11, Theorem 3.13 holds for $i = k + 1$. We prove the general case by induction.

By Corollary 3.10 and Theorem 3.11,

$$I^h(i + 1) > I^h(k + 1) > I^h(i - k) \quad \text{for } k + 1 \leq i \leq 2k - 1.$$

Theorem 3.13 now follows from Lemma 3.1. □

We next prove

Theorem 3.14. $I^h(2k) > I^h(i)$ *for all* $i > 2k$.

Proof: By the proved part of (2.3), Theorem 3.14 holds for $i = 2k + 1$. We prove the general case by induction on i.

By Corollary 3.6,

$$I_n^h(i) - I_n^h(2k) = \sum_{j=0}^{k-1} pq^j \left[I_{n-1-j}^h(i - 1 - j) - I_{n-1-j}^h(2k) \right] < 0$$

since each term is nonpositive and at most one term is zero by Theorem 3.13 and by induction. □

Corollary 3.15. $I^h(3k) > I^h(3k + 1)$.

Proof: Setting $i = 3k$ in Lemma 3.1. □

We summarize our chief finding for the half-line case:

$$I^h(1) < I^h(2) < \cdots < I^h(k - 1) < I^h(k + 1) < I^h(i) < I^h(2k) < I^h(k)$$

for all $i > k + 1$ and $i \neq 2k$.

4. The k = 3 case

We assume $k = 3$ throughout this section.

Lemma 4.1.

(i) *Suppose* $I_m((t-1)k+1) < I_m(tk+2)$ *and* $I_m(tk+1) < I_m(tk)$ *for all* $m < n$. *Then* $I_n(tk+1) < I_n(i)$ *for all* $i > tk+1$.

(ii) *Suppose* $I_m(tk+1) < I_m((t-1)k)$ *and* $I_m(tk-1) < I_m(tk)$ *for all* $m < n$. *Then* $I_n(tk) > I_n(i)$ *for all* $i > tk$.

Proof: (i) By Lemma 3.1 and the assumption $I_{n-k-1}(tk+2) > I_{n-k-1}((t-1)k+1)$, we have

$$I_{n-1}(tk+2) > I_{n-1}(tk+1).$$

By Corollary 3.6,

$$\begin{aligned} I_n(tk+3) - I_n(tk+1) = {} & p[I_{n-1}(tk+2) - I_{n-1}(tk+1)] + pq^2[I_{n-3}(tk) \\ & - I_{n-3}(tk+1)] > 0 \end{aligned}$$

by the assumption $I_{n-3}(tk) > I_{n-3}(tk+1)$ and what we just proved. We prove the general case by induction on $i \geq tk+4$.

$$\begin{aligned} I_n(i) - I_n(tk+1) = {} & p[I_{n-1}(i-1) - I_{n-1}(tk+1)] + pq[I_{n-2}(i-2) \\ & - I_{n-2}(tk+1)] + pq^2[I_{n-3}(i-3) - I_{n-3}(tk+1)] > 0. \end{aligned}$$

(ii) By Lemma 3.1 and the assumption $I_{n-k-1}(tk+1) < I_{n-k-1}((t-1)k)$, we have

$$I_{n-1}(tk+1) < I_{n-1}(tk).$$

By Corollary 3.6,

$$\begin{aligned} I_n(tk+2) - I_n(tk) = {} & p[I_{n-1}(tk+1) - I_{n-1}(tk)] + pq^2[I_{n-3}(tk-1) - I_{n-3}(tk)] \\ & < 0 \end{aligned}$$

by the assumption $I_{n-3}(tk-1) < I_{n-3}(tk)$ and what we just proved. We prove the general case by induction on $i \geq tk+3$.

$$\begin{aligned} I_n(i) - I_n(tk) = {} & p[I_{n-1}(i-1) - I_{n-1}(tk)] + pq[I_{n-2}(i-2) - I_{n-2}(tk)] \\ & + pq^2[I_{n-3}(i-3) - I_{n-3}(tk)] < 0. \end{aligned}$$

□

Corollary 4.2. *Lemma* 4.1 *also holds for the half-line case and the combinatorial case.*

Theorem 4.3. $I(k+1) < I(i)$ *for all* $i > k+1$.

