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Fault-Tolerant Hamiltonicity of Twisted Cubes[1]

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The twisted cube TQ_n, is derived by changing some connection of hypercube Q_n according to specific rules. Recently, many topological properties of this variation cube are studied. In this paper, we consider a faulty twisted n-cube with both edge and/or node faults. Let F be a subset of $V(TQ_n) \cap E(TQ_n)$, we prove that $TQ_n - F$ remains hamiltonian if $|F| \leqslant n-2$. Moreover, we prove that there exists a hamiltonian path in $TQ_n - F$ joining any two vertices u, v in $V(TQ_n) - F$ if $|F| \leqslant n-3$. The result is optimum in the sense that the fault-tolerant hamiltonicity (fault-tolerant hamiltonian connectivity respectively) of TQ_n is at most $n-2$ ($n-3$ respectively).

Key Words: hamiltonian; hamiltonian connected; fault-tolerant; twisted cube.

0. INTRODUCTION AND NOTATION

The architecture of an interconnection network is usually represented by a graph. We use graphs and networks interchangeably. There are a lot of mutually conflicting requirements in designing the topology of interconnection networks. It is almost impossible to design a network which is optimum from all aspects. One has to design a suitable network depending on the requirements and its properties. The hamiltonian property is one of the major requirements in designing the topology of networks. Fault tolerance is also desirable in massive parallel systems that have relatively high probability of failure.

A network is represented as an undirected graph in this paper. For the graph theoretic definition and notation we follow [5]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex* (or *node*) *set* and E is the *edge* (or *link*) *set*. Two nodes a and b are *adjacent* if $(a, b) \in E$. A *path* is a sequence of nodes such that two consecutive nodes are

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adjacent. A path is represented by $\langle v_0 \to v_1 \to v_2 \ldots \to v_{k-1}\rangle$. We also write the path $\langle v_0 \to v_1 \to v_2 \ldots \to v_{k-1}\rangle$ as $\langle v_0 \to P_1 \to v_i \to v_{i+1} \ldots \to v_j \to P_2 \to v_t \to v_{t+1} \ldots \to v_{k-1}\rangle$, where $P_1 = \langle v_0 \to v_1 \ldots \to v_i\rangle$ and $P_2 = \langle v_j \to v_{j+1} \ldots \to v_t\rangle$. A path is a *hamiltonian path* if its nodes are distinct and they span V. A *cycle* is a path with at least three nodes such that the first node is the same as the last node. A cycle is called a *hamiltonian cycle* if it traverses every node of G exactly once. A graph is *hamiltonian* if it has a hamiltonian cycle. Let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be two graphs. Following the definition [17], the *Cartesian product* of G_0 and G_1, denoted by $G_0 \times G_1$, is the graph with the vertex set $V_0 \times V_1$ such that $(x, y) \in E(G_0 \times G_1)$ with $x = (v_0^x, v_1^x)$ and $y = (v_0^y, v_1^y)$ if and only if either ($v_0^x = v_0^y$ and $(v_1^x, v_1^y) \in E(G_1)$) or ($v_1^y = v_1^y$ and $(v_0^x, v_0^y) \in E(G_0)$).

Since node faults and link faults may happen when a network is used, it is practically meaningful to consider faulty networks. The vertex fault-tolerant hamiltonicity and the edge fault-tolerant hamiltonicity, proposed by Hsieh *et al.* [10], measure the performance of the hamiltonian property in the faulty networks. The *vertex fault-tolerant hamiltonicity*, $\mathcal{H}_v(G)$, is defined to be the maximum integer k such that $G - F$ remains hamiltonian for every $F \subset V(G)$ with $|F| \leqslant k$ if G is hamiltonian and undefined if otherwise. Obviously, $\mathcal{H}_v(G) \leqslant \delta(G) - 2$ where $\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$. Similarly, the *edge fault-tolerant hamiltonicity*, $\mathcal{H}_e(G)$, is defined to be the maximum integer k such that $G - F$ remains hamiltonian for every $F \subset E(G)$ with $|F| \leqslant k$ if G is hamiltonian and undefined if otherwise. Again, it is obvious that $\mathcal{H}_e(G) \leqslant \delta(G) - 2$. Many topological properties of graphs have been studied [9, 10, 12, 13, 15, 16]. In [10], Hsieh *et al*, showed that an arrangement graph $A_{n,k}$ remains hamiltonian if the parameters n and k satisfy some conditions and the total number of edge and/or vertex faults is not more than a certain amount, for example, $k(n-k)-2, n-3$, or k. In [12], Latif *et al.* demonstrated that an n-dimensional hypercube with at most $n-2$ link faults is hamiltonian. In [13], Rowley and Bose showed that, with slight modification, a base-d undirected de Bruijn graph with at most $d-1$ edges faults is hamiltonian. In [15], Sung *et al.* demonstrated that a double loop network, which is a digraph with n nodes and $2n$ links, with a node or a link fault is hamiltonian. In [16], Tseng *et al.* proved that an n-dimensional star graph with at most $n-3$ edge faults is hamiltonian. In [9], Huang *et al.* proposed a preliminary result of our current study in this paper.

