Semiclassical quantization for the spherically symmetric systems under an Aharonov-Bohm magnetic flux

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The semiclassical quantization rule is derived for a system with a spherically symmetric potential $V(r) \sim r^{\nu}(-2<\nu<\infty)$ and an Aharonov-Bohm magnetic flux. Numerical results are presented and compared with known results for models with $\nu=-1,0,2,\infty$. It is shown that the results provided by our method are in good agreement with previous results. One expects that the semiclassical quantization rule shown in this paper will provide a good approximation for all principle quantum numbers, including the large principle quantum number $n \gg 1$. The rule is even derived in the large principal quantum number limit $n \gg 1$. We also discuss the power parameter ν dependence of the energy spectra pattern in this paper.

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I. INTRODUCTION

In the past 20 years, the Aharonov-Bohm (AB) effect, a topological nonlocal physical effect at the quantum level, has been of much interest in the studies of cosmic string [1], $(2+1)$-dimensional gravity theories [2], and especially in the context of anyon [3], which has shed light on the understanding of the phenomenon of the fractional quantum Hall effect [4–7], superconductivity [7,8], repulsive Bose gases [9], and so forth. There are only a few models coupled to different potentials along with an AB magnetic flux that can be solved exactly. For the system with both an AB magnetic flux and a spherically symmetric potential of the form $V(r) =\lambda r^{\nu}(-2<\nu<\infty)$, the solvable models known to us include the cases with the parameter $\nu=-1,0,2,\infty$ [10–12,16]. Here λ is a constant parameter. Note that when $\nu=-1$, it is a system with both an AB magnetic flux and a Coulomb potential (ABC) [10–12]. This system describes the relative motion of two charged particles, with one of them carrying electric charge and magnetic flux $(-q,-\Phi/Z)$, while the other one carries (Zq,Φ). Here $Z(\neq 0)$ is a nonvanishing real number. This system is of much interest in many different areas [12].

In the past three decades, much progress has been made in the semiclassical methods toward the understanding of these systems. These kinds of semiclassical methods provide us with a powerful approximation tool in different areas in order to extract useful information from various unsolved problems including the quantization of the classical chaotic systems [13], deformed atomic nuclei, asymmetric fission nuclei [14], semiclassical quantum dots, and weak localization in mesoscopic systems [15]. In this paper, we will consider a generalized system with both an AB magnetic flux and a spherically symmetric potential of the form mentioned above. The set of the parameters (λ,ν) will be discussed in the following ranges (i) $(\lambda<0,-2<\nu<0)$ and (ii) $(\lambda>0,\nu>0)$. We will derive a semiclassical quantization rule of the approximated energy spectra for this set of parameters. The distribution tendency of the energy spectra on different values of the parameter ν will also be given. By comparing with the known results, including the models with $\nu=-1,0,2$, we find that our method agrees with these exact results. In addition, for the exactly solvable model with $\nu=\infty$, the difference between the exact and semiclassical results will be shown to be very small from a numerical computation. Therefore, we are confident in that our formulas will also provide a good approximation for the two ranges of parameters mentioned above where $\nu\neq -1,0,2,\infty$.

This paper is organized as follows. In Sec. II, we will derive the semiclassical quantization rule of the AB effect under a spherically symmetric potential. In particular, we will first derive the nonintegrable phase factor of the Green's function due to the AB effect in a spherically symmetric system. The corresponding radial Schrödinger equation will also be derived accordingly. The semiclassical wave functions will also be derived according to the semiclassical consideration of the Bohr's corresponding principle. Consequently, the quantization rule can thus be obtained by comparing with the well-known WKB phase. We will also study the distribution dependence of energy spectra in various models in Sec. III. The effect of magnetic flux will also be discussed and emphasized in this section. Finally, in Sec. IV, some conclusions will be drawn. In order to provide a self-contained information, we will show the WKB matching condition of the semiclassical wave functions in the Appendix.

II. SEMICLASSICAL QUANTIZATION RULE OF THE AB EFFECT WITH A SPHERICALLY SYMMETRIC POTENTIAL

The fixed-energy Green's function $G^0(\mathbf{r},\mathbf{r}';E)$ for a charged particle with mass m propagating from $\mathbf{r}$ to $\mathbf{r}'$ satisfies the Schrödinger equation

$$\left[E-H_0\left(\mathbf{r},\frac{\hbar}{i}\boldsymbol{\nabla}\right)\right]G^0(\mathbf{r},\mathbf{r}';E)=\delta^3(\mathbf{r}-\mathbf{r}'), \tag{1}$$

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where the system Hamiltonian is given by $H_0 = -\hbar^2\nabla^2/2m + V(r)$ as usual. In the spherically symmetric cases, the angular decomposition of the Green's function can be written as

$$G^0(\mathbf{r},\mathbf{r}';E) = \sum_{l=0}^{\infty}\sum_{k=-l}^{l} G_l^0(r,r';E) Y_{lk}(\theta,\varphi) Y_{lk}^*(\theta',\varphi'), \tag{2}$$

with Y_{lk} the well-known spherical harmonics. As a result, the left- hand side of the Eq. (1) can be brought to the following form:

$$\left\{E - \sum_{l=0}^{\infty}\sum_{k=-l}^{l}\left[-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{l(l+1)\hbar^2}{2mr^2}\right]-V(r)\right\} \times G_l^0(r,r';E) Y_{lk}(\theta,\varphi) Y_{lk}^*(\theta',\varphi'). \tag{3}$$

For a charged particle in a magnetic field, the Green's function G is related to G^0 by the following equation:

$$G(\mathbf{r},\mathbf{r}';E) = G^0(\mathbf{r},\mathbf{r}';E)\exp\left[\frac{ie}{\hbar c}\int_{\mathbf{r}'}^{\mathbf{r}}\mathbf{A}(\tilde{\mathbf{r}})\cdot d\tilde{\mathbf{r}}\right], \tag{4}$$

with a globally path-dependent nonintegrable phase factor [17,18] given above. Here we have used the vector potential $\mathbf{A}(\tilde{\mathbf{r}})$ to represent the magnetic field. For the Aharonov-Bohm magnetic flux under consideration, the vector potential can be written as

$$\mathbf{A}(\mathbf{x}) = \begin{cases} \frac{1}{2}B\rho\hat{e}_\varphi & (\rho < \epsilon) \\ \frac{1}{2}B\frac{\epsilon^2}{\rho}\hat{e}_\varphi = \frac{\Phi}{2\pi\rho}\hat{e}_\varphi & (\rho > \epsilon), \end{cases} \tag{5}$$

