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Note

A necessary condition for a graph to be the visibility graph of a simple polygon ☆

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Abstract

Two vertices of a simple polygon P are *visible* if the line segment joining them does not intersect the exterior of P. The *visibility graph* of P is a graph G obtained by representing each vertex of P by a vertex of G and two vertices of G are joined by an edge if and only if their corresponding vertices in P are visible. A graph G is a *visibility graph* if there exists a simple polygon P such that G is isomorphic to the visibility graph of P. No characterization of visibility graphs is available. Coullard and Lubiw derived a necessary condition for a graph to be the visibility graph of a simple polygon. They proved that any 3-connected component of a visibility graph has a 3-clique ordering starting from any triangle. However, Coullard and Lubiw insisted on the non-standard definition of visibility (the line segment joining two visible vertices may not go through intermediate vertices) and they said they do not know if their result will hold under the standard definition of visibility. The purpose of this paper is to prove that Coullard and Lubiw's result still holds under the standard definition of visibility.

Keywords: Computational geometry; Visibility problem; Visibility graph; Polygon

1. Introduction

Our terminology and notation in visibility problem are standard; see [10], except as indicated. Two vertices v_i and v_j of a simple polygon P are *visible* (i.e., v_i and v_j can *see* each other) if the line segment v_iv_j does not intersect the exterior of P. The

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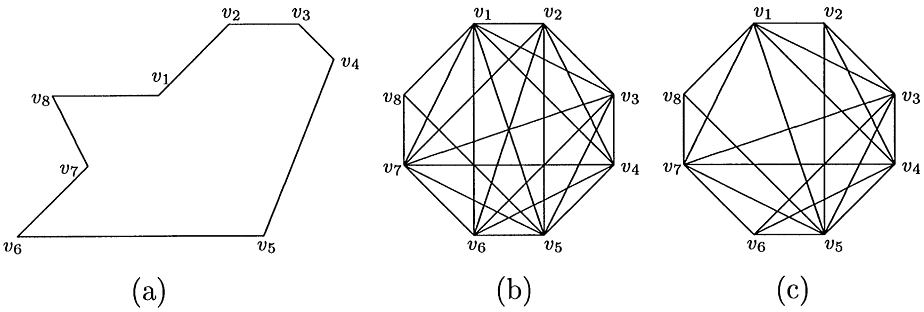

Fig. 1. (a) the polygon P (note that v_6, v_7, v_1, v_2 are collinear), (b) the visibility graph of P under the standard definition of visibility, (c) the visibility graph of P under the non-standard definition of visibility.

visibility graph of P is a graph G obtained by representing each vertex of P by a vertex of G and two vertices of G are joined by an edge if and only if their corresponding vertices in P are visible. See Fig. 1(a) and (b). A graph G is a *visibility graph* if there exists a simple polygon P such that G is isomorphic to the visibility graph of P. For a survey of the visibility problem, refer to [10, 11].

Our terminology and notation in graphs are standard; see [3], except as indicated. Graphs discussed in this paper are assumed simple and finite. Suppose G is a graph and $A \subseteq V(G)$. $G[A]$ is used to denote the subgraph of G induced by A. Let k be a positive integer. A *k-clique* is a complete graph with k vertices. Suppose $[v_1, v_2, \ldots, v_n]$ is a vertex ordering of G; then, A_j is used to denote the set of vertices in $\{v_1, v_2, \ldots, v_{j-1}\}$ that are adjacent to v_j. A *k-clique ordering* of a graph G is a vertex ordering such that the first k vertices form a k-clique and for any other vertex v, the subgraph of G induced by the vertices adjacent to v that precede v in the ordering contains a k-clique. More precisely, a *k-clique ordering* of a graph G is a vertex ordering $[v_1, v_2, \ldots, v_n]$ such that $\{v_1, v_2, \ldots, v_k\}$ form a k-clique and for any other vertex v_j, $G[A_j]$ contains a k-clique. For example, consider the vertex ordering $[v_1, v_2, \ldots, v_8]$ of the graph in Fig. 1(b). Then, $A_4 = \{v_1, v_2, v_3\}$, $A_5 = \{v_1, v_2, v_3, v_4\}$, $A_6 = \{v_1, v_2, v_3, v_4, v_5\}$, $A_7 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $A_8 = \{v_1, v_5, v_7\}$. It is not difficult to verify that $[v_1, v_2, \ldots, v_8]$ is a 3-clique ordering.