In this paper, we consider a more general parameter. The *fault-tolerant hamiltonicity*, $\mathcal{H}_f(G)$, is defined to be the maximum integer k such that $G - F$ remains hamiltonian for every $F \subset V(G) \cup E(G)$ with $|F| \leqslant k$ if G is hamiltonian and undefined if otherwise. Obviously, $\mathcal{H}_f(G) \leqslant \min\{\mathcal{H}_v(G), \mathcal{H}_e(G)\} \leqslant \delta(G) - 2$. For technical reasons, we also introduce the term *fault-tolerant hamiltonian connectivity*. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. The *fault-tolerant hamiltonian connectivity*, $\mathcal{H}_f^\kappa(G)$, is defined to be the maximum integer k such that $G - F$ remains hamiltonian connected for every $F \subset V(G) \cap E(G)$ with $|F| \leqslant k$ if G is hamiltonian connected and undefined if otherwise. Obviously, $\mathcal{H}_f^\kappa(G) \leqslant \delta(G) - 3$. A graph G is called *k-fault-tolerant hamiltonian (k-fault-tolerant hamiltonian connected*, respectively) or simply *k-hamiltonian* (*k-hamiltonian connected*, respectively) if it remains hamiltonian (hamiltonian connected, respectively), after removing at most k vertices and/or edges.

Among all interconnection networks proposed in the literature, the hypercube Q_n is one of the most popular topologies. Twisted cube [8], TQ_n, is derived by changing some connections of hypercube Q_n according to specific rules. Recently, many topological properties of this variation cube have been studied: In [8], Hilbers *et al.* first defined the twisted cubes. In [1], Abraham and Padmanabhan proved that the twisted cube supported a better performance than that of the hypercube, although it is an asymmetry network. In [2], Abuelrub and Bettayeb demonstrated that a complete binary tree can be embedded in the twisted cube. In [6], Chang *et al.* showed that the connectivity of the twisted cube TQ_n, is n, the wide diameter and the fault diameter are $\lceil \frac{n}{2} \rceil + 2$, and the twisted cube is a pancyclic network. All these results indicate that the performance of TQ_n is better than that of Q_n in the conditions mentioned in those papers.

In this paper, we prove that TQ_n still remains hamiltonian (hamiltonian connected, respectively), even if it has up to $n-2$ ($n-3$, respectively) edge and/or node faults. This result is optimum in the sense that the fault-tolerant hamiltonicity (fault-tolerant hamiltonian connectivity, respectively) of TQ_n is at most $n-2$ ($n-3$, respectively). Therefore, $\mathscr{H}_f(TQ_n) = n-2$ and $\mathscr{H}^\kappa_f(TQ_n) = n-3$, for $n \geqslant 3$ and n is odd. In contrast with the hypercube, the grid, the mesh, and the torus, the fault-tolerant hamiltonicity property of the twisted cubes is much better. For hypercube network Q_n, it is proved in [12, 14] that the vertex fault-tolerant hamiltonicity of Q_n is equal to 0 and the edge fault-tolerant hamiltonicity of Q_n is equal to $n-2$. Thus, the fault-tolerant hamiltonicity of Q_n is equal to 0 if $n \geqslant 2$. For the grid [3], the mesh [11], and the torus [4, 7] with 2^n vertices, because they are bipartite graphs, there are no hamiltonian cycles even if there is only one vertex fault in these graphs. Therefore, the vertex fault-tolerant hamiltonicity of these graphs is equal to 0. So the fault-tolerant hamiltonicity of these graphs with 2^n vertices is equal to 0.