where the two-dimensional radial length is defined as $\rho^2 = x^2 + y^2$ as usual. Moreover, $\hat{e}_\varphi$ is the unit vector of coordinate φ and ϵ is the radius of region where magnetic field exists. Hence the total magnetic flux is given by $\Phi = \pi\epsilon^2 B$. Note that the associated magnetic field lines are confined inside a tube, with radius ϵ, along the z axis. Along the region without magnetic field, the path-dependent nonintegrable phase factor is given by

$$\exp\left[-i\mu_0\int_P^{\lambda} d\lambda'\,\dot{\varphi}(\lambda')\right], \tag{6}$$

where we have used the subscript P to represent the path-dependent nature of phase factor and we have denoted $\dot{\varphi}(\lambda') = d\varphi/d\lambda'$. Also, $\mu_0 = -2eg/\hbar c$ is a dimensionless number defined by $\Phi = 4\pi g$. The minus sign is a matter of convention. According to the discussion in Ref. [18], only phase factors with closed-loop contour are considered where the description of electromagnetic phenomenon are complete. Hence, we have

$$n = \frac{1}{2\pi}\int_P^{\lambda} d\lambda'\,\dot{\varphi}(\lambda'), \tag{7}$$

with integer values n corresponding the winding number. The magnetic interaction is therefore purely topological. Therefore the nonintegrable phase factor becomes

$$e^{-i\mu_0(2n\pi)}. \tag{8}$$

With the help of the equality between the associated Legendre polynomial $P_\nu^\mu(z)$ and the Jacobi function $P_n^{(\alpha,\beta)}(z)$ [19,20], we find that

$$P_l^k(\cos\theta) = (-1)^k\frac{\Gamma(l+k+1)}{\Gamma(l+1)}\left(\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)^k P_{l-k}^{(k,k)}(\cos\theta). \tag{9}$$

Therefore the angular part of the Green's function in the expression (3) can be turned into the following form:

$$\begin{aligned}\sum_{k=-l}^{l} Y_{lk}(\theta,\varphi)Y_{lk}^*(\theta',\varphi') &= \sum_{k=-l}^{l}\frac{2l+1}{4\pi}\frac{\Gamma(l-k+1)}{\Gamma(l+k+1)} \\ &\quad\times P_l^k(\cos\theta)P_l^k(\cos\theta')e^{ik(\varphi-\varphi')} \\ &= \sum_{k=-l}^{l}\left[\frac{2l+1}{4\pi}\frac{\Gamma(l-k+1)\Gamma(l+k+1)}{\Gamma^2(l+1)}\right] \\ &\quad\times\left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^k \\ &\quad\times P_{l-k}^{(k,k)}(\cos\theta)P_{l-k}^{(k,k)}(\cos\theta')e^{ik}(\varphi-\varphi'). \end{aligned}\tag{10}$$

In order to include the nonintegrable phase factor due to the AB effect, we will change the index l into q related by the definition $l-k=q$. As a result one can rewrite the Eq. (3) as

$$\begin{aligned}&\left\{E - \sum_{q=0}^{\infty}\sum_{k=-\infty}^{\infty}\left[-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)\right.\right. \\ &\left.\left.+\frac{(q+k)(q+k+1)\hbar^2}{2mr^2}\right]-V(r)\right\}G_{q+k}^0(r,r';E) \\ &\times\left[\frac{2(q+k)+1}{4\pi}\frac{\Gamma(q+1)\Gamma(q+2k+1)}{\Gamma^2(q+k+1)}\right] \\ &\times\left(\cos\frac{\theta}{2}\cos\frac{\theta^{0\prime}}{2}\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^k \\ &\times P_q^{(k,k)}(\cos\theta)P_q^{(k,k)}(\cos\theta')e^{ik(\varphi-\varphi')}. \end{aligned}\tag{11}$$

In addition, the nonintegrable phase in Eq. (8) can now be included with the help of the Poisson's summation formula (p. 124, [21])

$$\sum_{k=-\infty}^{\infty} f(k) = \int_{-\infty}^{\infty} dy \sum_{n=-\infty}^{\infty} e^{2\pi nyi} f(y). \tag{12}$$

Therefore, the expression (11) can be written as

$$\left\{E-\sum_{q=0}^{\infty}\int dz\sum_{k=-\infty}^{\infty}\left[-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{(q+z)(q+z+1)\hbar^2}{2mr^2}\right]-V(r)\right\}$$
$$\times G_{q+z}(r,r';E)\left[\frac{2(q+z)+1}{4\pi}\frac{\Gamma(q+1)\Gamma(q+2z+1)}{\Gamma^2(q+z+1)}\right]\left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^z P_q^{(z,z)}$$
$$\times(\cos\theta)P_q^{(z,z)}(\cos\theta')\exp[i(z-\mu_0)(\varphi+2k\pi-\varphi')], \tag{13}$$

where the superscript 0 in G^0_{q+k} has been suppressed to reflect the inclusion of the AB effect. The summation over all indices k forces $z=\mu_0$ modulo an arbitrary integer number. Therefore, one has

$$\left\{E-\sum_{q=0}^{\infty}\sum_{k=-\infty}^{\infty}\left[-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{(q+|k+\mu_0|)(q+|k+\mu_0|+1)\hbar^2}{2mr^2}\right]-V(r)\right\}$$
$$\times G_{q+|k+\mu_0|}(r,r';E)\left\{\frac{[2(q+|k+\mu_0|)+1]}{4\pi}\frac{\Gamma(q+1)\Gamma(2|k+\mu_0|+q+1)}{\Gamma^2(|k+\mu_0|+q+1)}\right\}e^{ik(\varphi-\varphi')}$$
$$\times(\cos\theta/2\cos\theta'/2\sin\theta/2\sin\theta'/2)^{|k+\mu_0|}P_q^{(|k+\mu_0|,|k+\mu_0|)}(\cos\theta)P_q^{(|k+\mu_0|,|k+\mu_0|)}(\cos\theta'). \tag{14}$$

Note that the influence of the AB effect to the radial Green's function is to replace the integer quantum number l with a fractional quantum number $q+|k+\mu_0|$. Analogously the same procedure can be applied to the delta function $\delta^3(\mathbf{r}-\mathbf{r}')$ in the rhs of the Eq. (1) with the help of the following solid angle representation of the δ function:

$$\delta(\Omega-\Omega')=\sum_{l=0}^{\infty}\sum_{k=-l}^{l}Y_{lk}(\theta,\varphi)Y^*_{lk}(\theta',\varphi'). \tag{15}$$

