Performance of Service-Node-Based Mobile Prepaid Service

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***Abstract*—In the recent years, mobile prepaid service has become an important mobile application with rapid growth of subscription rate. The most widely deployed prepaid solution today is the service node approach that deducts and updates the prepaid credit during a phone call. Implementation of the service node approach may generate large number of credit checks that significantly degrades the performance of a service node. We investigate how the number of credit checks affects the workload of the service node and the bad debt that a service provider may bear. We propose an analytic model to derive the optimal credit checking/updating frequency for the service node approach. The analytic analysis is validated against simulation experiments. Our study indicates that the number of credit checks increases rapidly when the call pattern is irregular. We also observe that in order to reduce the checking cost of the service node, the prepaid service provider should encourage the customer to make long calls by giving them discounts.**

***Index Terms*—Global system for mobile communications (GSM), mobile phone network, prepaid service, service node.**

I. Introduction

RECENTLY, the mobile prepaid service has become popular. In USA, the prepaid calling market grew 56% to about two billion US dollars in 1998 and is expected to maintain a high growth rate to 2005 [1]. By 2001, it is predicted that more than 40% global system for mobile communications (GSM) customers will subscribe to prepaid service [1]. At the end of 2003, mobile prepaid service will account for 62% of the cellular user base [2]. In Australia, Telstra started prepaid service with 100 000 customers and had exceeded the system capacity in early 1999 [3]. In Taiwan, FarEastone reported that more than 40% of their customers subscribed to prepaid service in May 1999.

From the customer's point of view, prepaid service provides an immediate service without a long-term contract or regular bills. From the system provider's point of view, prepaid service enlarges the customer base and reduces operation overhead such as printing monthly bills and checking customer's credit before providing service. In prepaid service, the revenue is received typically one and half month earlier than the postpaid service. In summary, prepaid service improves the cash flow for the operators, reduces the time of cost reclamation and increases the capability of competition.

Prepaid service works as follows. To initiate the prepaid service, a customer purchases a prepaid card from the operator. The prepaid card includes an associated directory number and the credit. The prepaid service is usually activated immediately or within a certain number of days after the initialization. When the customer originates a prepaid call, the corresponding charge is decremented from the remaining prepaid credit. Most prepaid systems have configured credit thresholds [4]. If customer's balance is below the threshold, the customer will hear a whisper tone reminding this person to recharge while he/she is talking.

Once the balance reaches zero, the customer cannot originate calls, but may be allowed to receive phone calls for a period (e.g., six months). Prepaid service can be reactivated by purchasing a top-up card to recharge the prepaid credit. The top-up card is like a scratch card with a secret code inside it. The customer dials a toll-free number and follows the instructions of an interactive voice response (IVR) to input the phone number and the secret code. The prepaid system verifies the secret code and refreshes customer's account. On the other hand, if the balance remains zero for a period without being recharged, the prepaid system deletes the customer record and the customer cannot originate or receive any phone calls.

Four billing technologies are used in prepaid service: hot billing approach, service node approach, intelligent network (IN) approach and handset-based approach [5]. The hot billing approach uses call detail records (CDRs) produced by the wireless switch (i.e., mobile switching center) to process the prepaid usage. These records are generated after call completions and are transported from the mobile switching center (MSC) to the prepaid service center. Hot billing approach is cost effective because it does not require major changes in the network infrastructure [6]. However, since a CDR can only be transmitted until the call is completed, the prepaid credit may become negative at the end of a phone call. This incurs the one call exposure problem that may cause large loss to the service providers.

In the handset-based approach, the prepaid credit and balance information is stored in the SIM card of a handset. In GSM handset-based prepaid service, during the call set-up or tariff switching, the MSC provides tariff parameters to the handset through the GSM phase 2 supplementary message AOC (advice of charge) [5]. The handset converts the AOC message into a sequence of SIM commands that modify the balance information in the SIM card. During the conversation, the handset decrements the prepaid credit on a real-time basis.

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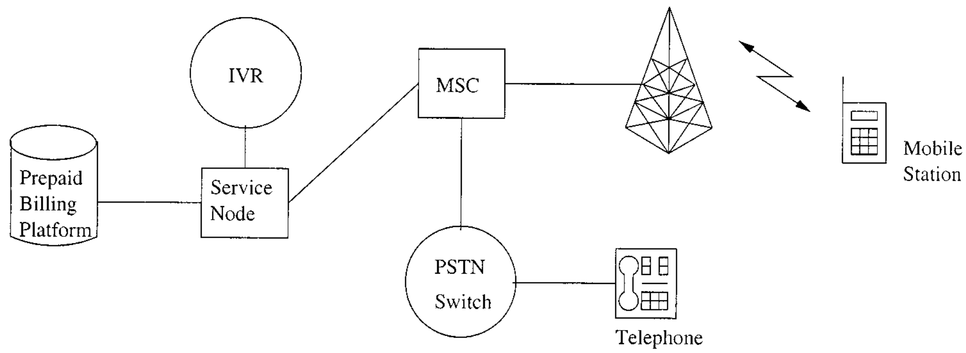

Fig. 1. Service node architecture for prepaid service.

Similar to the hot billing approach, the handset-based approach does not incur major modification to the carrier's infrastructure. However, the prepaid system requires GSM phase-II-compliant handsets. This requirement restricts the market penetration. In addition, security is a serious problem with this approach. To eliminate the possibility of fraud, network needs to act as a backup to keep track of the prepaid credit usage.

The intelligent network approach is considered as a complete solution for the prepaid service. It migrates the service control and service development functions from the MSCs to the prepaid service control point (P-SCP). The SCP contains service logic programs (SLPs) and associated data to provide IN services [7]. When an MSC encounters a prepaid call, it communicates with the P-SCP through SS7 links, asking the P-SCP to decide how the call should be processed. The P-SCP performs the service control functions (e.g., checking credit and activating a countdown timer) based on the customer's credit and sends a response message back to the MSC. After receiving the message, the MSC performs the P-SCP instructions to accept or reject the prepaid call. Since the P-SCP is not on the voice path, the intelligent network solution allows real-time call control with low capacity expansion cost. However, not all carriers are interested in implementing this approach because of the investment on P-SCP and the necessity for software modifications in all MSCs.

The service node approach is the most widely deployed prepaid solution today and is viewed as a stepping-stone to the intelligent network approach. Compared with the intelligent network approach, the service node approach integrates the functions of the MSC and service control point (SCP) in a closed configuration [8]. The service node usually collocates with an MSC and is connected to the MSC using high-speed T1/E1 trunks. For each prepaid call, the MSC routes the call to the service node for call processing. After the service node performs the service control functions, the prepaid call is routed back to the MSC and then to the called party. If the called party is a wire-line telephone, then the MSC sets up a trunk to the switch (i.e., central office) in the PSTN. The PSTN switch performs call switching and connects to the wire-line telephone. Thus, to set up a mobile prepaid call, it requires two ports on the service node and four ports on the MSC. Since the service node is on the voice path, this approach allows real-time call control. However, the cost of capacity expansion in the service node approach is higher than that of the intelligent network approach. On the other hand, a new signaling protocol is required in the IN approach to support prepaid services. Upgrading all MSCs to support the new signaling protocol is expensive. Such a new protocol is not required in the service node approach.

We have studied the hot billing approach in [9]. The comparison of the four prepaid approaches can be found in [10]. This paper investigates the performance of the service node approach and is organized as follows. Section II describes the call origination procedures of the service node approach. Section III presents the analytic model for the service node approach. Numeric results are presented in Section IV, and the conclusions are given in Section V.

II. The Service Node Approach

We use the GSM network as an example to illustrate the service node architecture and the prepaid call origination procedure. Fig. 1 depicts the architecture of the service node approach. The service node, which controls the call processing for prepaid calls, can be viewed as an extended platform from the existing telecommunication network. Customer billing and account information is stored in the prepaid billing platform (PBP) to support real-time call rating. When a customer subscribes to the prepaid service, the prepaid billing platform creates the subscriber record including the MS identities, amount of the prepaid credit, the date of initialization and related authentication information. Prepaid service is activated within a short time after the service is subscribed. In the service node approach, the customers can interact with the IVR for service query and credit recharging. The IVR can also communicate with a customer when the prepaid credit is low.

The prepaid call origination procedure is illustrated in Fig. 2 and is described in the following steps.

Step 1) A prepaid customer originates a prepaid call by dialing the called party's phone number.

Step 2) The MSC identifies that the call is a prepaid call and routes the call to the service node.

Step 3) The service node requests the prepaid billing platform to verify if the customer has sufficient credit to make this call.

Step 4) If the call is granted, the service node activates a countdown timer for charging and sets up a trunk back to the MSC. Eventually, the call is routed from the MSC to its destination. During the conversation, the prepaid credit is decremented in real time at the

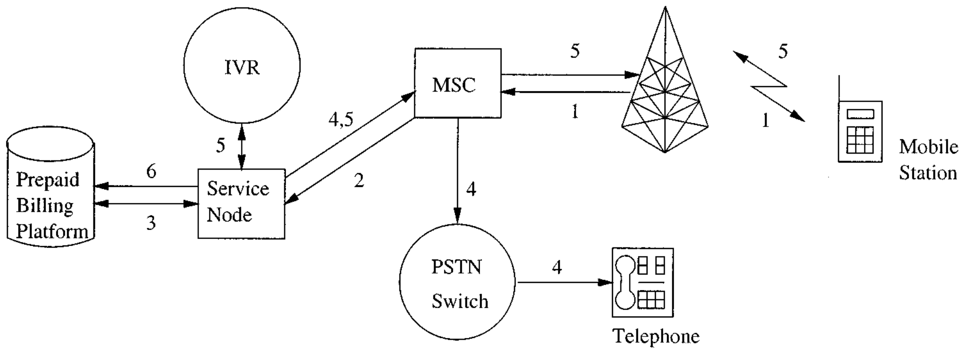

Fig. 2. The prepaid call origination procedure.