1. TWISTED CUBE AND ITS PROPERTIES

The vertex set of the twisted n-cube TQ_n, is the set of all binary strings of length n. Let $u = u_{n-1}u_{n-2}\ldots u_1u_0$ be any vertex in TQ_n. For $0 \leqslant i \leqslant n-1$, we define the *ith parity function* $P_i(u) = u_i \oplus u_{i-1} \oplus \cdots \oplus u_0$, where $\oplus$ is the exclusive-or operation. When twisted cube was first defined by Hibers *et al.* [8], the authors only considered twisted n-cubes TQ_n for odd values of n exclusively. Following the definition in [8], we can recursively define TQ_n as follows: A twisted 1-cube, TQ_1, is a complete graph with two vertices 0 and 1. Suppose that $n \geqslant 3$. We can decompose the vertices of TQ_n into four sets, $TQ^{0,0}_{n-2}$, $TQ^{0,1}_{n-2}$, $TQ^{1,0}_{n-2}$ and $TQ^{1,1}_{n-2}$ where $TQ^{i,j}_{n-2}$ consists of those vertices u with $u_{n-1} = i$ and $u_{n-2} = j$. For each $(i, j) \in \{(0,0), (0,1), (1,0), (1,1)\}$, the induced subgraph of $TQ^{i,j}_{n-2}$ in TQ_n is isomorphic to TQ_{n-2}. Edges which connect these four subtwisted cubes can be described as follows: Any node $U = u_{n-1}u_{n-2}\cdots u_1u_0$ with $P_{n-3}(U) = 0$ is connected to $V = v_{n-1}v_{n-2}\cdots v_1v_0$, where $V = \bar{u}_{n-1}\bar{u}_{n-2}u_{n-3}\cdots u_1u_0$ or $V = \bar{u}_{n-1}u_{n-2}u_{n-3}\cdots u_1u_0$; a node U with $P_{n-3}(U) = 1$ is connected to V, where $V = u_{n-1}\bar{u}_{n-2}u_{n-3}\cdots u_1u_0$ or $V = \bar{u}_{n-1}u_{n-2}u_{n-3}\cdots u_1u_0$. TQ_3 and TQ_5 are shown in Figs. 1 and 2, respectively.

From the definition, we have the following lemma.

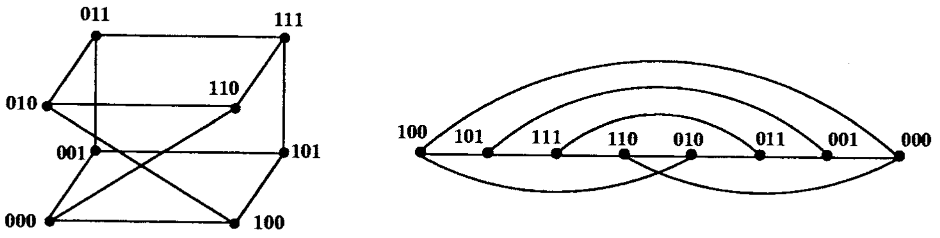

FIG. 1. Twisted 3-cube TQ_3.

LEMMA 1.1. *Both the subgraph induced by $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$ and the subgraph induced by $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$ are isomorphic to $TQ_{n-2} \times K_2$ where K_2 is the complete graph with two vertices. Moreover, the edges joining $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$ and $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$ form a perfect matching of TQ_n.*

2. HAMILTONIAN CYCLES IN FAULTY TWISTED CUBE

Let G^0 and G^1 be two graphs with the same number of nodes, and let M be an arbitrary perfect matching between the nodes of G^0 and G^1; i.e., M is a set of edges connecting the nodes of G^0 and G^1 in a one to one fashion.In this paper, we define a connection graph $G^0 \oplus_M G^1$ as follows, where $G^0 = G^1 = G$. It has two copies of G

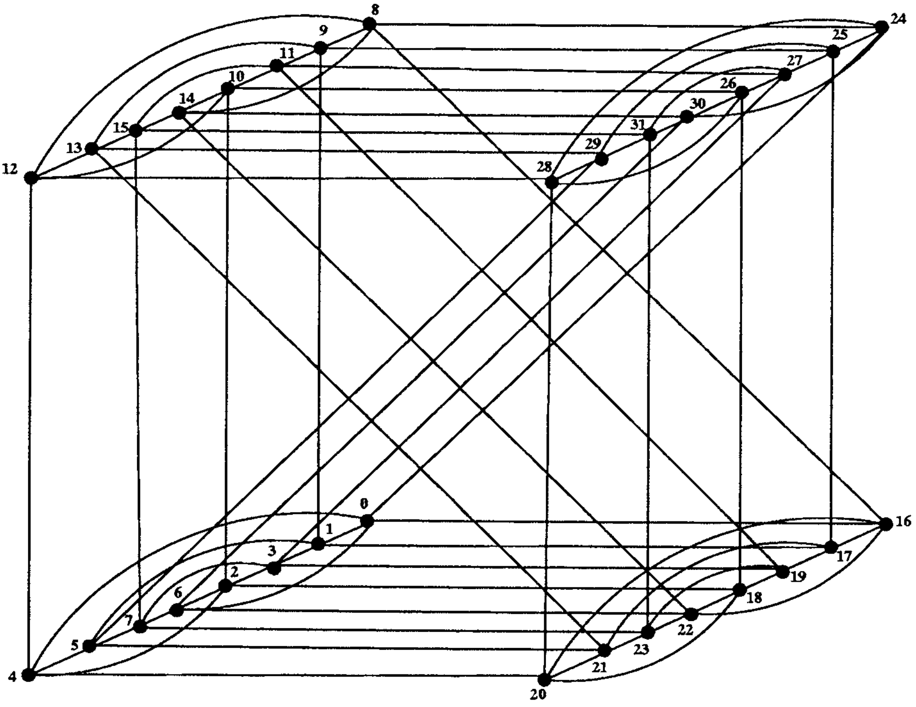

FIG. 2. Twisted 5-cube TQ_5 labeled with decimal number.