Therefore, for the set of the fixed quantum numbers (q,k) one can show that the radial Green's function satisfies

$$\left\{E-\left[-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{(q+|k+\mu_0|)(q+|k+\mu_0|+1)\hbar^2}{2mr^2}\right]-V(r)\right\}$$
$$\times G_{q+|k+\mu_0|}(r,r';E)=\delta(r-r'). \tag{16}$$

As a result, the corresponding radial wave equation reads

$$\frac{\hbar^2}{2m}\frac{d^2}{dr^2}u_\gamma(r)+\left[E-\left(V(r)+\frac{\hbar^2}{2m}\frac{\gamma(\gamma+1)}{r^2}\right)\right]u_\gamma(r)=0, \tag{17}$$

where we have set $\gamma=q+|k+\mu_0|$, and $u_\gamma(r)\equiv rR_{\tilde{n}\gamma}(r)$. Obviously, $R_{\tilde{n}\gamma}$ satisfies the spherical Bessel equation

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\left(\kappa^2-U(r)-\frac{\gamma(\gamma+1)}{r^2}\right)\right]R_{\tilde{n}\gamma}(r)=0 \tag{18}$$

with the definitions $\kappa=\sqrt{2mE/\hbar^2}$ and the reduced potential $U(r)=2mV(r)/\hbar^2$. For simplicity, we have written $R_{\tilde{n}\gamma}(r)$ instead of $R_{\tilde{n},q,k}(r)$ in which each set $(\tilde{n},q,k)$ denotes a quantum state. Hence the AB effect reflects itself by the coupling to the angular momentum in radial Green's function, which turns the integer quantum number into a fractional one.

To find the semiclassical quantization rule, let us first consider the asymptotic form of the bound-state wave functions of a charged particle moving in a spherically symmetric potential of the form $V(r)=\lambda r^\nu(\lambda<0,-2<\nu<0)$ under an AB magnetic flux for the energy limit $E\to 0$. Due to the Bohr corresponding principle, this stands for the semiclassical approximation since there are infinitely densed energy levels near $E\to 0^-$. According to Eq. (17), the asymptotic wave equation reads, in the $E\to 0^-$ limit,

$$\frac{\hbar^2}{2m}\frac{d^2}{dr^2}u_\gamma(r)-\left(\lambda r^\nu+\frac{\hbar^2}{2m}\frac{\gamma(\gamma+1)}{r^2}\right)u_\gamma(r)=0. \tag{19}$$

We can also perform the following transformations:

$$\rho=r\left(\frac{2m|\lambda|}{\hbar^2}\right)^{1/(\nu+2)},\qquad u(r)=W(\rho). \tag{20}$$

Consequently, Eq. (19) yields

$$\frac{d^2W}{d\rho^2}+\left[\rho^\nu-\frac{\gamma(\gamma+1)}{\rho^2}\right]W=0, \tag{21}$$

which can be further reduced with the help of the following change of variables:

$$z=\frac{2}{\nu+2}\rho^{(\nu+2)/2},\quad W(\rho)=z^{1/(\nu+2)}v(z). \tag{22}$$

As a result, Eq. (21) becomes

$$\frac{d^2v}{dz^2}+\frac{1}{z}\frac{dv}{dz}+\left[1-\left(\frac{2\gamma+1}{\nu+2}\right)^2\frac{1}{z^2}\right]v=0. \tag{23}$$

This is exactly the Bessel's equation of integral order $\nu_1 \equiv (2\gamma+1)/(\nu+2)$. The boundary condition (BC) of the function $u(r)$ in the Eq. (17) is simply $u(0)=0$. Therefore, the corresponding BC of $v(z)$ in the Eq. (23) is $v(0)=0$. The Bessel function of the first kind is known to be the solution of the Bessel's equation. Therefore, by imposing the BC appropriately, one can show that

$$v(z)=J_{\nu_1}(z) \tag{24}$$

is the solution of the Eq. (23) with the prescribed boundary condition. Therefore, the solution of the radial wave equation near $E\to 0$ becomes

$$u(r)=W(\rho)=z^{1/(\nu+2)}J_{\nu_1}(z). \tag{25}$$

From the asymptotic behavior of the Bessel function near $r\to 0$, or equivalently $\rho\to 0$ and $z\to 0$, one can show that

$$u(r)\sim z^{1/(\nu+2)}z^{(2\gamma+1)/(\nu+2)}\sim z^{(2\gamma+2)/(\nu+2)}\sim r^{\gamma+1}. \tag{26}$$

On the other hand, from the asymptotic behavior of the Bessel function approaching $r\to\infty$,

$$J_\alpha(z)\to\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right), \tag{27}$$

one can show that

$$\begin{aligned}u(r)&\sim z^{1/(\nu+2)}\sqrt{\frac{2}{\pi z}}\cos\left(z-\nu_1\frac{\pi}{2}-\frac{\pi}{4}\right)\\&\sim\rho^{-\nu/4}\cos\left(\frac{2}{\nu+2}\rho^{(\nu+2)/2}-\nu_1\frac{\pi}{2}-\frac{\pi}{4}\right).\end{aligned} \tag{28}$$

Note that one can also compute the following integral, near the limit $E\to 0$, and show that the following identities hold:

$$\begin{aligned}\int_0^r\sqrt{\frac{2m}{\hbar^2}[E-V(r)]}dr&=\left(\frac{2m|\lambda|}{\hbar^2}\right)^{1/2}\int_0^r r^{\nu/2}dr\\&=\left(\frac{2m|\lambda|}{\hbar^2}\right)^{1/2}\frac{2}{\nu+2}r^{(\nu+2)/2}\\&=\frac{2}{\nu+2}\rho^{(\nu+2)/2}.\end{aligned} \tag{29}$$

It follows that, in the limit $E\to 0$ and $r\to\infty$,

$$\begin{aligned}u(r)&\sim r^{-\nu/4}\cos\left(\int_0^r\sqrt{\frac{2m}{\hbar^2}[E-V(r)]}-\nu_1\frac{\pi}{2}-\frac{\pi}{4}\right)\\&\sim r^{-\nu/4}\sin\left(\int_0^r\sqrt{\frac{2m}{\hbar^2}[E-V(r)]}-\nu_1\frac{\pi}{2}+\frac{\pi}{4}\right),\end{aligned} \tag{30}$$

where $V(r)=\lambda r^\nu(\lambda<0,-2<\nu<0)$. If we take the integration upper bound r as the classical turning point r_c, where $V(r_c)=E$, the phase of $u(r)$ can be shown to be the WKB phase (see the Appendix for details)

$$u(r_c)\propto\sin\left[\left(n+\frac{3}{4}\right)\pi\right]. \tag{31}$$

Consequently, from comparing the Eqs. (30) and (31), one can extract the following quantization condition

$$\int_0^{r_c}\sqrt{2m[E-V(r)]}=\left[n+\frac{2\gamma+\nu+3}{2(\nu+2)}\right]\pi\hbar,\qquad \text{for}\quad n=0,1,2,3\ldots. \tag{32}$$