The visibility graph problem is to determine, given a graph, whether the graph is a visibility graph. No characterization of visibility graphs is available. It is even not known whether the visibility graph problem is in NP. Some known results are [1, 2, 4–15]. In particular, Coullard and Lubiw [4] derived a necessary condition for a graph to be a visibility graph. They proved that any 3-connected component of a visibility graph has a 3-clique ordering starting from any triangle. The advantage of the 3-clique ordering property over other known properties of visibility graphs is that it can be tested in polynomial time and Coullard and Lubiw [4] had proposed an algorithm for doing it. Coullard and Lubiw [4] also used the 3-clique ordering property to solve the distance visibility graph problem; for details, see their paper. Abello et al. [1] also used the 3-clique ordering property to prove that any 3-connected planar visibility graph is maximal planar and any 4-connected visibility graph is non-planar.

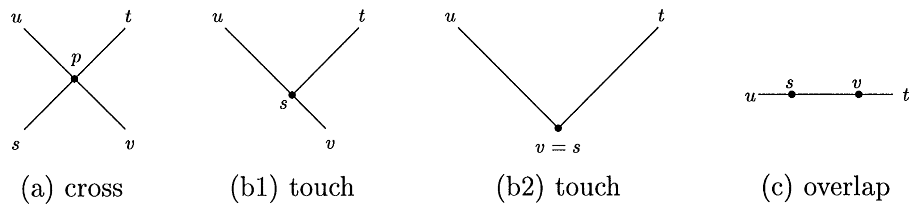

Fig. 2. Three types of intersection.

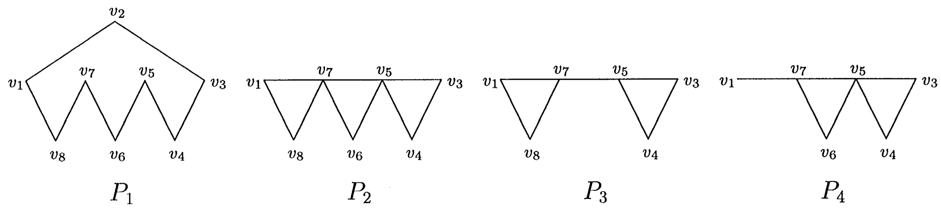

Fig. 3. $P_1 = [v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$, $P_2 = [v_1, v_3, v_4, v_5, v_6, v_7, v_8]$, $P_3 = [v_1, v_3, v_4, v_5, v_7, v_8]$, $P_4 = [v_1, v_3, v_4, v_5, v_6, v_7]$.

The motivation of this paper is: Coullard and Lubiw [4] insisted on the non-standard definition of visibility (the line segment joining two visible vertices may not go through intermediate vertices; see Fig. 1(c)). They said they do not know if their result will hold under the standard definition of visibility. The purpose of this paper is to prove that Coullard and Lubiw's result still holds under the standard definition of visibility.

2. Definitions

We begin with differentiating three type of intersection. Two line segments uv and st *cross* if they intersect at a point $p \neq u, v, s, t$; see Fig. 2(a). Two line segments uv and st *touch* if they intersect at u or v or s or t; see Fig. 2(b1) and (b2). Two line segments uv and st *overlap* if their intersection is a line segment; see Fig. 2(c).

A *polygon* is a cyclically ordered sequence of n distinct vertices $v_1, v_2, \ldots, v_n$ and n edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$ such that no pair of non-consecutive edges cross. Note that we allow a polygon

(1) to have two non-consecutive edges touching or overlapping and

(2) to have two consecutive edges overlapping.