Proof: Follows immediately from Lemma 4.1 (i) and (2.2). □

Theorem 4.4. $I(2k+1) < I(i)$ *for all* $i > 2k+1$.

Proof: $I(k) > I(2k+1)$ by (2.2). Hence by Lemma 3.1, $I(2k+1) < I(2k)$. Also, $I(k+1) < I(2k+2)$ by Theorem 4.3. Theorem 4.4 now follows from Lemma 4.1 (i). □

Corollary 4.5. $I(3k+1) < I(3k+2)$.

Proof: Set $i = 3k+1$ in Lemma 3.1. □

Theorem 4.6. $I^h(3k+1) < I^h(i)$ *for all* $i > 3k+1$.

Proof: $I^h(2k) > I^h(3k+1)$ by Theorem 3.14. Hence by Lemma 3.1, $I^h(3k+1) < I^h(3k)$. Also, $I(2k+1) < I(3k+2)$ by Theorem 4.4. Theorem 4.6 now follows from Lemma 4.1 (i). □

Corollary 4.7. $I^h(4k+1) < I^h(4k+2)$.

Proof: Set $i = 4k+1$ in Lemma 3.1. □

Lemma 4.8.
(i) $I^c(2k-1) < I^c(2k+1)$ *for* $n \geq 14$.
(ii) $I^c(2k+2) > I^c(3k+1)$ *for* $n \geq 19$.

Proof: Lemma 4.8 is proved by induction on n. (i) is easily verified for $n = 14, 15, 16$, and so is (ii) for $n = 19, 20, 21$. The proof for general n follows immediately from the last equality of Corollary 3.6. □

We quote a result from Lin et al. (1999), which can be obtained by setting $p = 1/2$ in Lemma 3.4 (i).

Lemma 4.9. $I_n^c(i+1) - I_n^c(i) = (1/2)^{k+1}[I_{n-k-1}^c(i) - I_{n-k-1}^c(i-k)]$.

We now prove

Lemma 4.10. $I^c(3k-1) < I^c(3k)$.

Proof: By Corollary 3.12 and Lemma 4.8

$$I^c(3k-1) \equiv I^c(2k+2) > I^c(2k+1) > I^c(2k-1).$$

Lemma 4.10 follows from Lemma 4.9 immediately. □

In Theorem 3.14 we proved $I^h(2k) > I^h(i)$ for all $i > 2k$. Here we show for $k = 3$ under the combinatorial case, we have a better upper bound.

Theorem 4.11. *$I^c(3k) > I^c(i)$ for all $i > 3k$.*

Proof: Since

$$\begin{aligned} I^c(3k+1) &< I^c(2k) \quad \text{by Theorem 3.14, and} \\ I^c(3k-1) &< I^c(3k) \quad \text{by Lemma 4.10,} \end{aligned}$$

Theorem 4.11 follows from Lemma 4.1 (ii) immediately. □

The ordering for $k = 3$ under the combinatorial case given in Lin et al. (1999) is now completely proved:

$$I^c(1) < I^c(2) < I^c(4) < I^c(5) < I^c(7) < I^c(10) < I^c(i) < I^c(9) < I^c(6) < I^c(3),$$

where $i = 8$ or $i \geq 11$, except $I^c_{13}(7) < I^c_{13}(5)$.

5. Nonexistence of plausible results and conjectures

Let $f_{k,n}$ denote the nth Fibonacci number of order k. Namely,

$$f_{k,n} = \begin{cases} 0 & 1 \leq n \leq k-1 \\ 1 & n = k \\ \sum_{j=n-k}^{n-1} f_{k,j} & n \geq k+1 \end{cases}.$$

Lin et al. proved:

$$I^c_{k,n}(i) = (1/2)^{n-1}(2 f_{k,i+k} f_{k,n-i+k+1} - f_{k,n+k+1}).$$

We used this formula to compute $I^c_{k,n}(i)$ for $1 \leq i \leq n, 3 \leq k \leq 8, 1 \leq n \leq 50$ and also $n = \{60, 70, \ldots, 200\}$. We found that the reason why the results in Sections 2–4 are kind of restrictive is that their generalizations are usually wrong. We give the following counterexamples (smallest k or t selected).