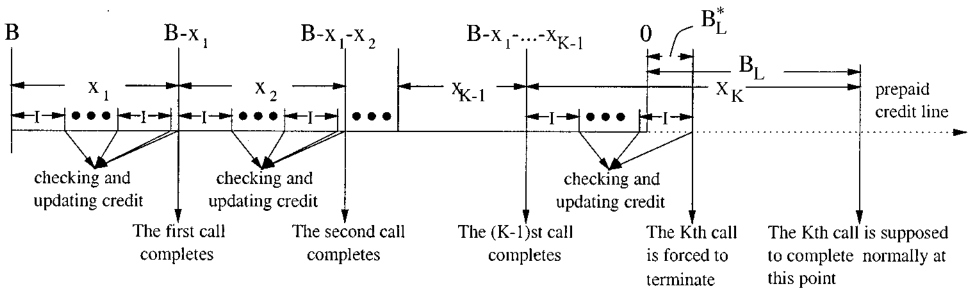

Fig. 3. Case i): The charges for prepaid calls where the last call is forced to terminate by the service node.

service node according to the carrier-defined rate plan.

Step 5) If the prepaid credit becomes negative before the end of the conversation, the call is forced to terminate by the service node. In this case, the call may be rerouted to the IVR to play announcements reminding the customer to recharge the prepaid credit.

Step 6) After the prepaid call completes, the credit is updated at the prepaid billing platform. In the low credit case, the IVR may also be instructed to play a warning message to the customer.

The service node checks and decrements the prepaid credit periodically during the conversation to avoid potential large bad debt. However, the capability of the service node may be limited since all service control and call-switching functions are implemented on the service node. Theoretically by upgrading the processing power of the switch or service node, the service node will permit a real-time credit monitoring. However, in a real mobile phone network operation, the processing budget for a service node should be accurately planned. An example of processing budget planning for a telecommunication node can be found in [11]. In real operation, a service node may process over 10 000 prepaid calls simultaneously. To support real-time monitoring for so many simultaneously calls, the cost for upgrading processing power is too high, and is not justified. Thus, the operators always ask the following question: "What is the credit checking frequency so that the sum of the credit checking cost and the bad debt is minimized?" This paper utilizes analytic and simulation models to investigate the performance of the service node, and answers the above question. After the answer is found, the operator can choose the appropriate processing power for the service node so that it can support, say, over 10 000 simultaneously prepaid calls at the selected credit checking frequency.

III. The Analytic Model

In this section, we propose an analytic model to derive the expected number of credit checks ($E[N^*_{ch}]$) and the expected bad debt ($E[B^*_L]$) in the service node approach. Let B be the prepaid credit and K be the number of calls that a customer has made when the prepaid credit runs out. We assume that a customer will consume all the prepaid credit before he/she gives up the prepaid service. When a customer is in conversation, the service node periodically decrements the customer's credit by the amount I until either the call completes or the credit becomes negative.

Let x_i $(i = 1, 2, \ldots K-1)$ be the charge of the ith call. The last (i.e., the Kth) call terminates in one of the two cases:

Case i) The service node discovers that the prepaid credit runs out by periodic checks and the call is forced to terminate (see Fig. 3).

Case ii) The last call completes before the service node discovers that the credit becomes negative (see Fig. 4).

In Figs. 3 and 4, the horizontal line is the "prepaid credit line" that illustrates the decrement of the prepaid credit due to periodic credit checks during the calls (the vertical lines). For the derivation purpose, let x_K be the charge of the last call if the service node would not terminate the call when the credit becomes negative. We assume that x_i $(i = 1, 2, \ldots, K)$ are independent and identical random variables and the expected value $E[x_i] = 1/\gamma$. Let B^*_L be the loss of the service provider and B_L be the corresponding value if the last call were allowed to complete (i.e., the amount between when the credit is exhausted and when the call completes). Note that B^*_L equals to B_L in Case ii).

Since the call holding times are independent and identically distributed, the call completion in the service node approach can be modeled by a renewal process [12]. Let $E[N_{ch}]$ be the

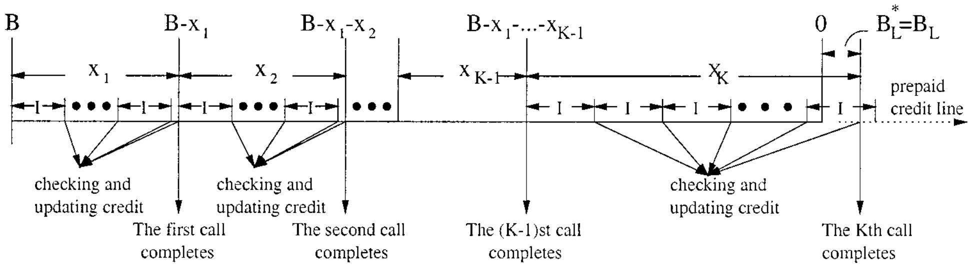

Fig. 4. Case ii): The charges for prepaid calls where the last call completes before the service node discovers that the credit becomes negative.

expected number of credit checks assuming that the total credit is $B + B_L$. In this case, the Kth call is allowed to complete normally. Let $E[n_{ch}]$ be the expected number of credit checks for a call. Based on the Wald's equation [12], $E[N_{ch}]$ can be expressed as

$$E[N_{ch}] = E[K]E[n_{ch}]. \tag{1}$$

Let $E[N^*_{ch}]$ be the expected number of credit checks in the service node approach (i.e., the expected number of credit checks when the total prepaid credit is B). From (1), $E[N^*_{ch}]$ can be approximated as

$$\begin{aligned} E[N^*_{ch}] &\approx E[N_{ch}] - \frac{E[B_L]}{I} \\ &= E[K]E[n_{ch}] - \frac{E[B_L]}{I}. \end{aligned} \tag{2}$$

Let y_n be the accumulated charge of the first n calls. That is, $y_n = \sum_{i=1}^{n} x_i$. Let $f_n(y_n)$ and $F_n(y) = \Pr\{y_n < y\}$ be the density and distribution functions of y_n, respectively. From the renewal theory [12, p. 100], the expected value $E[K]$ can be derived as

$$E[K] = \sum_{n=1}^{\infty} F_n(B) + 1. \tag{3}$$

We consider two cases for prepaid credit B: fixed credit and recharged credit. In the fixed credit case, B is fixed. In the recharged credit case, a customer may recharge the prepaid credit several times before the customer gives up the prepaid service.

A. Fixed Credit Case

In the fixed credit case, the prepaid credit is a constant. In PCS services, the call holding times are usually assumed to be exponentially or Erlang distributed [13]–[15]. Erlang distribution is used so that we can obtain close form for the analytical model. Furthermore, Erlang distribution is more general than the exponential distribution, and it can be used to investigate the case when the variance of the call holding time distribution is small. Note that the effect of call dropped due to handover is already considered in the call holding times in our model. In a real PCS network, the measured call holding times include both complete calls and dropped calls.

Since the charge of a call is proportional to its call holding time, x_i can be assumed to have an Erlang density function $f(x_i)$ with mean $E[x_i] = 1/\gamma$ and variance $Var[x_i] = 1/m_1\gamma^2$ (i.e., $f(x_i) = [(m_1\gamma)^{m_1}/(m_1 - 1)!] x_i^{m_1-1}e^{-m_1\gamma x_i}$ and $f_n(y_n) = [(m_1\gamma)^{m_1 n}/(m_1 n - 1)!] y_n^{m_1 n-1}e^{-m_1\gamma y_n}$). Then, $E[n_{ch}]$ can be expressed as

$$\begin{aligned} E[n_{ch}] &= \sum_{j=1}^{\infty} \int_{x_i=(j-1)I}^{jI} j f(x_i)\, dx_i \\ &= \int_{x_i=0}^{I} \left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right] x_i^{m_1-1} e^{-m_1\gamma x_i}\, dx_i \\ &\quad + \int_{x_i=I}^{2I} 2\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right] x_i^{m_1-1} e^{-m_1\gamma x_i}\, dx_i + \cdots \\ &= 1 + \sum_{j=1}^{\infty}\sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma jI)^k e^{-m_1\gamma jI}}{k!}\right]. \end{aligned} \tag{4}$$

From (3), $E[K]$ can be expressed as

$$E[K] = \sum_{n=1}^{\infty}\left\{1 - \sum_{j=0}^{m_1 n-1}\left[\frac{(m_1\gamma B)^j e^{-m_1\gamma B}}{j!}\right]\right\} + 1.$$

When B is sufficiently large, an approximation for $E[K]$ has been derived in [9]

$$E[K] \approx \frac{2m_1\gamma B + m_1 + 1}{2m_1}. \tag{5}$$

From (5) and the Wald's equation, with B sufficiently large, $E[B_L]$ can be expressed as

$$E[B_L] = E[x_i]E[K] - B \approx \frac{m_1+1}{2m_1\gamma}. \tag{6}$$

From (2) and (4)–(6), $E[N^*_{ch}]$ is approximated as

$$\begin{aligned} E[N^*_{ch}] \approx &\left\{1 + \sum_{j=1}^{\infty}\sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma jI)^k e^{-m_1\gamma jI}}{k!}\right]\right\} \\ &\times\left(\frac{2m_1\gamma B + m_1 + 1}{2m_1}\right) - \frac{m_1+1}{2m_1\gamma I}. \end{aligned}$$

Next, we derive the expected loss of the service provider $E[B^*_L]$ as follows. Let p_1 and p_2 be the probability that Case i) and Case ii) occur, respectively. Then, $E[B^*_L]$ is expressed as

$$E[B^*_L] = p_1 E[B^*_L|\text{ Case i)}] + p_2 E[B^*_L|\text{ Case ii)}]. \tag{7}$$