A polygon is *simple* if both (1) and (2) are not allowed. For example, each of P_1, P_2, P_3, P_4 in Fig. 3 is a polygon, but only P_1 is a simple polygon. Given a polygon, we can traverse its boundary clockwise or counterclockwise; in this paper, we assume the *clockwise traversal*, except as indicated.

The *exterior* of a polygon P is the open region of the plane outside P. A *chord* of P is a line segment joining two visible vertices v_i and v_j of P such that v_iv_j is not an edge of P. For example, in Fig. 3, v_1v_3, v_1v_7, v_2v_5, v_2v_6 are chords of P_1. Two chords ab and xy *interlace* if when we traverse the boundary of P clockwise, the ordering of a, b, x, y is a, x, b, y. For example, in P_1 of Fig. 3, v_1v_5 and v_2v_6 interlace (they also cross); v_1v_5 and v_2v_7 interlace (they also touch); v_1v_5 and v_3v_7 interlace (they also overlap). A *subpolygon* of P is a polygon such that its vertices are vertices of P, its edges are edges or chords of P, and it does not intersect the exterior of P. For example, in Fig. 3, P_2, P_3 and P_4 are subpolygons of P_1; P_3 and P_4 are subpolygons of P_2.

In Coullard and Lubiw's proof [4], every subpolygon is a simple subpolygon, and the 3-clique ordering of visibility graphs has the property that any prefix of the ordering corresponds to a subpolygon of the original polygon. Therefore, they also called the 3-clique ordering a *subpolygon ordering*. They need subpolygons to be simple to make the proof go through. Consider the graph in Fig. 1(b); it is the visibility graph of the polygon if Fig. 1(a). $[v_1, v_2, v_6, v_7, v_5, v_8, v_3, v_4]$ is a 3-clique ordering of it starting from the triangle v_1, v_2, v_6. But under Coullard and Lubiw's definition, this ordering is certainly not a subpolygon ordering, since v_1, v_2, v_6 do not form a simple subpolygon; also, v_1, v_2, v_6, v_7 do not form a simple subpolygon.

It is obvious that a "simple subpolygon" is a " subpolygon" and the converse is not true. It is also obvious that if two chords "cross" then they "interlace" and the converse is not true. The key points of overcoming the difficulty encountered by Coullard and Lubiw [4] are to replace "simple subpolygons" by "subpolygons" and to replace "finding a chord crossed by another chord" by "finding a chord interlaced by another chord." Note that we still follow the definition that a graph G is a *visibility graph* if there exists a "simple polygon" P such that G is isomorphic to the visibility graph of P. But to prove the 3-clique ordering property of visibility graphs, we allow subpolygons to be non-simple.

3. The main result

Lemma 1 (see also [4]). *Let P be a simple polygon such that its visibility graph G is* 3*-connected, and let S be a proper subpolygon of P with at least three vertices. Then there exists an edge ab of S such that* (i) *ab is a chord of P and* (ii) *ab is interlaced by a chord xy of P with $x \notin S$ and $y \in S$.*

Proof. For convenience, we will use the same name for a vertex of P and for its corresponding vertex in G. That is, if v is a vertex of P, then v is also used to denote its corresponding vertex in G. See Fig. 4 for an illustration.

Since S is a proper subpolygon of P, S has an edge ab that is a chord of P. Assume that a occurs before b when we traverse S clockwise. Let X be the set of vertices of P encountered after a and before b, and let Y be the set of vertices of P encountered after b and before a. Note that all of the vertices of S are in Y and X is non-empty. Let $X'(Y')$ be the corresponding set of vertices of $X(Y)$ in G. Since G is 3-connected,

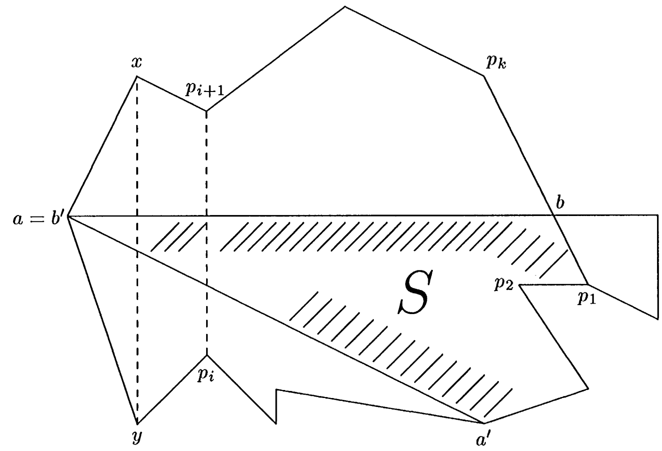

Fig. 4. An illustration of the proof of Lemma 1.