(2.3) does not hold even for the combinatorial case.

Example 1. $k = 3, n = 50$, $I^c(8k) = 0.00817316129022 < I^c(8k+1) = 0.00817316129208$.

Example 2. $k = 5, n = 70$, $I^c(6k) = 0.04822129731578 < I^c(6k+1) = 0.04822129731696$.

For the uniform case, t can be as small as 4.

Example 3. $k = 7, n = 60, p = 0.3, I(4k) = 0.048158527164 < I(4k+1) = 0.048158528555$.

(2.4) does not hold for the uniform case in general.

Example 4. $k = 6, n = 100, p = 0.2, I(7k+2) = 0.000042881153 < I(7k+1) = 0.000042881154$.

(2.5) does not hold even for the combinatorial case.

Example 5. $k = 3, n = 35, I^c(6k-1) = 0.02871431695530 > I^c(6k) = 0.02871431043604$.

Example 6. $k = 4, n = 31, I^c(4k-1) = 0.09231528453529 > I^c(4k) = 0.09231519699097$.

(2.12) does not hold in general.

Example 7. $k = 3, n = 13, I^c(2k-1) = 0.18017578125000 > I^c(2k+1) = 0.17968750000000$.

However, our data show that this is the only exception for $k = 3$.
For $4 \leq k \leq 8$, our data show $I^c(2k-1) > I^c(2k+1)$ always. Thus $k = 3$ seems to be a special case.
(2.8) cannot be extended to $I^c((t+1)k) < I^c(tk)$ for all $t \geq 1$.

Example 8. $k = 3, n = 47, I^c(7k) = 0.01050828216397 < I^c(8k) = 0.01050828216627$.

In particular, not for $t \geq 5$.

Example 9. $k = 5, n = 60, I^c(5k) = 0.05725561176026 < I^c(6k) = 0.05725561183744$.

We also make the following conjectures based on our computations.

Conjecture 1. $I^c(tk+1) < I^c(i)$ *for all* $i > tk+1$.

Conjecture 2. $I^c(2k+1) < I^c(2k+2) < \cdots < I^c(3k-1)$.

Conjecture 2 cannot be extended to

$$I^c(3k+1) < I^c(3k+2) < \cdots < I^c(4k-1).$$

Example 10. $k = 6, n = 48, I^c(3k+4) = 0.06068431501809 > I^c(3k+5) = 0.06068431266068$.

Conjecture 3. $I^c(2k+1) < I^c(2k-1)$ *for all* $k \geq 4$.

We also show that results held for the special case among I^c, I^h and I may not extend to the more general case.

Theorem 3.9 does not extend to the uniform case.

Example 11. $k = 3, n = 7, p = 0.3, I(k-1) = 0.24357900 > I(k+1) = 0.21564900$.

Theorem 3.13 does not extend to the uniform case.

Example 12. $k = 4, n = 16, p = 0.3, I(2k) = 0.12337090 < I(2k-1) = 0.12376224$.

Conjecture 1 does not extend to the uniform case, in fact, not even for the weaker statement $I((t+1)k+1) > I(tk+1)$.

Example 13. $k = 6, n = 100, p = 0.2, I(6k+1) = 0.000042881164 > I(7k+1) = 0.000042881154 > I(8k+1) = 0.000042881149$.

Conjecture 2 does not extend to the uniform case.

Example 14. $k = 5, n = 29, p = 0.3, I(2k+4) = 0.06846597 < I(2k+3) = 0.06847519$.

Conjecture 3 does not extend to the half-line case.

Example 15. $k = 4, n = 17, p = 0.7, I(2k-1) = 0.06541376 < I(2k+1) = 0.06543711$.

Finally we conjecture that (2.8), (2.9), (2.10), (2.13), Theorems 3.11 and 4.11 (except for $n = 21$) hold for the uniform case, and also Theorem 3.8 hold for all t.

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