Let l be the number of credit checks of the last call. The expected bad debt of the service provider in Case i) can be expressed as

$$\begin{aligned} E[B_L^*|\text{ Case i)}] &= \Phi_1(B) \\ &= \left(\frac{1}{p_1}\right)\Bigg\{\int_{x_1=\lceil B/I\rceil I}^{\infty}\left(\left\lceil\frac{B}{I}\right\rceil I - B\right) \\ &\quad \times f(x_1)\,dx_1 + \sum_{n=1}^{\infty}\sum_{l=1}^{\lceil B/I\rceil-1}\int_{x_{n+1}=lI}^{\infty} \\ &\quad \times \int_{y_n=B-lI}^{B-(l-1)I}(lI+y_n-B)f_n(y_n) \\ &\quad \times f(x_{n+1})\,dy_n\,dx_{n+1} + \sum_{n=1}^{\infty}\int_{x_{n+1}=\lceil B/I\rceil I}^{\infty} \\ &\quad \times \int_{y_n=0}^{B-(\lceil B/I\rceil-1)I}\left(\left\lceil\frac{B}{I}\right\rceil I + y_n - B\right) \\ &\quad \times f_n(y_n)\,f(x_{n+1})\,dy_n\,dx_{n+1}\Bigg\} \\ &= A_1 + A_2 + A_3 \end{aligned} \tag{8}$$

where $f_n(y_n)$ is the density function of the charge of n accumulated calls and

$$A_1 = \left(\frac{1}{p_1}\right)\int_{x_1=\lceil B/I\rceil I}^{\infty}\left(\left\lceil\frac{B}{I}\right\rceil I - B\right)f(x_1)\,dx_1 \tag{9}$$

$$\begin{aligned} A_2 = \left(\frac{1}{p_1}\right)&\sum_{n=1}^{\infty}\sum_{l=1}^{\lceil B/I\rceil-1}\int_{x_{n+1}=lI}^{\infty}\int_{y_n=B-lI}^{B-(l-1)I} \\ &\times (lI+y_n-B)f_n(y_n)f(x_{n+1})\,dy_n\,dx_{n+1} \end{aligned} \tag{10}$$

and

$$\begin{aligned} A_3 = \left(\frac{1}{p_1}\right)&\sum_{n=1}^{\infty}\int_{x_{n+1}=\lceil B/I\rceil I}^{\infty}\int_{y_n=0}^{B-(\lceil B/I\rceil-1)I} \\ &\times \left(\left\lceil\frac{B}{I}\right\rceil I + y_n - B\right) \\ &\times f_n(y_n)f(x_{n+1})\,dy_n\,dx_{n+1}. \end{aligned} \tag{11}$$

The term A_1 represents a trivial situation in Case i) where the first call consumes all prepaid credit (i.e., $x_1 \geq \lceil B/I\rceil I$). The loss of the service provider B_L^* is $\lceil B/I\rceil I - B$. The terms A_2 and A_3 represent the situation where a customer has made n $(n \geq 1)$ complete calls before prepaid credit runs out. Since the last call is forced to terminate at the lth credit check, the bad debt of the service provider B_L^* is $lI + y_n - B$. The term A_2 represents the situation in Case i) where $1 \leq l \leq \lceil B/I\rceil - 1$. The call charge of the last call x_{n+1} is larger than lI and the total charge of prior n calls y_n satisfies $B - lI \leq y_n < B-(l-1)I$. The term A_3 represents the special situation in Case i) where $l = \lceil B/I\rceil$ and the total charge of prior n calls y_n satisfies $0 < y_n < B - (\lceil B/I\rceil - 1)I$. From (28)–(30) in Appendix I, A_1, A_2 and A_3 are expressed as

$$\begin{aligned} A_1 = \left(\frac{1}{p_1}\right)\left(\left\lceil\frac{B}{I}\right\rceil I - B\right)\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right] \\ \times e^{-m_1\gamma\lceil B/I\rceil I} \end{aligned} \tag{12}$$

$$\begin{aligned} A_2 = \left(\frac{1}{p_1}\right)&\sum_{n=1}^{\infty}\sum_{l=1}^{\lceil B/I\rceil-1}\Bigg\{-\sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma lI)^k}{k!}\right] \\ &\times \sum_{j=0}^{m_1n-1}\left[\frac{(m_1\gamma)^j(B-lI)^{j+1}}{j!}\right]e^{-m_1\gamma B} \\ &+ (B-lI)\sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma lI)^k}{k!}\right] \\ &\times \sum_{j=0}^{m_1n-1}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^j}{j!}\right\}e^{-m_1\gamma(B+I)} \\ &+ \left(\frac{n}{\gamma}\right)\sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma lI)^k}{k!}\right] \\ &\times \sum_{j=0}^{m_1n}\left\{\frac{[m_1\gamma(B-lI)]^j}{j!}\right\}e^{-m_1\gamma B} - \left(\frac{n}{\gamma}\right) \\ &\times \sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma lI)^k}{k!}\right] \\ &\times \sum_{j=0}^{m_1n}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^j}{j!}\right\}e^{-m_1\gamma(B+I)}\Bigg\} \end{aligned} \tag{13}$$

$$\begin{aligned} A_3 = \left(\frac{1}{p_1}\right)&\sum_{n=1}^{\infty}\Bigg\{\left(\left\lceil\frac{B}{I}\right\rceil I - B + \frac{n}{\gamma}\right) \\ &\times \sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right]e^{-m_1\gamma\lceil B/I\rceil I} \\ &+ \left(B - \left\lceil\frac{B}{I}\right\rceil I\right)\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right] \\ &\times \sum_{j=0}^{m_1n-1}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^j}{j!}\right\} \\ &\times e^{-m_1\gamma(B+I)} - \left(\frac{n}{\gamma}\right)\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right] \\ &\times \sum_{j=0}^{m_1n}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^j}{j!}\right\} \\ &\times e^{-m_1\gamma(B+I)}\Bigg\}. \end{aligned} \tag{14}$$

Using a similar approach as above, the expected bad debt in Case ii) can be expressed as

$$\begin{aligned}
E[B_L^*|\text{ Case ii)}] &= \Phi_2(B)\\
&= \left(\frac{1}{p_2}\right)\left\{\int_{x_1=B}^{\lceil B/I\rceil I}(x_1-B)f(x_1)\,dx_1\right.\\
&\quad+\sum_{n=1}^{\infty}\sum_{l=1}^{\lceil B/I\rceil-1}\int_{x_{n+1}=(l-1)I}^{lI}\int_{y_n=B-x_{n+1}}^{B-(l-1)I}\\
&\quad\times(x_{n+1}+y_n-B)f_n(y_n)f(x_{n+1})\,dy_n\\
&\quad\times dx_{n+1}+\sum_{n=1}^{\infty}\int_{x_{n+1}=(\lceil B/I\rceil-1)I}^{\lceil B/I\rceil I}\\
&\quad\times\int_{y_n=0}^{B-(\lceil B/I\rceil-1)I}(x_{n+1}+y_n-B)\\
&\quad\left.\times\, f_n(y_n)f(x_{n+1})\,dy_n\,dx_{n+1}\right\}\\
&= B_1+B_2+B_3
\end{aligned}\tag{15}$$

where B_1, B_2 and B_3 are

$$B_1=\left(\frac{1}{p_2}\right)\int_{x_1=B}^{\lceil B/I\rceil I}(x_1-B)f(x_1)\,dx_1\tag{16}$$

$$\begin{aligned}
B_2=\left(\frac{1}{p_2}\right)&\sum_{n=1}^{\infty}\sum_{l=1}^{\lceil B/I\rceil-1}\int_{x_{n+1}=(l-1)I}^{lI}\int_{y_n=B-x_{n+1}}^{B-(l-1)I}\\
&\times(x_{n+1}+y_n-B)f_n(y_n)f(x_{n+1})\,dy_n\,dx_{n+1}
\end{aligned}\tag{17}$$

and

$$\begin{aligned}
B_3=\left(\frac{1}{p_2}\right)&\sum_{n=1}^{\infty}\int_{x_{n+1}=(\lceil B/I\rceil-1)I}^{\lceil B/I\rceil I}\int_{y_n=0}^{B-(\lceil B/I\rceil-1)I}\\
&\times(x_{n+1}+y_n-B)f_n(y_n)f(x_{n+1})\,dy_n\,dx_{n+1}.
\end{aligned}\tag{18}$$

From (31)–(33) in Appendix II, B_1, B_2, and B_3 are expressed as

$$\begin{aligned}
B_1=\left(\frac{1}{p_2\gamma}\right)&\left\{\sum_{k=0}^{m_1}\left[\frac{(m_1\gamma B)^k}{k!}\right]e^{-m_1\gamma B}\right.\\
&\left.-\sum_{k=0}^{m_1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right]e^{-m_1\gamma\lceil B/I\rceil I}\right\}\\
&-\left(\frac{B}{p_2}\right)\left\{\sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma B)^k}{k!}\right]e^{-m_1\gamma B}\right.\\
&\left.-\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right]e^{-m_1\gamma\lceil B/I\rceil I}\right\}
\end{aligned}\tag{19}$$