G has an edge xy such that $x \in X'$ and $y \in Y'$. Then x sees y in P and xy is a chord of P. It is clear that ab is interlaced by xy and $x \notin S$. If $y \in S$, then we are done. In the following, assume that $y \notin S$.

Let $a'(b')$ be the vertex of S that is closest to y when we traverse S clockwise (counterclockwise). Then $a'b'$ is an edge of S and is a chord of P. Since S contains at least three vertices, it is not possible that $a' = b$ and $b' = a$. Without loss of generality, assume that $a' \neq b$.

Let P' be the subpolygon of P formed by the chord yx, the vertices of P from x to b, the vertices of S from b to a', and the vertices of P from a' to y. Traverse P' clockwise starting from b; let $p_1, p_2, \ldots, p_k$ be the sequence of vertices of P' that are visible from b, encountered during the traversal. Then p_i and p_{i+1} are visible for all i, $1 \leqslant i \leqslant k-1$. Let U be the set of vertices of P' from b to y, and let V be the set of vertices of P' from x to b. Then there exists an index i such that $p_i \in U$ and $p_{i+1} \in V$ (note that it is possible that $p_i = y$ and it is possible that $p_{i+1} = x$). Note that both $p_{i+1}p_i$ and p_ib are chords of P. If $p_i \in S$, then ab is interlaced by $p_{i+1}p_i$ with $P_{i+1} \notin S$ and $p_i \in S$. If $p_i \notin S$, then $a'b'$ is interlaced by p_ib with $p_i \notin S$ and $b \in S$. □

Theorem 2. *Let P be a simple polygon such that its visibility graph G is* 3-*connected. If S is a proper subpolygon of P with at least three vertices, then there exists a vertex $p \in P - S$ and there exist three vertices $a, b, w \in S$ such that* (i) *a, b, w can see each other* (*i.e., any two of them are visible*), (ii) *p can see all of a, b, w,* (iii) *$p \cup S$* (*with the ordering inherited from P*) *form a subpolygon of P.*

Proof. By Lemma 1, there exists an edge ab of S such that ab is a chord of P and ab is interlaced by a chord xy of P with $x \notin S$ and $y \in S$. Let a, x, b, y be the order of the four vertices a, b, x, y when we traverse P clockwise. For convenience, let $V(S)$

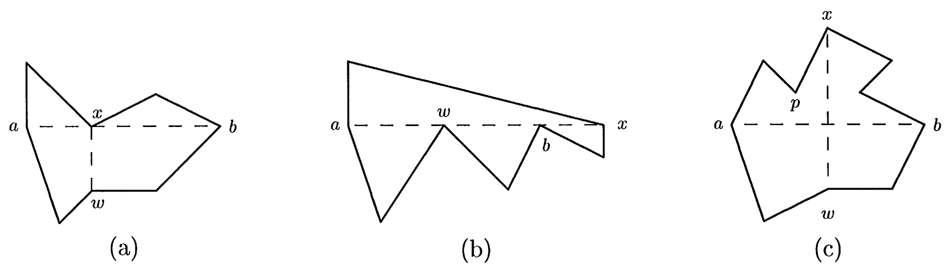

Fig. 5. An illustration of the proof of Theorem 2.

denote the set of vertices of S and let $|uv|$ denote the length of an edge (or a chord) uv of P.