Here n is the radial quantum number. Although Eq. (30) is obtained in the limit $E\to 0$, or equivalently in the large quantum number where $n\gg 1$, above result can still be extended to all possible values of n. In fact, the integral in Eq. (32) can be written in an analytic form. Indeed, with the help of the following change of variables:

$$\frac{\lambda}{E}r^\nu=\csc^2\xi, \tag{33}$$

one can rewrite the above integral as

$$\int_0^{r_c}\sqrt{2m(E-V(r))}=-\frac{2}{\nu}\left(\frac{E}{\lambda}\right)^{1/\nu}\sqrt{2m|E|}\int_0^{\pi/2}\times\cos^2\xi(\sin\xi)^{-2/\nu-2}d\xi. \tag{34}$$

In addition, with the help of the following formula (see, for example, Ref. [23], p. 8):

$$\int_0^{\pi/2}\cos^{2q-1}z\sin^{2p-1}z\,dz=\frac{\Gamma(p)\Gamma(q)}{2\Gamma(p+q)}, \tag{35}$$

one has

$$\int_0^{r_c}\sqrt{2m[E-V(r)]}=-\frac{2}{\nu}\left(\frac{E}{\lambda}\right)^{1/\nu}\sqrt{2m|E|}\times\frac{\sqrt{\pi}}{4}\frac{\Gamma\left(-\frac{1}{\nu}-\frac{1}{2}\right)}{\Gamma\left(1-\frac{1}{\nu}\right)}. \tag{36}$$

Inserting the result of the Eq. (36) into Eq. (32), one has

$$E=-|\lambda|^{2/(\nu+2)}\left(\frac{\hbar^2}{2m}\right)^{\nu/(\nu+2)}\times\left[2|\nu|\sqrt{\pi}\left(n+\frac{2(q+|k+\mu_0|)+\nu+3}{2\nu+4}\right)\times\frac{\Gamma\left(1-\frac{1}{\nu}\right)}{\Gamma\left(-\frac{1}{\nu}-\frac{1}{2}\right)}\right]^{2\nu/(\nu+2)}, \tag{37}$$

where the ranges of the parameters are $\lambda,E<0, -2<\nu<0, n,q=0,1,2,\ldots$, and $-\infty<k<\infty$. For example, with the potential of the form $V(r)=-e^2/r$, one has

$$E_{n,q,k}=-mc^2\frac{\alpha^2}{2[n+q+|k+\mu_0|+1]^2}. \tag{38}$$

Here $\alpha=e^2/\hbar c$ denotes the fine structure constant. This agrees with the exact result given in Ref. [11]. We see that the AB effect has changed the splitting of energy levels although the electron moves in the absence of the magnetic field. In addition, when the flux is quantized, namely, $4\pi g=(2\pi\hbar c/e)\times$ integer, $|k+\mu_0|$ is an integer and hence the spectrum is the same as the energy spectrum of the pure hydrogen atom.

To obtain the semiclassical quantization rule for all positive powers $\nu>0$ of the potential $V(r)=\lambda r^{\nu}$, one can perform the following change of variable:

$$\rho=r^{\alpha},\ u(r)=\rho^{\beta}v(\rho) \tag{39}$$

and can show that Eq. (17) becomes

$$\frac{d^2u}{dr^2}=\alpha^2\rho^{2+\beta-2/\alpha}\frac{d^2v(\rho)}{d\rho^2}+\alpha^2\left(2\beta+1-\frac{1}{\alpha}\right)\times\rho^{1+\beta-2/\alpha}\frac{dv(\rho)}{d\rho}+\alpha^2\beta\left(\beta-\frac{1}{\alpha}\right)\rho^{\beta-2/\alpha}v(\rho). \tag{40}$$

Note that the different ranges $\nu>0$ and $\nu<0$ can be properly adjusted when the parameters α and β are chosen appropriately [22]. In addition, if we set

$$\alpha=-\frac{\nu}{\nu'},\qquad \beta=-\frac{1}{2}\left(1+\frac{\nu'}{\nu}\right), \tag{41}$$

the term $dv/d\rho$ in Eq. (40) disappears. Inserting this back into Eq. (39) and then Eq. (17), one has

$$\frac{\hbar^2}{2m}\rho^{2+\nu'+2\nu'/\nu}\frac{d^2v}{d\rho^2}+\left[-\lambda\left(\frac{\nu'}{\nu}\right)^2+E\left(\frac{\nu'}{\nu}\right)^2\rho^{\nu'}\right]v-\frac{\hbar^2}{2m}\rho^{-2}\left[\gamma(\gamma+1)\left(\frac{\nu'}{\nu}\right)^2+\frac{1}{4}\left(\frac{\nu'}{\nu}\right)^2-\frac{1}{4}\right]\times\rho^{2+\nu'+2\nu'/\nu}v=0. \tag{42}$$

If we choose $\nu'=-2\nu/(2+\nu)$, the above equation reduces to

$$\frac{\hbar^2}{2m}\frac{d^2v}{d\rho^2}+\left[E'-\lambda'\rho^{\nu'}-\gamma'(\gamma'+1)\frac{\hbar^2}{2m\rho^2}\right]v=0, \tag{43}$$

with the following relations linking different parameters:

$$\nu'=-\frac{2\nu}{(2+\nu)},$$

$$E'=-\lambda\left(\frac{\nu'}{\nu}\right)^2,$$

$$\lambda'=-E\left(\frac{\nu'}{\nu}\right)^2,$$

$$\gamma'=-\left(\gamma+\frac{1}{2}\right)\frac{\nu'}{\nu}-\frac{1}{2}=\frac{2\gamma+1}{\nu+2}-\frac{1}{2}. \tag{44}$$