$$\begin{aligned}
B_2=\left(\frac{1}{p_2}\right)&\sum_{n=1}^{\infty}\sum_{l=1}^{\lceil B/I\rceil-1}\left\{\sum_{k=0}^{m_1n-1}\left\{\left[\frac{(m_1\gamma)^{m_1+k}}{(m_1-1)!k!}\right]\right.\right.\\
&\times e^{-m_1\gamma B}\sum_{j=0}^{k+1}(-1)^{k+j}\binom{k+1}{j}\\
&\left.\times\left[\frac{B^j(lI)^{m_1+k-j+1}}{m_1+k-j+1}\right]\right\}-\sum_{k=0}^{m_1n-1}\left\{\left[\frac{(m_1\gamma)^{m_1+k}}{(m_1-1)!k!}\right]\right.\\
&\times e^{-m_1\gamma B}\sum_{j=0}^{k+1}(-1)^{k+j}\binom{k+1}{j}\\
&\left.\times\left\{\frac{B^j[(l-1)I]^{m_1+k-j+1}}{m_1+k-j+1}\right\}\right\}-\left(\frac{1}{\gamma}\right)\\
&\times\sum_{k=0}^{m_1n-1}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}e^{-m_1\gamma B}\\
&\times\sum_{j=0}^{m_1}\frac{[m_1\gamma(l-1)I]^j}{j!}+\left(\frac{1}{\gamma}\right)\\
&\times\sum_{k=0}^{m_1n-1}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}e^{-m_1\gamma(B+I)}\\
&\times\sum_{j=0}^{m_1}\frac{(m_1\gamma lI)^j}{j!}+B\\
&\times\sum_{k=0}^{m_1n-1}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}e^{-m_1\gamma B}\\
&\times\sum_{j=0}^{m_1-1}\frac{[m_1\gamma(l-1)I]^j}{j!}-B\\
&\times\sum_{k=0}^{m_1n-1}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}e^{-m_1\gamma(B+I)}\\
&\times\sum_{j=0}^{m_1-1}\frac{(m_1\gamma lI)^j}{j!}+\left(\frac{n}{\gamma}\right)\sum_{k=0}^{m_1n}\left\{\left[\frac{(m_1\gamma)^{m_1+k}}{(m_1-1)!k!}\right]\right.\\
&\left.\times e^{-m_1\gamma B}\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}\left[\frac{B^j(lI)^{m_1+k-j}}{m_1+k-j}\right]\right\}\\
&-\left(\frac{n}{\gamma}\right)\sum_{k=0}^{m_1n}\left\{\left[\frac{(m_1\gamma)^{m_1+k}}{(m_1-1)!k!}\right]e^{-m_1\gamma B}\sum_{j=0}^{k}(-1)^{k+j}\right.\\
&\left.\times\binom{k}{j}\left[\frac{B^j[(l-1)I]^{m_1+k-j}}{m_1+k-j}\right]\right\}-\left(\frac{n}{\gamma}\right)\\
&\times\sum_{k=0}^{m_1n}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}e^{-m_1\gamma B}\\
&\times\sum_{j=0}^{m_1-1}\frac{[m_1\gamma(l-1)I]^j}{j!}+\left(\frac{n}{\gamma}\right)\\
&\times\sum_{k=0}^{m_1n}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}e^{-m_1\gamma(B+I)}\\
&\left.\times\sum_{j=0}^{m_1-1}\frac{(m_1\gamma lI)^j}{j!}\right\}
\end{aligned}\tag{20}$$

$$
\begin{aligned}
B_3 = {} & \left(\frac{1}{p_2}\right)\sum_{n=1}^{\infty}\Bigg\{\left(\frac{1}{\gamma}\right)\sum_{k=0}^{m_1}\left\{\frac{\left[m_1\gamma\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^k}{k!}\right\}\\
& \times e^{-m_1\gamma(\lceil B/I\rceil-1)I}-\left(\frac{1}{\gamma}\right)\sum_{k=0}^{m_1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right]\\
& \times e^{-m_1\gamma\lceil B/I\rceil I}-B\sum_{k=0}^{m_1-1}\left\{\frac{\left[m_1\gamma\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^k}{k!}\right\}\\
& \times e^{-m_1\gamma(\lceil B/I\rceil-1)I}+B\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right]\\
& \times e^{-m_1\gamma\lceil B/I\rceil I}-\left(\frac{1}{\gamma}\right)\\
& \times\sum_{k=0}^{m_1n-1}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^k}{k!}\right\}e^{-m_1\gamma B}\\
& \times\sum_{j=0}^{m_1}\frac{\left[m_1\gamma\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^j}{j!}+\left(\frac{1}{\gamma}\right)\\
& \times\sum_{k=0}^{m_1n-1}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^k}{k!}\right\}\\
& \times e^{-m_1\gamma(B+I)}\sum_{j=0}^{m_1}\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^j}{j!}+B\\
& \times\sum_{k=0}^{m_1n-1}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^k}{k!}\right\}e^{-m_1\gamma B}\\
& \times\sum_{j=0}^{m_1-1}\frac{\left[m_1\gamma\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^j}{j!}-B\\
& \times\sum_{k=0}^{m_1n-1}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^k}{k!}\right\}\\
& \times e^{-m_1\gamma(B+I)}\\
& \times\sum_{j=0}^{m_1-1}\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^j}{j!}+\left(\frac{n}{\gamma}\right)\\
& \times\sum_{k=0}^{m_1-1}\left\{\frac{\left[m_1\gamma\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^k}{k!}\right\}\\
& \times e^{-m_1\gamma(\lceil B/I\rceil-1)I}\\
& -\left(\frac{n}{\gamma}\right)\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right]\\
& \times e^{-m_1\gamma\lceil B/I\rceil I}-\left(\frac{n}{\gamma}\right)\\
& \times\sum_{k=0}^{m_1n}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^k}{k!}\right\}e^{-m_1\gamma B}\\
& \times\sum_{j=0}^{m_1-1}\frac{\left[m_1\gamma\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^j}{j!}+\left(\frac{n}{\gamma}\right)\\
& \times\sum_{k=0}^{m_1n}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^k}{k!}\right\}\\
& \times e^{-m_1\gamma(B+I)}\\
& \times\sum_{j=0}^{m_1-1}\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^j}{j!}\Bigg\}.
\end{aligned}
\tag{21}
$$

When $m_1 = 1$, x_i is exponentially distributed. From (7), (8), (12)–(15), (19)–(21) with $m_1 = 1$, $E[B_L^*]$ can be expressed as

$$
\begin{aligned}
E[B_L^*] = {} & \left(\frac{1}{\gamma}\right)e^{-\gamma B}-\left(\frac{1}{\gamma}\right)e^{-\gamma\lceil B/I\rceil I}\\
& +\sum_{n=1}^{\infty}\Bigg\{\sum_{l=1}^{\lceil B/I\rceil-1}\\
& \times\left\{\sum_{k=0}^{n-1}\left\{\frac{\gamma^{k-1}[B-(l-1)I]^k}{k!}\right\}e^{-\gamma(B+I)}\right.\\
& \left.-\sum_{k=0}^{n-1}\left[\frac{\gamma^k(B-lI)^{k+1}}{(k+1)!}\right]e^{-\gamma B}\right\}\\
& +\left\{\frac{\gamma^{n-1}[B-(l-1)I]^n}{n!}\right\}e^{-\gamma B}-\left(\frac{1}{\gamma}\right)e^{-\gamma B}\Bigg\}\\
& +\sum_{n=1}^{\infty}\Bigg\{\left[\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I-B+\frac{n+1}{\gamma}\right]\\
& \times e^{-\gamma(\lceil B/I\rceil-1)I}-\left(\frac{1}{\gamma}\right)e^{-\gamma\lceil B/I\rceil I}
\end{aligned}
$$

$$+\sum_{k=0}^{n-1}\left\{\frac{\gamma^k\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^{k+1}}{k!}\right\}e^{-\gamma B}$$
$$-\sum_{k=0}^{n-1}\left\{\frac{\gamma^{k-1}\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^{k}}{k!}\right\}e^{-\gamma B}$$
$$+\sum_{k=0}^{n-1}\left\{\frac{\gamma^{k-1}\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^{k}}{k!}\right\}e^{-\gamma(B+I)}$$
$$-\sum_{k=0}^{n}\left\{\frac{n\gamma^{k-1}\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^{k}}{k!}\right\}e^{-\gamma B}\Bigg\}. \tag{22}$$

B. Recharged Credit Case

In the recharged credit case, a customer may recharge his/her prepaid card before the credit runs out. At the beginning, a customer purchases an initial credit B_I and then recharges his/her credit several times. Let B_r be the amount of single recharged credit and N_R be the number of recharges that a customer has made before he/she gives up the prepaid service. We assume that N_R is a geometric random variable with the parameter p (i.e., p is the recharge probability). Then, the probability mass function of N_R is expressed as

$$\Pr\{N_R=n_r\}=(1-p)p^{n_r}, \qquad n_r=0,1,2,\ldots,\infty.$$

The prepaid credit B equals to $B_I+n_rB_r$ and its expected value $E[B]$ equals to $B_I+(p/(1-p))B_r$. First, we consider the case where the call charge x_i has an Erlang distribution with mean $E[x_i]=1/\gamma$ and variance $Var[x_i]=1/m_1\gamma^2$. To derive $E[K|N_R]$, we assume that the last call is allowed to complete normally. The probability that n calls are completed before total credit runs out given $N_R=n_r$ can be expressed as

$$\Pr\{y_n<B_I+n_rB_r<y_n+x_{n+1}|N_R=n_r\}$$
$$=\int_{y_n=0}^{B_I+n_rB_r}\int_{x_{n+1}=B_I+n_rB_r-y_n}^{\infty}\times f_n(y_n)f(x_{n+1})\,dx_{n+1}\,dy_n. \tag{23}$$

From (36) in Appendix III, (23) can be expressed as

$$\Pr\{y_n<B_I+n_rB_r<y_n+x_{n+1}|N_R=n_r\}$$
$$=\sum_{k=0}^{m_1-1}\left\{\frac{[m_1\gamma(B_I+n_rB_r)]^{m_1n+k}}{(m_1n+k)!}\right\}e^{-m_1\gamma(B_I+n_rB_r)}.$$

Thus, $E[K|N_R=n_r]$ can be expressed as

$$E[K|N_R=n_r]$$
$$=\sum_{n=0}^{\infty}\left\{(n+1)\left\{\sum_{k=0}^{m_1-1}\left\{\frac{[m_1\gamma(B_I+n_rB_r)]^{m_1n+k}}{(m_1n+k)!}\right\}\times e^{-m_1\gamma(B_I+n_rB_r)}\right\}\right\}. \tag{24}$$

From (24), $E[B_L]$ can be expressed as

$$E[B_L]=\sum_{n_r=0}^{\infty}\{E[K|N_R=n_r]E[x_i]-E[B_I+n_rB_r]\}$$
$$=\left(\frac{1}{\gamma}\right)\left\{\sum_{n=0}^{\infty}(n+1)\left\{\sum_{n_r=0}^{\infty}p^{n_r}(1-p)\times\left\{\sum_{k=0}^{m_1-1}\left\{\frac{[m_1\gamma(B_I+n_rB_r)]^{m_1n+k}}{(m_1n+k)!}\right\}\times e^{-m_1\gamma(B_I+n_rB_r)}\right\}\right\}\right\}-\left(B_I+\frac{p}{1-p}B_r\right). \tag{25}$$