We first find $w \in S$ such that a, b, w can see each other. Note that ab is a chord of P and therefore a can see b. If y can see both a and b, then let $w = y$; clearly, a, b, w can see each other. Now assume that y cannot see both a and b. Since collinear vertices can see each other, a, b, y are not collinear. Thus a, b, y form a triangle $\triangle aby$. Let

$$Q = \{z \in V(S) \backslash \{a, b, y\}\colon z \text{ is inside } \triangle\, aby \text{ or on the boundary of } \triangle\, aby\}.$$

Since y cannot see both a and b, there must exist a vertex of S that blocks y to see a or blocks y to see b; this vertex is inside $\triangle aby$. Therefore $Q \neq \emptyset$. Choose $w \in Q$ such that

$$|wa| + |wb| = \min_{z \in Q}\{|za| + |zb|\}.$$

Then w can see both a and b. Hence a, b, w can see each other.

We now find $p \in P - S$ such that p can see all of a, b, w. There are three cases:

Case 1: *a, x, b are collinear and w is not collinear with a, x, b.* See Fig. 5(a) for an illustration. Since collinear vertices are visible, a, x, b can see each other. We claim that x is on the line segment ab. Suppose this is not true, i.e., x is not on the line segment ab. Since a, x, b are collinear and ab is interlaced by xy, y must be collinear with a, x, b. Since collinear vertices can see each other, y can see both a and b. In our method of choosing w, if y sees both a and b, then we will choose $w = y$. Then w is collinear with a, x, b; this contradicts the assumption that w is not collinear with a, x, b. From the above, x is on the line segment ab. Let $p = x$. Then $p \in P - S$ and p can see all of a, b, w.

Case 2: *a, x, b are collinear and w is also collinear with a, x, b.* See Fig. 5(b) for an illustration. In this case, let $p = x$. Then $p \in P - S$. Since collinear vertices can see each other, p can see all of a, b, w.

Case 3: *a, x, b are not collinear.* See Fig. 5(c) for an illustration. In this case, a, x, b from a triangle $\triangle axb$. Let X be the set of vertices of P encountered after a and before b during a clockwise traversal of P. Then $X \cap V(S) = \emptyset$. Let

$$Q' = \{z \in X\colon\ z \text{ is inside } \triangle\, axb \text{ or on the boundary of } \triangle\, axb\}.$$

Since $x \in Q'$, $Q' \neq \emptyset$. Choose $p \in Q'$ such that

$$|pa| + |pb| = \min_{z \in Q'}\{|za| + |zb|\}.$$

Then p can see all of a, b, w. Since $p \in X$ and $X \cap V(S) = \emptyset$, $p \in P - S$.

From the above, there exists a vertex $p \in P - S$ and there exist three vertices $a, b, w \in S$ such that a, b, w can see each other and p can see all of a, b, w. Since a, b, w can see each other and p can see all of a, b, w, we know that $p \cup S$ (with the ordering inherited from P) form a subpolygon of P. □

The following is the main result.

Theorem 3. *Any* 3*-connected visibility graph has a* 3*-clique ordering starting from any triangle.*

Proof. Let G be a 3-connected visibility graph and let P be its corresponding simple polygon. For convenience, we will use the same name for a vertex of G and for its corresponding vertex in P. Let v_1, v_2, v_3 be an arbitrary triangle of G. Since v_1, v_2, v_3 can see each other, they form a subpolygon of P. Call this subpolygon S_3. Since G is 3-connected, G has at least four vertices. Thus P has at least four vertices and therefore S_3 is a proper subpolygon of P. Vertex v_i and subpolygon S_i (for $i = 4, 5, \ldots, n$) are found in the following way: Since S_{i-1} is a proper subpolygon of P with at least three vertices, by Theorem 2, there exists a vertex $p \in P - S_{i-1}$ and there exist three vertices $a, b, w \in S_{i-1}$ such that (i) a, b, w can see each other, (ii) p can see all of a, b, w, and (iii) $p \cup S_{i-1}$ (with the ordering inherited from P) form a subpolygon of P; let $v_i = p$ and let $S_i = p \cup S_{i-1}$. It is not difficult to see that $[v_1, v_2, \ldots, v_n]$ is a 3-clique ordering of G. □