Note that the structure of the Eqs. (17) and (43) is similar except the signs of the parameters. Accordingly, the eigen solutions for $\lambda,\nu,E>0$ can be found from $\lambda',\nu',E'<0$. Inserting the relations (44) into Eq. (37), one thus finds that

$$E=\lambda^{2/(\nu+2)}\left(\frac{\hbar^2}{2m}\right)^{\nu/(\nu+2)}\left[2\nu\sqrt{\pi}\left(n+\frac{(q+|k+\mu_0|)}{2}+\frac{3}{4}\right)\times\frac{\Gamma\left(\frac{1}{\nu}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{\nu}\right)}\right]^{2\nu/(\nu+2)}, \tag{45}$$

with the ranges of the parameters $\lambda,\nu,E>0, n,q=0,1,2,\ldots$, and $-\infty<k<\infty$. As a realization, the three-dimensional simple harmonic oscillator moving in the presence of the AB magnetic flux can be described by the model with the parameters $\nu=2$ and $\lambda=m\omega^2/2$. Hence we can calculate the energy eigenvalue from the Eq. (45) that gives us the following result:

$$E_{n,q,k}=[2n+(q+|k+\mu_0|)+\tfrac{3}{2}]\hbar\omega. \tag{46}$$

Another example is given by the model with an infinitely deep potential

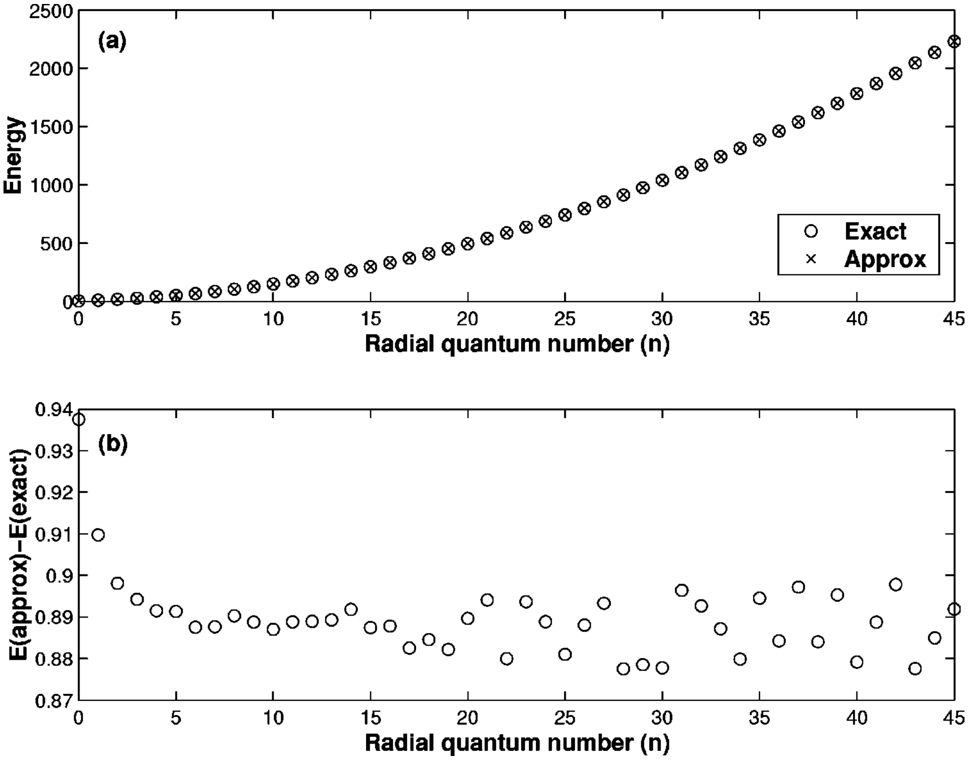

FIG. 1. Comparison of the exact and approximate energy eigenvalue as a function of the radial quantum number n. (a) The exact and approximate energy eigenvalues are shown in (a). Their difference is shown in (b). Here we have set $q+|k+\mu_0|=2.5$. See Eqs. (48) and (49) for details.

$$V(r)=\lambda r^{\nu}=\begin{cases}\infty & \text{for} \quad r\geqslant a\\ 0 & \text{for} \quad r<a.\end{cases} \tag{47}$$

Similarly, Eq. (45) implies the following energy spectra:

$$E_{n,q,k}=\frac{\hbar^2\pi^2}{2ma^2}\left[n+\frac{q+|k+\mu_0|}{2}+1\right]^2. \tag{48}$$

Here we have replaced $(n+\gamma/2+3/4)$ with $(n+\gamma/2+1)$ according to the matching condition of the WKB approximation given in the Appendix. The analytic energy spectra of this system are then given by the zeros of the modified Bessel function in Eq.(3.28) of the Ref. [16]

$$I_{q+|k+\mu_0|+1/2}\left(\frac{\sqrt{-2mE}}{\hbar}a\right)=0. \tag{49}$$

The numerical analysis shown in Fig. 1(a) (for $q+|k+\mu_0|=2.5$) indicates that the result (48) is in good agreement with the exact result (49). In addition, Fig. 1(b) exhibits the difference between the exact and approximate results.

III. THE ν DEPENDENCE OF THE DISTRIBUTION OF THE ENERGY SPECTRA

Note that Eq. (45) indicates that

$$E_{n,q,k}\propto\left(n+\frac{q+|k+\mu_0|}{2}+\frac{3}{4}\right)^{2\nu/(\nu+2)}. \tag{50}$$

For example, for the model with an infinitely deep potential (i.e., $\nu\to\infty$), one has

$$E_{n,q,k}\propto\left(n+\frac{q+|k+\mu_0|}{2}+1\right)^2. \tag{51}$$

On the other hand, from Eq. (37) one has (when $-2<\nu<0$)

$$E_{n,q,k}\propto-\left[\left(n+\frac{2(q+|k+\mu_0|)+\nu+3}{2\nu+4}\right)\right]^{2\nu/(\nu+2)}. \tag{52}$$

In addition, we can calculate their derivatives with respect to n and find that

$$\frac{\partial E_{n,q,k}}{\partial n}>0 \tag{53}$$

for all considered models. Thus one expects that the energy levels $E_{n,q,k}$ will monotonically increase as n increases monotonically. The q and $|k+\mu_0|$ dependence of the energy eigenvalue $E_{n,q,k}$ can be found by the Hellmann-Feynman formula (e.g., [24])

$$\frac{\partial E_{n,q,k}}{\partial q}=\left\langle \Psi_{n,q,k}\left|\frac{\partial H}{\partial q}\right|\Psi_{n,q,k}\right\rangle, \tag{54}$$

where the Hamiltonian is given by

$$H=-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\left(\lambda r^{\nu}+\frac{\hbar^2}{2m}\times\frac{(q+|k+\mu_0|)(q+|k+\mu_0|+1)}{r^2}\right). \tag{55}$$

Thus, we can derive the following results:

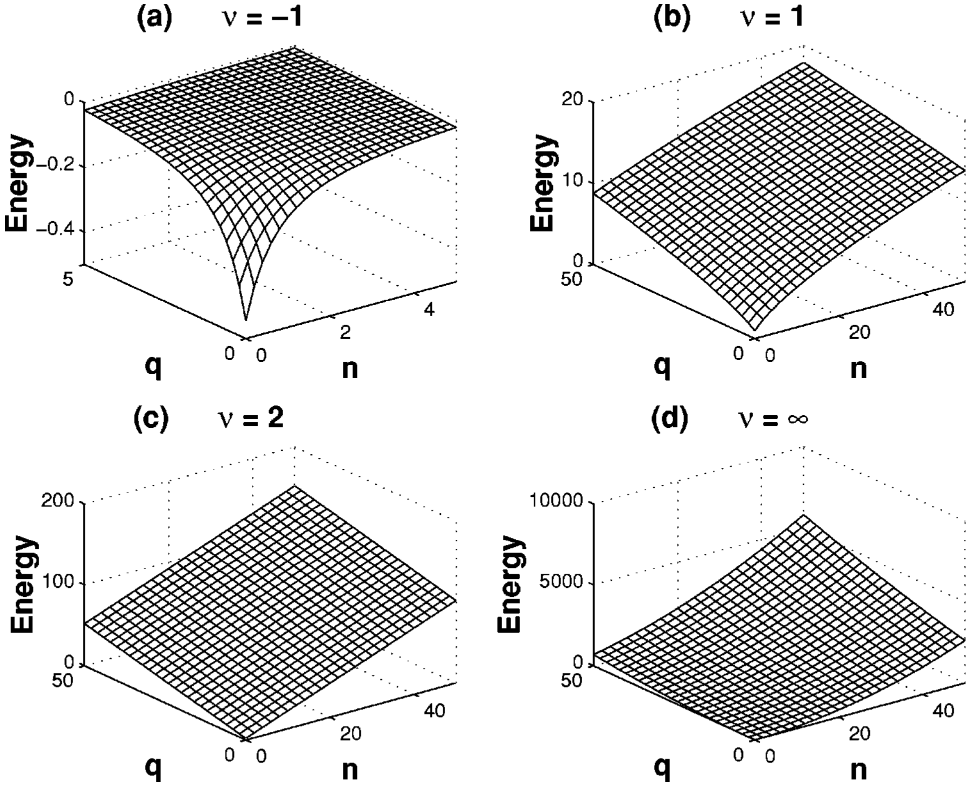

FIG. 2. Energy as a function of q and n for four different ν's is shown. Here we have chosen $|k+\mu_0|=0.5$. The unit of the energy eigenvalue is set as $mc^2\alpha^2/2$, $(9\pi^2\lambda^2\hbar^2/8m)^{1/3}$, $\hbar\omega$, and $\hbar^2\pi^2/2ma^2$ for (a), (b), (c), and (d), respectively.

$$\frac{\partial E_{n,q,k}}{\partial q}=\left\langle \Psi_{n,q,k}\left|\frac{[2(q+|k+\mu_0|)+1]\hbar^2}{2mr^2}\right|\Psi_{n,q,k}\right\rangle>0, \tag{56}$$

$$\frac{\partial E_{n,q,k}}{\partial |k+\mu_0|}=\left\langle \Psi_{n,q,k}\left|\frac{[2(q+|k+\mu_0|)+1]\hbar^2}{2mr^2}\right|\Psi_{n,q,k}\right\rangle>0. \tag{57}$$

This means that the energy spectra $E_{n,q,k}$ will monotonically increase as any one of the quantum numbers in the set (n,q,k) increases monotonically. Therefore the ground state will be given by $n=q=k=0$. The details can be obtained by analyzing the tendency of $E_{n,q,k}$ with respect to the change of the parameter ν.

A. Distribution tendency of the energy spectra for $\nu=-1$

The energy spectra for a charged particle moving in the Coulomb potential and an AB flux is given by Eq. (38). Its first- and second-order derivatives with respect to the parameters $(n,q,|k+\mu_0|)$ are

$$\frac{\partial E_{n,q,k}}{\partial n}=mc^2\alpha^2\frac{1}{(n+q+|k+\mu_0|+1)^3}>0,$$

$$\frac{\partial^2 E_{n,q,k}}{\partial n^2}=mc^2\alpha^2\frac{-3}{(n+q+|k+\mu_0|+1)^4}<0, \tag{58}$$

$$\frac{\partial E_{n,q,k}}{\partial q}=mc^2\alpha^2\frac{1}{(n+q+|k+\mu_0|+1)^3}>0,$$

$$\frac{\partial^2 E_{n,q,k}}{\partial q^2}=mc^2\alpha^2\frac{-3}{(n+q+|k+\mu_0|+1)^4}<0, \tag{59}$$

$$\frac{\partial E_{n,q,k}}{\partial |k+\mu_0|}=mc^2\alpha^2\frac{1}{(n+q+|k+\mu_0|+1)^3}>0,$$

$$\frac{\partial^2 E_{n,q,k}}{\partial |k+\mu_0|^2}=mc^2\alpha^2\frac{-3}{(n+q+|k+\mu_0|+1)^4}<0. \tag{60}$$

Consequently, $E_{n,q,k}$ tends to increase and saturate gradually as anyone of the parameters in the set $(n,q,|k+\mu_0|)$ increases. It implies the bending curve as shown in Fig. 2(a). The unit of the energy eigenvalue in Fig. 2(a) is chosen as $mc^2\alpha^2/2$.

B. Distribution tendency of the energy spectra for $\nu=1$

The energy levels for the model with $\nu=1$ are given by Eq. (45),

$$E_{n,q,k}=\left(\frac{\lambda^2\hbar^2}{2m}\right)^{1/3}\left[\frac{3\pi}{2}\left(n+\frac{(q+|k+\mu_0|)}{2}+\frac{3}{4}\right)\right]^{2/3}. \tag{61}$$

Their derivatives with respect to the parameters $(n,q,|k+\mu_0|)$ yield

$$\frac{\partial E_{n,q,k}}{\partial n}=\left(\frac{\lambda^2\hbar^2}{2m}\right)^{1/3}\pi\left[\frac{3\pi}{2}\left(n+\frac{(q+|k+\mu_0|)}{2}+\frac{3}{4}\right)\right]^{-1/3}>0,$$