From (2), (4), (24), and (25), $E[N_{ch}^*]$ can be approximated as

$$E[N_{ch}^*]\approx\left\{1+\sum_{j=1}^{\infty}\sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma jI)^k e^{-m_1\gamma jI}}{k!}\right]\right\}$$
$$\times\left\{\sum_{n=0}^{\infty}(n+1)\left\{\sum_{n_r=0}^{\infty}p^{n_r}(1-p)\times\sum_{k=0}^{m_1-1}\left\{\frac{[m_1\gamma(B_I+n_rB_r)]^{m_1n+k}}{(m_1n+k)!}\right\}\times e^{-m_1\gamma(B_I+n_rB_r)}\right\}\right\}-\left(\frac{1}{I}\right)$$
$$\times\left\{\left(\frac{1}{\gamma}\right)\left\{\sum_{n=0}^{\infty}(n+1)\left\{\sum_{n_r=0}^{\infty}p^{n_r}(1-p)\times\sum_{k=0}^{m_1-1}\left\{\frac{[m_1\gamma(B_I+n_rB_r)]^{m_1n+k}}{(m_1n+k)!}\right\}\times e^{-m_1\gamma(B_I+n_rB_r)}\right\}\right\}-\left(B_I+\frac{p}{1-p}B_r\right)\right\}.$$

Using a similar approach as the fixed credit case [see (8)], the expected bad debt in Case i) can be expressed as

$$E[B_L^*|\text{ Case i)}]=p^{n_r}(1-p)\Phi_1(B_I+n_rB_r). \tag{26}$$

Similar to the fixed credit case [see (15)], the expected bad debt in Case ii) can be expressed as

$$E[B_L^*|\text{ Case (ii)}]=p^{n_r}(1-p)\Phi_2(B_I+n_rB_r). \tag{27}$$

TABLE I
COMPARISON OF ANALYTIC AND SIMULATION MODELS (FIXED CREDIT CASE, $E[x_i]$ = NT\$36, $Var[x_i] = 1296$, I = NT\$12)

	$E[N^*_{ch}]$			$E[B^*_L]$		
B	Simulation	Analytic	Error	Simulation	Analytic	Error
100	10.37	10.33	0.39%	5.78	5.78	0.0%
300	29.95	29.93	0.07%	5.67	5.67	0.0%
400	39.75	39.73	0.05%	5.67	5.67	0.0%
500	49.55	49.52	0.06%	5.67	5.67	0.0%

TABLE II
COMPARISON OF ANALYTIC AND SIMULATION MODELS (RECHARGED CREDIT CASE, $E[B]$ = NT\$500, B_I = NT\$100, $p = 2/3$, $E[x_i]$ = NT\$36, $Var[x_i] = 1296$)

	$E[N^*_{ch}]$			$E[B^*_L]$		
I	Simulation	Analytic	Error	Simulation	Analytic	Error
18	35.88	35.84	0.11%	8.25	8.25	0.0%
12	49.55	49.52	0.06%	5.71	5.71	0.0%
0.2	2507.05	2507.45	0.02%	0.098	0.098	0.0%

From (7), (8), and (15) and (26) and (27), $E[B^*_L]$ can be expressed as

$$E[B^*_L] = \sum_{n_r=0}^{\infty} p^{n_r}(1-p) \times [p_1\Phi_1(B_I + n_rB_r) + p_2\Phi_2(B_I + n_rB_r)].$$

IV. NUMERIC EXAMPLES

This section investigates the performance of the service node approach based on the analytic model developed in the previous section. Simulation experiments have been conducted to validate the analytic results. Each simulation experiment was repeated 500 000 times to ensure stable results. To reflect the situation of prepaid service in Taiwan, the expected charge of a call is assumed to be NT\$36, and the expected prepaid credit Bs are NT\$100, NT\$300, NT\$400 and NT\$500. Tables I and II compare the results of analytic and simulation models. The tables indicate that the analytic results are consistent with the simulation results.

A. Effects of the Variation of Call Charges

This subsection studies the effect of the variation of call charges on $E[N^*_{ch}]$ and $E[B^*_L]$ for fixed credit and recharged credit cases. The call charge is assumed to have a Gamma distribution. The Gamma distribution is selected because it has been widely used in the PCS studies [14] and can be shaped to represent many distributions. A Gamma distribution has the density function

$$f(t) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, t^{\alpha-1}e^{-\beta t}, \qquad \text{for } t > 0$$

where α (>0) is the shape parameter, β (>0) is the scale parameter and $\Gamma(q) = \int_{z=0}^{\infty} z^{q-1}e^{-z}\,dz$. The mean of Gamma distribution is α/β and the standard derivation is $\sqrt{\alpha}/\beta$. Let C_x be the coefficient of the variation of call charge; C_x equals to the ratio of the standard derivation to the mean of the distribution. For Gamma distribution, $C_x = 1/\sqrt{\alpha}$. In our experiment, C_x ranges from 10^{-3} to 10. A large C_x represents that there are more short calls and long calls.

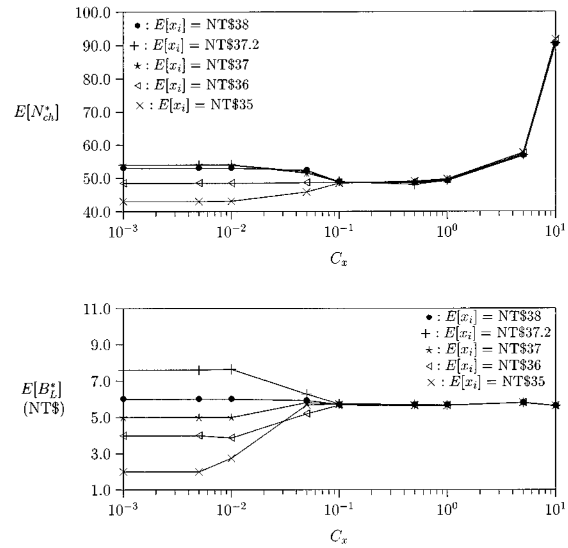

Fig. 5. Effects of C_x in the fixed credit case (B = NT\$500, I = NT\$12).

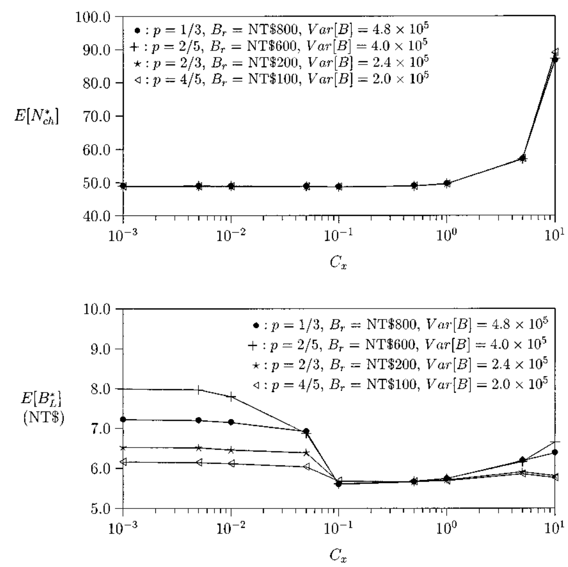

Fig. 6. Effects of C_x in the recharged credit case ($E[B]$ = NT\$500, B_I = NT\$100, $E[x_i]$ = NT\$36 and I = NT\$12).

In both fixed credit (see Fig. 5) and recharged credit (see Fig. 6) cases, the service node periodically checks and updates prepaid credit with I = NT\$12. The coefficient of variation C_x ranges from 10^{-3} to 10^1. In the fixed credit case, the prepaid credit equals to NT\$500. The call charges have a gamma distribution with mean $E[x_i]$ = NT\$38, NT\$37.2, NT\$37, NT\$36 and NT\$35, respectively. In the recharged credit case, the initial prepaid credit B_I = NT\$100 and the mean of prepaid credit $E[B]$ is NT\$500. The number of recharges is assumed to have a geometric distribution and the recharge probability p varies as 1/3, 2/5, 2/3 and 4/5. In this experiment, we only present the results where the call charges have a gamma distribution with mean $E[x_i]$ = NT\$36. Similar conclusions can be drawn for x_i with various means.

Both figures show that for $C_x < 5 \times 10^{-3}$, $E[N^*_{ch}]$ and $E[B^*_L]$ are sensitive to $E[x_i]$, but insensitive to C_x. We explain this phenomenon in Appendix III. The figure also shows that

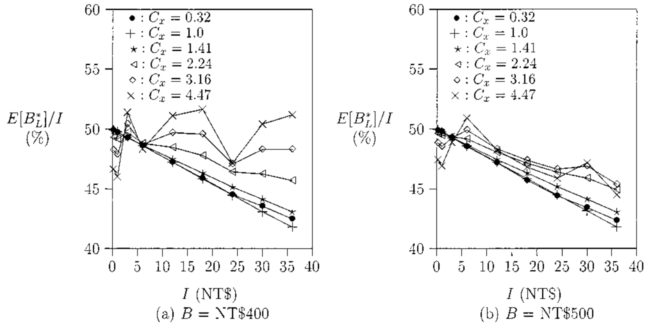

Fig. 7. Effects of I on $E[B_L^*]/I$ ($E[x_i]$ = NT$36).

when $C_x > 5$, $E[N_{ch}^*]$ increases sharply in both fixed credit and recharged credit cases. The reason is that when C_x increases, the number of short calls, whose call charges are less than I, also increases. For each short call, a small amount of credit ($x_i < I$) is consumed and a credit check is required. As a result, the number of credit checks increases as the number of short calls increases. We call this the short call effect. To avoid this effect, the prepaid service provider can implement billing policies to discourage short calls (e.g., higher rates in the first minute of a prepaid call and lower rates for the remaining call holding time).

B. Effect of I on $E[B_L^*]/I$

Fig. 7 plots $E[B_L^*]/I$ as a function of I in the fixed credit case. The mean of call charges $E[x_i]$ is NT$36 and I ranges from NT$0.2 to NT$36. In this experiment, we consider two scenarios where B = NT$400 and NT$500, respectively.