A pair of vertices u, v of a connected graph G is a *separating pair* if deleting u, v from G disconnects G. Note that any visibility graph is 2-connected, since it has a hamiltonian cycle corresponding to the polygon boundary. We now prove that:

Corollary 4. *Any* 3*-connected component of a visibility graph has a* 3*-clique ordering starting from any triangle.*

Proof. Let G be a visibility graph and let P be its corresponding simple polygon. If G is 3-connected, then by Theorem 3, we are done. In the following, assume that G is not 3-connected. For convenience, we will use the same name for a vertex of G and for its corresponding vertex in P. Since G is not 3-connected, G has a separating pair u, v. Since G has a hamiltonian cycle, $G \backslash \{u, v\}$ consists of exactly two connected components C_1 and C_2. Set $V_1 = V(C_1)$ and $V_2 = V(C_2)$ for easy writing. Let $V_1'(V_2')$ be the corresponding set of vertices of V_1 (V_2) in P. Also, let $G_1 = G[V_1 \cup \{u, v\}]$ and let $G_2 = G[V_2 \cup \{u, v\}]$. To prove this corollary, it is sufficient to prove that both G_1 and G_2 are visibility graphs.

We claim that no vertex of P is on the line segment uv, except u,v. Suppose this is not true and x is on the line segment uv. Since u,v is a separating pair, $V_1' \neq \emptyset$ and $V_2' \neq \emptyset$. Without loss of generality, assume that $x \in V_1'$. Since P is a simple polygon and $V_2' \neq \emptyset$, x must see a vertex $y \in V_2'$. Hence xy is an edge of G with $x \in V_1$ and $y \in V_2$. This contradicts the assumption that u,v is a separating pair. From the above, no vertex of P is on the line segment uv, except u,v. Thus cutting P along the line segment uv divides P into two simple subpolygons P_1 and P_2. Without loss of generality, assume that x is in P_1. Then it is not difficult to see that G_1 is the visibility graph of P_1 and G_2 is the visibility graph of P_2. We are done. □

References

[1] J. Abello, H. Lin, S. Pisupati, On visibility graphs of simple polygons, Congr. Numer. 90 (1992) 119–128.
[2] T. Andreae, Some results on visibility graphs, Discrete Appl. Math. 40 (1992) 5–17.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, North-Holland, New York, 1976.
[4] C. Coullard, A. Lubiw, Distance visibility graphs, Int. J. Comput. Geom. Appl. 2 (4) (1992) 349–362.
[5] H. ElGindy, Hierarchical decomposition of polygon with applications, Ph.D. Thesis, School of Computer Science, McGill University, Montreal, Quebec, 1985.
[6] H. Everett, Visibility graphs of recognition, Ph.D. Thesis, University of Toronto, Toronto, Ontario, 1990.
[7] H. Everett, D.G. Corneil, Recognizing visibility graphs of spiral polygons, J. Algorithms 11 (1990) 1–26.
[8] S.K. Ghosh, On recognizing and characterizing visibility graphs of simple polygons, Report JHU/EECS-86/14, Department of Electrical Engineering and Computer Science, The Johns Hopkins University, Baltimore, MD, 1986.
[9] F. Luccio, S. Mazzone, C.K. Wong, A note on visibility graphs, Discrete Math. 64 (1987) 209–219.
[10] J. O'Rourke, Art Gallery Theorems and Algorithms, Oxford University Press Inc., New York, 1987.
[11] J. O'Rourke, Computational geometry column 18, Int. J. Comput. Geom. Appl. 3 (1) (1993) 107–113.
[12] J. O'Rourke, J. Rippel, Two segment classes with Hamiltonian visibility graphs, Comput. Geom. Theory Appl. 4 (1994) 209–218.
[13] X. Shen, Q. Hu, A note on minimal visibility graphs, Inform. Process. Lett. 46 (1993) 101.
[14] T.C. Shermer, Recent results in art galleries, Proc. IEEE 80 (1992) 1384–1399.
[15] G. Srinivasaraghavan, A. Mukhopadhyay, A new necessary condition for the vertex visibility graphs of simple polygons, Discrete Comput. Geom. 12 (1994) 65–82.