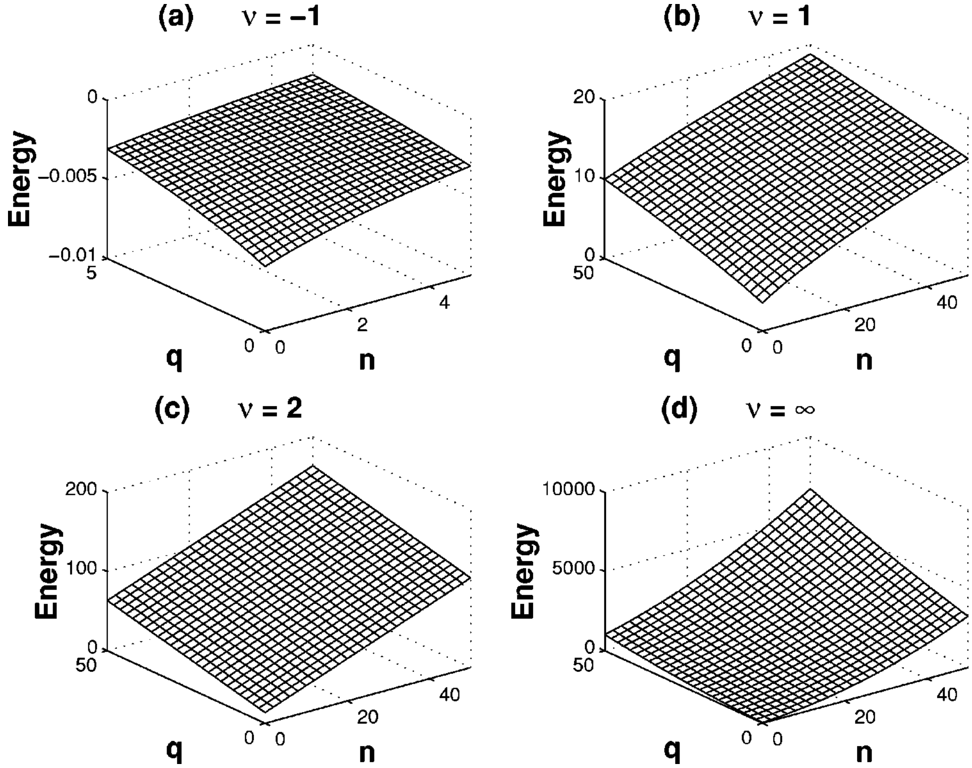

FIG. 3. Energy as a function of q and n for four different ν's. Here we choose $|k+\mu_0|=12$.

$$\frac{\partial^2 E_{n,q,k}}{\partial n^2}=-\left(\frac{\lambda^2\hbar^2}{2m}\right)^{1/3}\left(\frac{\pi^2}{2}\right)\times\left[\frac{3\pi}{2}\left(n+\frac{(q+|k+\mu_0|)}{2}+\frac{3}{4}\right)\right]^{-4/3}<0, \tag{62}$$

$$\frac{\partial E_{n,q,k}}{\partial q}=\left(\frac{\lambda^2\hbar^2}{2m}\right)^{1/3}\frac{\pi}{2}\left[\frac{3\pi}{2}\left(n+\frac{(q+|k+\mu_0|)}{2}+\frac{3}{4}\right)\right]^{-1/3}>0,$$

$$\frac{\partial^2 E_{n,q,k}}{\partial q^2}=-\left(\frac{\lambda^2\hbar^2}{2m}\right)^{1/3}\left(\frac{\pi^2}{8}\right)\times\left[\frac{3\pi}{2}\left(n+\frac{(q+|k+\mu_0|)}{2}+\frac{3}{4}\right)\right]^{-4/3}<0, \tag{63}$$

and

$$\frac{\partial E_{n,q,k}}{\partial|k+\mu_0|}=\left(\frac{\lambda^2\hbar^2}{2m}\right)^{1/3}\left(\frac{\pi}{2}\right)\times\left[\frac{3\pi}{2}\left(n+\frac{(q+|k+\mu_0|)}{2}+\frac{3}{4}\right)\right]^{-1/3}>0,$$

$$\frac{\partial^2 E_{n,q,k}}{\partial|k+\mu_0|^2}=-\left(\frac{\lambda^2\hbar^2}{2m}\right)^{1/3}\left(\frac{\pi^2}{8}\right)\times\left[\frac{3\pi}{2}\left(n+\frac{(q+|k+\mu_0|)}{2}+\frac{3}{4}\right)\right]^{-4/3}<0. \tag{64}$$

It is obvious that $E_{n,q,k}$ will monotonically increase when the value of any parameter of the set $(n,q,|k+\mu_0|)$ increases as shown in Fig. 2(b). Note that the slope is much more smooth than the model with $\nu=-1$. The unit of energy in Fig.2(b) is chosen as $(9\pi^2\lambda^2\hbar^2/8m)^{1/3}$.

C. Distribution tendency of the energy spectra for $\nu=2$

The energy spectra for a charged particle moving in the three-dimensional harmonic potential and an AB flux is given by Eq. (46). Its first- and second-order derivatives with respect to the set of parameters $(n,q,|k+\mu_0|)$ read

$$\frac{\partial E_{n,q,k}}{\partial n}=2\hbar\omega(\mathrm{const}),\quad \frac{\partial^2 E_{n,q,k}}{\partial n^2}=0, \tag{65}$$

$$\frac{\partial E_{n,q,k}}{\partial q}=\hbar\omega(\mathrm{const}),\quad \frac{\partial^2 E_{n,q,k}}{\partial q^2}=0, \tag{66}$$

and

$$\frac{\partial E_{n,q,k}}{\partial|k+\mu_0|}=\hbar\omega\ \ (\mathrm{const}),\quad \frac{\partial^2 E_{n,q,k}}{\partial|k+\mu_0|^2}=0. \tag{67}$$

This means that $E_{n,q,k}$ will linearly increase as any one of the parameters in the set $(n,q,|k+\mu_0|)$ increases. The details is shown in Fig. 2(c) with the unit of energy given by $\hbar\omega$.

D. Distribution tendency of the energy spectra for $\nu=\infty$

According to Eq. (48), we obtain the first- and second-order derivatives with respect to $E_{n,q,k}$

$$\frac{\partial E_{n,q,k}}{\partial n}=\frac{\pi^2\hbar^2}{ma^2}\left[n+\frac{(q+|k+\mu_0|)}{2}+1\right]>0,$$

$$\frac{\partial^2 E_{n,q,k}}{\partial n^2}=\frac{\pi^2\hbar^2}{ma^2}>0, \tag{68}$$

$$\frac{\partial E_{n,q,k}}{\partial q}=\frac{\pi^2\hbar^2}{2ma^2}\left[n+\frac{(q+|k+\mu_0|)}{2}+1\right]>0,$$

$$\frac{\partial^2 E_{n,q,k}}{\partial q^2}=\frac{\pi^2\hbar^2}{4ma^2}>0, \tag{69}$$

and

$$\frac{\partial E_{n,q,k}}{\partial|k+\mu_0|}=\frac{\pi^2\hbar^2}{2ma^2}\left[n+\frac{(q+|k+\mu_0|)}{2}+1\right]>0,$$

$$\frac{\partial^2 E_{n,q,k}}{\partial|k+\mu_0|^2}=\frac{\pi^2\hbar^2}{4ma^2}>0 \tag{70}$$

for the model with $\nu\to\infty$. Note that $E_{n,q,k}$ will increase monotonically when any one of the parameters in the set $(n,q,|k+\mu_0|)$ increases. The rate of increase is, however, faster than the model $\nu=2$ since the curve climbs up as shown in Fig. 2(d) with the unit chosen as $\hbar^2\pi^2/2ma^2$.