Intuition suggests that $E[B_L^*]$ would be equal to $I/2$. However, the figure shows that $E[B_L^*]/I$ is equal to 1/2 when $C_x < 1.41$ and I is sufficiently small (e.g., I = NT$0.2). It is interesting to note that when $C_x < 1.41$, $E[B_L^*]/I$ almost linearly decreases as I increases. The reason is that as I increases, the probability that the Kth call terminates normally (rather than be forced to terminate by periodical checks) increases. Thus, the expected loss $E[B_L^*]$ becomes smaller than $I/2$. We also observe that when C_x is large (e.g., $C_x > 2.24$), $E[B_L^*]/I$ appears to vary with an irregular pattern. As C_x increases, the number of small and large x_i also increases. When C_x is large, the probability that the last call depletes all or most of the credit becomes large. For the same C_x, the bad debt B_L^* depends on the values of B and I. We can see that the pattern of variation in Fig. 7(a) and (b) are different when $C_x > 2.24$.

C. The Cost Function

Two costs are associated with the service node: the credit checking/updating cost and the bad debt. The credit checking cost and the bad debt are two conflicting factors, since smaller I represents smaller $E[B_L^*]$ and larger $E[N_{ch}^*]$. Consider a cost function $C = E[B_L^*] + \phi E[N_{ch}^*]$, where ϕ is the credit checking cost of the service node. The cost C provides the net effect of credit checking cost and bad debt. Fig. 8 plots C as a function of ϕ and I, where $E[x_i]$ = NT$36 and $Var[x_i] = 1296$. Both fixed credit and recharged credit cases are considered in this experiment. The triangle in the curves represents the cost for the optimal I.

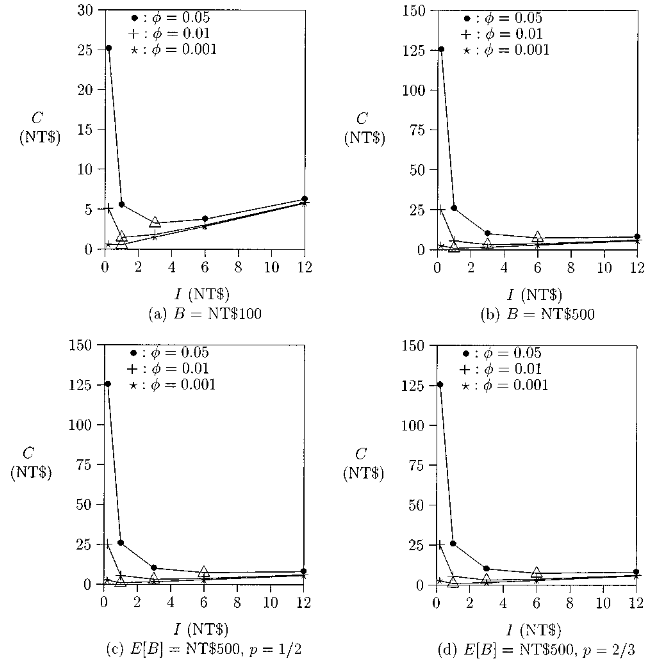

Fig. 8. The cost function C ($E[x_i]$ = NT$36, $Var[x_i] = 1296$).

Consider the fixed credit case where B = NT$500. For $\phi = 0.05$, the credit checking cost is high and I = NT$6 should be selected. For $\phi = 0.001$, the credit checking cost is low and I = NT$1 should be selected. In addition, for the same ϕ, the value of optimal I increases as B increases. Although the above results are intuitive, our analysis quantitatively computes the prepaid service overhead to select the optimal checking interval I according to the capability of the service node. For the examples in Fig. 8 (which are consistent with the real network operation), acceptable I values range from NT$1 to NT$6.

V. CONCLUSION

This paper studied the service-node based approach for the prepaid service. We described the system architecture and the procedures for call origination. An analytical model was proposed to analyze the performance in the fixed credit and the recharged credit cases. The analytic results were validated by simulation experiments. We observed the following results:

- If the call pattern of a prepaid customer is very irregular (i.e., many short calls and many long calls), it is desirable that more credit checks will be needed on the service node. To avoid large number of credit checks on the service node, the service provider can implement billing policies to discourage short calls (e.g., higher rates in the first minute of the prepaid call and lower rates for the remaining call holding time).
- Intuition suggests that the expected bad debt approximates to half of the amount of one credit check. However, our results show that it is incorrect when the variation of call charge is high or the amount of single credit check is large.
- A cost function was used to determine the minimal cost for the service-node-based approach. The minimal cost can be achieved by properly setting the credit checking/updating interval to balance the workload of the service node with

the bad debt. This optimal interval of credit checking can be determined by using our modeling technique.

APPENDIX I
DERIVING $E[B_L^*|$ CASE I)] FOR THE FIXED CREDIT AND ERLANG CALL CHARGE CASE

This appendix derives $E[B_L^*|$ Case i)] for the fixed credit and Erlang call charge case. From (9), A_1 is expressed as

$$A_1 = \left(\frac{1}{p_1}\right)\int_{x_1=\lceil B/I\rceil I}^{\infty}\left(\left\lceil\frac{B}{I}\right\rceil I - B\right) \times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_1^{m_1-1}e^{-m_1\gamma x_1}\right\}dx_1$$
$$= \left(\frac{1}{p_1}\right)\left(\left\lceil\frac{B}{I}\right\rceil I - B\right)\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right] \times e^{-m_1\gamma\lceil B/I\rceil I}. \quad (28)$$

From (10), A_2 is expressed as

$$A_2 = \left(\frac{1}{p_1}\right)\sum_{n=1}^{\infty}\sum_{l=1}^{\lceil B/I\rceil-1} \times\left\{\int_{x_{n+1}=lI}^{\infty}\int_{y_n=B-lI}^{B-(l-1)I}(lI-B) \times\left\{\left[\frac{(m_1\gamma)^{m_1n}}{(m_1n-1)!}\right]y_n^{m_1n-1}e^{-m_1\gamma y_n}\right\} \times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\}dy_n\,dx_{n+1} + \int_{x_{n+1}=lI}^{\infty}\int_{y_n=B-lI}^{B-(l-1)I} \times\left\{\left[\frac{(m_1\gamma)^{m_1n}}{(m_1n-1)!}\right]y_n^{m_1n}e^{-m_1\gamma y_n}\right\} \times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\}dy_n dx_{n+1}\right\}$$
$$= \left(\frac{1}{p_1}\right)\sum_{n=1}^{\infty}\sum_{l=1}^{\lceil B/I\rceil-1} \times\left\{\int_{x_{n+1}=lI}^{\infty}(lI-B) \times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\} \times\left\{\sum_{j=0}^{m_1n-1}\left\{\frac{[m_1\gamma(B-lI)]^j}{j!}\right\}e^{-m_1\gamma(B-lI)} - \sum_{j=0}^{m_1n-1}\left\{\frac{\{m_1\gamma[B-(l-1)I)]\}^j}{j!}\right. \times e^{-m_1\gamma[B-(l-1)I]}\right\}dx_{n+1} + \left(\frac{n}{\gamma}\right) \times\int_{x_{n+1}=lI}^{\infty}\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\} \times\left\{\sum_{j=0}^{m_1n}\frac{[m_1\gamma(B-lI)]^je^{-m_1\gamma(B-lI)}}{j!} - \sum_{j=0}^{m_1n}\left\{\frac{\{m_1\gamma[B-(l-1)I)]\}^j}{j!}\right\} \times e^{-m_1\gamma[B-(l-1)I]}\right\}dx_{n+1}\right\}$$
$$= \left(\frac{1}{p_1}\right)\sum_{n=1}^{\infty}\sum_{l=1}^{\lceil B/I\rceil-1} \times\left\{-\sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma lI)^k}{k!}\right] \times\sum_{j=0}^{m_1n-1}\left[\frac{(m_1\gamma)^j(B-lI)^{j+1}}{j!}\right]e^{-m_1\gamma B} + (B-lI)\sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma lI)^k}{k!}\right] \times\sum_{j=0}^{m_1n-1}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^j}{j!}\right\}e^{-m_1\gamma(B+I)} + \left(\frac{n}{\gamma}\right)\sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma lI)^k}{k!}\right] \times\sum_{j=0}^{m_1n}\left\{\frac{[m_1\gamma(B-lI)]^j}{j!}\right\}e^{-m_1\gamma B} - \left(\frac{n}{\gamma}\right) \times\sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma lI)^k}{k!}\right] \times\sum_{j=0}^{m_1n}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^j}{j!}\right\}e^{-m_1\gamma(B+I)}\right\}. \quad (29)$$

From (11), A_3 is expressed as

$$A_3 = \left(\frac{1}{p_1}\right)\sum_{n=1}^{\infty} \times\left\{\int_{x_{n+1}=\lceil B/I\rceil I}^{\infty}\int_{y_n=0}^{B-(\lceil B/I\rceil-1)I}\left(\left\lceil\frac{B}{I}\right\rceil I - B\right) \times\left\{\left[\frac{(m_1\gamma)^{m_1n}}{(m_1n-1)!}\right]y_n^{m_1n-1}e^{-m_1\gamma y_n}\right\} \times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\}dy_n\,dx_{n+1} + \int_{x_{n+1}=\lceil B/I\rceil I}^{\infty}\int_{y_n=0}^{B-(\lceil B/I\rceil-1)I} \times\left\{\left[\frac{(m_1\gamma)^{m_1n}}{(m_1n-1)!}\right]y_n^{m_1n}e^{-m_1\gamma y_n}\right\} \times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\}\right.$$

$$
\begin{aligned}
&\times dy_n\, dx_{n+1}\Bigg\} \\
&= \left(\frac{1}{p_1}\right)\sum_{n=1}^{\infty}\Bigg\{\int_{x_{n+1}=\lceil B/I\rceil I}^{\infty}\left(\left\lceil\frac{B}{I}\right\rceil I - B\right) \\
&\quad\times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\} \\
&\quad\times\left\{1-\sum_{j=0}^{m_1 n-1}\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^j}{j!}\right. \\
&\quad\left.\times e^{-m_1\gamma[B-(\lceil B/I\rceil-1)I]}\right\} dx_{n+1} \\
&\quad+\left(\frac{n}{\gamma}\right)\int_{x_{n+1}=\lceil B/I\rceil I}^{\infty} \\
&\quad\times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\} \\
&\quad\times\left\{1-\sum_{j=0}^{m_1 n}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^j}{j!}\right\}\right. \\
&\quad\left.\times e^{-m_1\gamma[B-(\lceil B/I\rceil-1)I]}\right\} dx_{n+1}\Bigg\} \\
&= \left(\frac{1}{p_1}\right)\sum_{n=1}^{\infty}\Bigg\{\left(\left\lceil\frac{B}{I}\right\rceil I - B + \frac{n}{\gamma}\right) \\
&\quad\times\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right]e^{-m_1\gamma\lceil B/I\rceil I} \\
&\quad+\left(B-\left\lceil\frac{B}{I}\right\rceil I\right)\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right] \\
&\quad\times\sum_{j=0}^{m_1 n-1}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^j}{j!}\right\} \\
&\quad\times e^{-m_1\gamma(B+I)} \\
&\quad-\left(\frac{n}{\gamma}\right)\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right] \\
&\quad\times\sum_{j=0}^{m_1 n}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^j}{j!}\right\} \\
&\quad\times e^{-m_1\gamma(B+I)}\Bigg\}. \qquad (30)
\end{aligned}
$$

From (8) and (28)–(30), $E[B_L^*|\text{ Case i})]$ can be obtained.

Appendix II
Deriving $E[B_L^*|$ Case ii)] for the Fixed Credit and Erlang Call Charge Case

This appendix derives $E[B_L^*|\text{ Case ii})]$ for the fixed credit and Erlang call charge case. From (16), B_1 is expressed as

$$
\begin{aligned}
B_1 &= \left(\frac{1}{p_2}\right)\int_{x_1=B}^{\lceil B/I\rceil I}(x_1-B) \\
&\quad\times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_1^{m_1-1}e^{-m_1\gamma x_1}\right\}dx_1 \\
&= \left(\frac{1}{p_2\gamma}\right)\Bigg\{\sum_{k=0}^{m_1}\left[\frac{(m_1\gamma B)^k}{k!}\right]e^{-m_1\gamma B} \\
&\quad-\sum_{k=0}^{m_1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right]e^{-m_1\gamma\lceil B/I\rceil I}\Bigg\}-\left(\frac{B}{p_2}\right) \\
&\quad\times\Bigg\{\sum_{k=0}^{m_1-1}\left[\frac{(m_1\gamma B)^k}{k!}\right]e^{-m_1\gamma B} \\
&\quad-\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right]e^{-m_1\gamma\lceil B/I\rceil I}\Bigg\}. \qquad (31)
\end{aligned}
$$

From (17), B_2 is expressed as

$$
\begin{aligned}
B_2 &= \left(\frac{1}{p_2}\right)\sum_{n=1}^{\infty}\sum_{l=1}^{\lceil B/I\rceil-1} \\
&\quad\times\Bigg\{\int_{x_{n+1}=(l-1)I}^{lI}\int_{y_n=B-x_{n+1}}^{B-(l-1)I} \\
&\quad\times(x_{n+1}-B)\left\{\left[\frac{(m_1\gamma)^{m_1 n}}{(m_1 n-1)!}\right]y_n^{m_1 n-1}e^{-m_1\gamma y_n}\right\} \\
&\quad\times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\}dy_n\,dx_{n+1} \\
&\quad+\int_{x_{n+1}=(l-1)I}^{lI}\int_{y_n=B-x_{n+1}}^{B-(l-1)I} \\
&\quad\times\left\{\left[\frac{(m_1\gamma)^{m_1 n}}{(m_1 n-1)!}\right]y_n^{m_1 n}e^{-m_1\gamma y_n}\right\}
\end{aligned}
$$

$$\times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\}$$
$$\times\, dy_n\, dx_{n+1}\Bigg\}$$
$$=\left(\frac{1}{p_2}\right)\sum_{n=1}^{\infty}\sum_{l=1}^{\lceil B/I\rceil-1}\left\{\int_{x_{n+1}=(l-1)I}^{lI}(x_{n+1}-B)\right.$$
$$\times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\}$$
$$\times\left\{\sum_{k=0}^{m_1n-1}\frac{[m_1\gamma(B-x_{n+1})]^k e^{-m_1\gamma(B-x_{n+1})}}{k!}\right.$$
$$\left.-\sum_{k=0}^{m_1n-1}\frac{\{m_1\gamma[B-(l-1)I]\}^k e^{-m_1\gamma[B-(l-1)I]}}{k!}\right\}$$
$$\times\, dx_{n+1}+\left(\frac{n}{\gamma}\right)\int_{x_{n+1}=(l-1)I}^{lI}$$
$$\times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\}$$
$$\times\left\{\sum_{k=0}^{m_1n}\left\{\frac{[m_1\gamma(B-x_{n+1})]^k}{k!}\right\}\right.$$
$$\times\, e^{-m_1\gamma(B-x_{n+1})}-\sum_{k=0}^{m_1n}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}$$
$$\left.\left.\times\; e^{-m_1\gamma[B-(l-1)I]}\right\}dx_{n+1}\right\}$$
$$=\left(\frac{1}{p_2}\right)\sum_{n=1}^{\infty}\sum_{l=1}^{\lceil B/I\rceil-1}$$
$$\times\left\{\sum_{k=0}^{m_1n-1}\left\{\left[\frac{(m_1\gamma)^{m_1+k}}{(m_1-1)!k!}\right]\right.\right.$$
$$\times\, e^{-m_1\gamma B}\sum_{j=0}^{k+1}(-1)^{k+j}\binom{k+1}{j}$$
$$\left.\times\left[\frac{B^j(lI)^{m_1+k-j+1}}{m_1+k-j+1}\right]\right\}$$
$$-\sum_{k=0}^{m_1n-1}\left\{\left[\frac{(m_1\gamma)^{m_1+k}}{(m_1-1)!k!}\right]\right.$$
$$\times\, e^{-m_1\gamma B}\sum_{j=0}^{k+1}(-1)^{k+j}\binom{k+1}{j}$$
$$\left.\times\left\{\frac{B^j[(l-1)I]^{m_1+k-j+1}}{m_1+k-j+1}\right\}\right\}-\left(\frac{1}{\gamma}\right)$$
$$\times\sum_{k=0}^{m_1n-1}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}e^{-m_1\gamma B}$$
$$\times\sum_{j=0}^{m_1}\frac{[m_1\gamma(l-1)I]^j}{j!}+\left(\frac{1}{\gamma}\right)$$
$$\times\sum_{k=0}^{m_1n-1}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}e^{-m_1\gamma(B+I)}$$
$$\times\sum_{j=0}^{m_1}\frac{(m_1\gamma lI)^j}{j!}+B$$
$$\times\sum_{k=0}^{m_1n-1}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}e^{-m_1\gamma B}$$
$$\times\sum_{j=0}^{m_1-1}\frac{[m_1\gamma(l-1)I]^j}{j!}-B$$
$$\times\sum_{k=0}^{m_1n-1}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}e^{-m_1\gamma(B+I)}$$
$$\times\sum_{j=0}^{m_1-1}\frac{(m_1\gamma lI)^j}{j!}+\left(\frac{n}{\gamma}\right)$$
$$\times\sum_{k=0}^{m_1n}\left\{\left[\frac{(m_1\gamma)^{m_1+k}}{(m_1-1)!k!}\right]e^{-m_1\gamma B}\right.$$
$$\left.\times\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}\left[\frac{B^j(lI)^{m_1+k-j}}{m_1+k-j}\right]\right\}-\left(\frac{n}{\gamma}\right)$$
$$\times\sum_{k=0}^{m_1n}\left\{\left[\frac{(m_1\gamma)^{m_1+k}}{(m_1-1)!k!}\right]e^{-m_1\gamma B}\right.$$
$$\left.\times\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}\left[\frac{B^j[(l-1)I]^{m_1+k-j}}{m_1+k-j}\right]\right\}-\left(\frac{n}{\gamma}\right)$$
$$\times\sum_{k=0}^{m_1n}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}e^{-m_1\gamma B}$$
$$\times\sum_{j=0}^{m_1-1}\frac{[m_1\gamma(l-1)I]^j}{j!}+\left(\frac{n}{\gamma}\right)$$
$$\times\sum_{k=0}^{m_1n}\left\{\frac{\{m_1\gamma[B-(l-1)I]\}^k}{k!}\right\}e^{-m_1\gamma(B+I)}$$
$$\left.\times\sum_{j=0}^{m_1-1}\frac{(m_1\gamma lI)^j}{j!}\right\}. \tag{32}$$

From (18), B_3 is expressed as

$$B_3=\left(\frac{1}{p_2}\right)\sum_{n=1}^{\infty}\left\{\int_{x_{n+1}=(\lceil B/I\rceil-1)I}^{\lceil B/I\rceil I}\int_{y_n=0}^{B-(\lceil B/I\rceil-1)I}\right.$$
$$\times\,(x_{n+1}-B)\left\{\left[\frac{(m_1\gamma)^{m_1n}}{(m_1n-1)!}\right]y_n^{m_1n-1}e^{-m_1\gamma y_n}\right\}$$
$$\times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\}dy_n\,dx_{n+1}$$
$$+\int_{x_{n+1}=(\lceil B/I\rceil-1)I}^{\lceil B/I\rceil I}\int_{y_n=0}^{B-(\lceil B/I\rceil-1)I}$$
$$\times\left\{\left[\frac{(m_1\gamma)^{m_1n}}{(m_1n-1)!}\right]y_n^{m_1n}e^{-m_1\gamma y_n}\right\}$$
$$\times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\}$$
$$\left.\times\, dy_n dx_{n+1}\right\}$$