In summary, all these results imply the following rules for a charged particle moving in the spherically symmetric potential $V(r)=\lambda r^\nu(-2<\nu<\infty)$ and an AB magnetic flux.

(a) The energy spectra of the bound states depend on the quantum number (n,q,k) and increase monotonically as any one of the quantum numbers increases.

(b) When $\nu=2$, the energy spectra $E_{n,q,k}$ depend linearly on any parameter in the set (n,q,k); when $\nu>2$, the energy curve bends up as any one of the quantum numbers (n,q,k) increases. On the other hand, when $\nu<2$, the curve bends down as any one of the quantum numbers increases.

(c) When $\nu=2$, we have $\partial E/\partial n:\partial E/\partial q=2:1$, $\partial E/\partial n:\partial E/\partial|k+\mu_0|=2:1$, and $\partial E/\partial q:\partial E/\partial|k+\mu_0|=1:1$, which are related to the closeness of the classical orbits and whether the model is exactly solvable or not. For the case with positive power of ν,

$$E\sim\left[\left(n+\frac{(q+|k+\mu_0|)}{2}+\frac{3}{4}\right)\right]^{2\nu/(\nu+2)}.$$

Although we still have the same ratio of derivatives, the above relation does not hold for the exact solution.

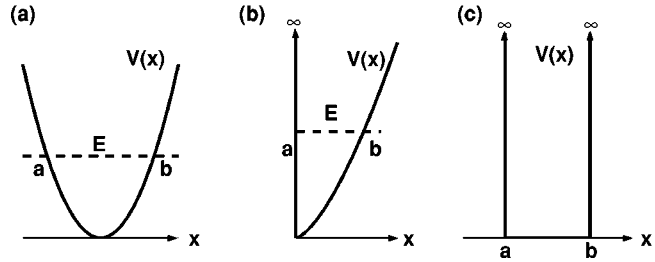

FIG. 4. WKB wave function matching boundary conditions for three cases of potentials.

(d) When $\nu=-1$, its energy spectra have the properties, $\partial E/\partial n:\partial E/\partial q=1:1$, $\partial E/\partial n:\partial E/\partial|k+\mu_0|=1:1$, and $\partial E/\partial q:\partial E/\partial|k+\mu_0|=1:1$. They are also related to the closeness of the classical orbits. For models with negative power of $\nu,(-2<\nu<0)$, the WKB approximation given by the Eq. (37) implies that

$$E\sim\left[\left(n+\frac{2(q+|k+\mu_0|)+\nu+3}{2\nu+4}\right)\right]^{2\nu/(\nu+2)}.$$

This hence implies that $\partial E/\partial n:\partial E/\partial q=\nu+2$,$\partial E/\partial n:\partial E/\partial|k+\mu_0|=\nu+2$, and $\partial E/\partial q:\partial E/\partial|k+\mu_0|=\nu+2$ are all equal. This relation does not hold for the exact result for the same reason.

(e) The increase in the intensity of the magnetic flux will change the slope of the energy distribution in both the models with $-2<\nu<0$ and the models with $0<\nu<\infty$. More explicitly, when $\nu<2$, increasing the flux will depress the slope; whereas when $\nu>2$, increasing the flux will lead to the increase in the slope. In addition, the model with $\nu=2$ is marginal in the sense that the slope of the energy distribution will not be affected by the change of the flux. For details, see the difference shown in the Figs. 2 and 3. Note that $|k+\mu_0|$ is set as 0.5 and 12 in Figs. 2 and 3, respectively.

IV. CONCLUSION

The semiclassical quantization rule is presented for a charged particle moving in a system with a general central force described by the potential $V(r)=\lambda r^\nu$, with $-2<\nu<\infty$, and an AB magnetic flux. The formulas obtained in this paper are in good agreement with the energy levels with all known exactly solvable models with some specific values of ν. Furthermore, we have presented numerical results for $\nu=\infty$, which are also in good agreement with the exact result. Therefore, one expects that the semiclassical quantization rules will also be in good agreement with the models prescribed by a large ranges of ν even the results shown in this paper are more reliable for the case with large principle quantum number n.

APPENDIX

The WKB wave function for a charged particle moving in a smooth potential well near the neighborhood $x\sim a(x>a)$, where $x=a,b$ are the intersection points of the horizonal line $y=E$ and the curve $y=V(x)$ as shown in Fig. 4(a), can be expressed in terms of the classical momentum p as (see, for example, Ref. [24] for details)

$$\Psi(x)=\frac{C}{\sqrt{p}}\sin\left[\frac{1}{\hbar}\int_a^x p\,dx+\frac{\pi}{4}\right]\equiv\frac{C}{\sqrt{p}}\sin\alpha(x), \qquad \text{(A1)}$$

where C is constant. Analogously, near the neighborhood $x \sim b$ $(x<b)$ we have

$$\Psi(x)=\frac{C'}{\sqrt{p}}\sin\left[\frac{1}{\hbar}\int_x^b p\,dx+\frac{\pi}{4}\right]\equiv\frac{C'}{\sqrt{p}}\sin\beta(x). \qquad \text{(A2)}$$

These two wave functions must be consistent. This means that near the neighborhoods a,b of x,

$$\alpha(x)+\beta(x)=\frac{1}{\hbar}\int_a^b p\,dx+\frac{\pi}{2}=(n+1)\pi, n=0,1,2,3,\ldots. \qquad \text{(A3)}$$

Or equivalently,

$$\oint p\,dx=\left(n+\frac{1}{2}\right)h, n=0,1,2,3,\ldots. \qquad \text{(A4)}$$

For the half-infinite potential well as shown in Fig. 4(b), one has

$$\oint p\,dx=\left(n+\frac{3}{4}\right)h, n=0,1,2,3,\ldots. \qquad \text{(A5)}$$

Analogously, the matching rule of the wave functions gives the quantization rule for the system with an infinitely deep square-well potential as illustrated in Fig. 4(c). Indeed, one has

$$\oint p\,dx=(n+1)h, n=0,1,2,3,\ldots. \qquad \text{(A6)}$$

The argument leading to the same result for a more general condition beyond the above examples can be found with the help of the Maslov index shown in Ref. [25].

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