$$
\begin{aligned}
&= \left(\frac{1}{p_2}\right)\sum_{n=1}^{\infty}\Bigg\{\left(\frac{1}{\gamma}\right)\sum_{k=0}^{m_1}\left\{\frac{\left[m_1\gamma\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^k}{k!}\right\}\\
&\quad\times e^{-m_1\gamma(\lceil B/I\rceil-1)I}-\left(\frac{1}{\gamma}\right)\sum_{k=0}^{m_1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right]\\
&\quad\times e^{-m_1\gamma\lceil B/I\rceil I}-B\sum_{k=0}^{m_1-1}\left\{\frac{\left[m_1\gamma\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^k}{k!}\right\}\\
&\quad\times e^{-m_1\gamma(\lceil B/I\rceil-1)I}+B\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right]\\
&\quad\times e^{-m_1\gamma\lceil B/I\rceil I}-\left(\frac{1}{\gamma}\right)\\
&\quad\times\sum_{k=0}^{m_1n-1}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^k}{k!}\right\}e^{-m_1\gamma B}\\
&\quad\times\sum_{j=0}^{m_1}\frac{\left[m_1\gamma\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^j}{j!}+\left(\frac{1}{\gamma}\right)\\
&\quad\times\sum_{k=0}^{m_1n-1}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^k}{k!}\right\}\\
&\quad\times e^{-m_1\gamma(B+I)}\\
&\quad\times\sum_{j=0}^{m_1}\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^j}{j!}+B\\
&\quad\times\sum_{k=0}^{m_1n-1}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^k}{k!}\right\}e^{-m_1\gamma B}\\
&\quad\times\sum_{j=0}^{m_1-1}\frac{\left[m_1\gamma\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^j}{j!}-B\\
&\quad\times\sum_{k=0}^{m_1n-1}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^k}{k!}\right\}\\
&\quad\times e^{-m_1\gamma(B+I)}\\
&\quad\times\sum_{j=0}^{m_1-1}\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^j}{j!}+\left(\frac{n}{\gamma}\right)\\
&\quad\times\sum_{k=0}^{m_1-1}\left\{\frac{\left[m_1\gamma\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^k}{k!}\right\}e^{-m_1\gamma(\lceil B/I\rceil-1)I}\\
&\quad-\left(\frac{n}{\gamma}\right)\sum_{k=0}^{m_1-1}\left[\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^k}{k!}\right]e^{-m_1\gamma\lceil B/I\rceil I}\\
&\quad-\left(\frac{n}{\gamma}\right)\sum_{k=0}^{m_1n}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^k}{k!}\right\}\\
&\quad\times e^{-m_1\gamma B}\\
&\quad\times\sum_{j=0}^{m_1-1}\frac{\left[m_1\gamma\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]^j}{j!}+\left(\frac{n}{\gamma}\right)\\
&\quad\times\sum_{k=0}^{m_1n}\left\{\frac{\left\{m_1\gamma\left[B-\left(\left\lceil\frac{B}{I}\right\rceil-1\right)I\right]\right\}^k}{k!}\right\}e^{-m_1\gamma(B+I)}\\
&\quad\times\sum_{j=0}^{m_1-1}\frac{\left(m_1\gamma\left\lceil\frac{B}{I}\right\rceil I\right)^j}{j!}\Bigg\}. \qquad (33)
\end{aligned}
$$

From (15) and (31)–(33), $E[B_L^*|\text{ Case ii})]$ can be obtained.

Appendix III
Deriving $\Pr\{y_n < B_I + n_rB_r < y_n + x_{n+1}|N_R = n_r\}$ For the Recharged Credit Case

This appendix derives $\Pr\{y_n < B_I + n_rB_r < y_n + x_{n+1}|N_R = n_r\}$ for the recharged credit case. From (23), $\Pr\{y_n < B_I + n_rB_r < y_n + x_{n+1}|N_R = n_r\}$ is expressed as

$$
\begin{aligned}
&\Pr\{y_n < B_I + n_rB_r < y_n + x_{n+1}|N_R = n_r\}\\
&\quad=\int_{y_n=0}^{B_I+n_rB_r}\int_{x_{n+1}=B_I+n_rB_r-y_n}^{\infty}\\
&\qquad\times\left\{\left[\frac{(m_1\gamma)^{m_1n}}{(m_1n-1)!}\right]y_n^{m_1n-1}e^{-m_1\gamma y_n}\right\}\\
&\qquad\times\left\{\left[\frac{(m_1\gamma)^{m_1}}{(m_1-1)!}\right]x_{n+1}^{m_1-1}e^{-m_1\gamma x_{n+1}}\right\}dx_{n+1}\,dy_n\\
&\quad=\int_{y_n=0}^{B_I+n_rB_r}\left\{\left\{\left[\frac{(m_1\gamma)^{m_1n}}{(m_1n-1)!}\right]y_n^{m_1n-1}e^{-m_1\gamma y_n}\right\}\right.\\
&\qquad\times\sum_{k=0}^{m_1-1}\left\{\frac{[m_1\gamma(B_I+n_rB_r-y_n)]^k}{k!}\right\}
\end{aligned}
$$

$$\times\, e^{-m_1\gamma(B_I+n_rB_r-y_n)}\, dy_n \Bigg\}$$
$$= \sum_{k=0}^{m_1-1} \Bigg\{ \left[\frac{(m_1\gamma)^{m_1n+k}e^{-m_1\gamma(B_I+n_rB_r)}}{(m_1n-1)!k!}\right] \times \int_{y_n=0}^{B_I+n_rB_r} y_n^{m_1n-1}(B_I+n_rB_r-y_n)^k\, dy_n \Bigg\}$$
$$= \sum_{k=0}^{m_1-1} \Bigg\{ \left[\frac{(B_I+n_rB_r)^{m_1n+k}}{m_1n}\right] \times \left[\frac{(m_1\gamma)^{m_1n+k}e^{-m_1\gamma(B_I+n_rB_r)}}{(m_1n-1)!k!}\right] \times \left[\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{m_1n}{m_1n+j}\right] \Bigg\}. \tag{34}$$

From [9], [16]

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{m_1n}{m_1n+j} = \frac{(m_1n)!k!}{(m_1n+k)!}. \tag{35}$$

From (35), (34) can be rewritten as

$$\Pr\{y_n < B_I+n_rB_r < y_n+x_{n+1} | N_R = n_r\} = \sum_{k=0}^{m_1-1}\left\{\frac{[m_1\gamma(B_I+n_rB_r)]^{m_1n+k}}{(m_1n+k)!}\right\} \times e^{-m_1\gamma(B_I+n_rB_r)}. \tag{36}$$

APPENDIX IV
EXPLAINING THE PHENOMENON THAT $E[N_{ch}^*]$ AND $E[B_L^*]$ ARE INSENSITIVE TO THE VARIATION OF CALL CHARGE WHEN $C_x < 5 \times 10^{-3}$

This section explains that $E[N_{ch}^*]$ and $E[B_L^*]$ are insensitive to C_x in the fixed credit and recharged credit cases when $C_x < 5\times10^{-3}$ (i.e., see Figs. 5 and 6). First, we illustrate the effect of C_x on $E[N_{ch}^*]$ in the fixed credit case. When C_x is sufficiently small, the call charges can be considered as fixed. Let x_R be the remaining credit after a customer has made $K-1$ complete calls. Then, we observe that $E[N_{ch}^*]$ can be approximated by the following equation

$$E[N_{ch}^*] \approx E[n_{ch}] * (E[K]-1) + \left\lceil \frac{x_R}{I} \right\rceil. \tag{37}$$

When C_x is sufficiently small, $E[K]$ equals to $\lfloor B/E[x_i] \rfloor + 1$ and x_R can be approximated by $B-(E[K]-1)E[x_i]$. If I cannot divide $E[x_i]$, then $E[n_{ch}]$ can be approximated by $\lceil E[x_i]/I \rceil$. Otherwise, $E[n_{ch}]$ can be approximated by $E[x_i]/I + 0.5$. The reason is that when C_x is sufficiently small, the values of x_i fall in a small interval which is symmetric to $E[x_i]$. Half of the calls whose call charges are smaller than $E[x_i]$ require $E[x_i]/I$ credit checks. The other calls whose call charges are larger than $E[x_i]$ require $E[x_i]/I+1$ credit checks. From (37), $E[N_{ch}^*]$ can be approximated as

$$E[N_{ch}^*] \approx \begin{cases} \left\lceil \frac{E[x_i]}{I} \right\rceil \left\lfloor \frac{B}{E[x_i]} \right\rfloor + \left\lceil \frac{B - \left\lfloor \frac{B}{E[x_i]} \right\rfloor E[x_i]}{I} \right\rceil, & \text{if } I \text{ cannot divide } E[x_i] \\ \left(\frac{E[x_i]}{I} + 0.5\right) \left\lfloor \frac{B}{E[x_i]} \right\rfloor + \left\lceil \frac{B - \left\lfloor \frac{B}{E[x_i]} \right\rfloor E[x_i]}{I} \right\rceil, & \text{if } I \text{ divides } E[x_i]. \end{cases} \tag{38}$$

Next, we illustrate the effect of C_x on $E[B_L^*]$ in the fixed credit case. When $C_x < 5\times10^{-3}$, if I cannot divide x_R, $E[B_L^*]$ can be approximated by $\lceil x_R/I \rceil I - x_R$. Otherwise, if I divides x_R, the zero credit line falls into either just before the service node is going to check the credit or after a credit check has just occurred. The bad debt B_L^* approximates 0 or I, accordingly. Thus, $E[B_L^*]$ can be approximated as

$$E[B_L^*] \approx \begin{cases} \left\lceil \frac{x_R}{I} \right\rceil I - x_R, & \text{if } I \text{ cannot divide } x_R \\ \frac{I}{2}, & \text{if } I \text{ divides } x_R. \end{cases} \tag{39}$$

One can verify that the approximations in (38) and (39) are consistent with the results shown in Fig. 5.

In the recharged credit case, let the number of recharge of a customer be n_r and the number of credit checks be $N_{ch}^*(n_r)$. Then, the prepaid credit B equals to $B_I + n_rB_r$. When $C_x < 5\times10^{-3}$, $N_{ch}^*(n_r)$ can be approximated by (38). We observe that the expected number of credit checks in the recharged credit case can be approximated by $\sum_{n_r=0}^{\infty} p^{n_r}(1-p)N_{ch}^*(n_r)$.

Let $B_L^*(n_r)$ be the bad debt of the customer who has recharged for n_r times. When $C_x < 5\times10^{-3}$, $B_L^*(n_r)$ can be approximated by (39). Thus, in the recharged credit case, $E[B_L^*]$ can be approximated by $\sum_{n_r=0}^{\infty} p^{n_r}(1-p)B_L^*(n_r)$. These approximations are consistent with the results in Fig. 6.

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