connected by a matching M; these two copies of G, denoted by G^0 and G^1, are called two sides of $G^0 \oplus_M G^1$ which has vertex set $V(G^0 \oplus_M G^1) = V(G^0) \cup V(G^1)$ and edge set $E(G^0 \oplus_M G^1) = E(G^0) \cup E(G^1) \cup M$. The matching edges connecting G^0 and G^1 are called *crossing edges*. For each node u of G, u^0 and u^1 are used to denote its two copies in G^0 and G^1, and are called *corresponding nodes* of G^0 and G^1. We also say that the corresponding node of u^0 is u^1. Observe that the product graph $G \times K_2$ is also a connection graph $G \oplus_M G$. Using the above terminologies, $G \times K_2$ has two sides G^0 and G^1 connected by matching $M = \{(u^0, u^1) \mid u \in V(G)\}$. Also $TQ_n = (TQ_{n-2} \times K_2) \oplus_M (TQ_{n-2} \times K_2)$ for a specific perfect matching M.

We will prove that the twisted n-cube TQ_n, for $n \geqslant 3$, has a hamiltonian cycle even if it has up to $n-2$ vertex and/or edge faults. In fact, we will prove a stronger result: TQ_n is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected for $n \geqslant 3$. The basic idea of our proof is by induction on n, and the outline of our proof is as follows: First, we observe that TQ_3 is 1-hamiltonian and hamiltonian connected. Then, assuming the result is true for TQ_k, for $3 \leqslant k \leqslant n$, we show that $TQ_n \times K_2$ is $(n-1)$-hamiltonian and $(n-2)$-hamiltonian connected. Finally, we prove that $TQ_{n+2} = (TQ_n \times K_2) \oplus_M (TQ_n \times K_2)$ is n-hamiltonian and $(n-1)$-hamiltonian connected, and this completes the induction proof. To start our induction, let us look at the twisted 3-cube TQ_3.

In Figs. 3a and 3b, there are two different but equivalent layouts of TQ_3, where the binary node labels are represented by their corresponding decimal numbers. By the node symmetry of TQ_3, it is a simple matter to check that TQ_3 is indeed hamiltonian connected. For example, "0-1-3-2-4-5-7-6," "0-6-2-4-5-1-3-7," "0-1-3-7-6-2-4-5," and "0-1-3-2-6-7-5-4" are hamiltonian paths between nodes and 6, 0 and 7, 0 and 5, and 0 and 4, respectively.

Again by the symmetry of Fig. 3b, we can check that TQ_3 is 1-hamiltonian. For example, if node 1 is faulty, then "0-6-2-3-7-5-4-0" is a fault-free hamiltonian cycle. Moreover, "0-4-2-3-1-5-7-6-0" is a hamiltonian cycle not using edges (0, 1), (2, 6), (3, 7) and (4, 5). Therefore, we have the following lemma.

LEMMA 2.2. *Twisted 3-cube TQ_3 is hamiltonian connected and 1-hamiltonian.*

As another example, TQ_5 shown in Fig. 2 is labeled by a decimal number. We can show that TQ_5 is 2-hamiltonian connected and 3-hamiltonian by applying Theorem 2.2. However, to get some intuition about the results, we check some cases

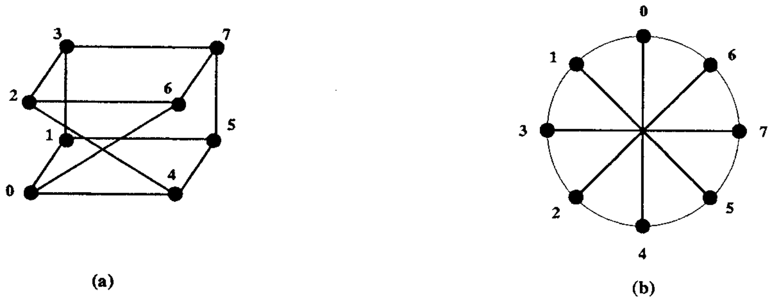

FIG. 3. Equivalent layout of TQ_3.

that TQ_5 is 2-hamiltonian connected. For example, if nodes 0 and 16 are faulty, "1-3-2-6-7-5-4-12-8-9-11-10-14-15-13-21-17-19-23-22-18-20-28-29-25-24-30-26-26-27-31" and "1-5-4-2-3-7-6-30-26-27-31-29-28-24-25-17-21-20-18-19-23-22-14-8-9-11-10-12-13-15" are fault-free hamiltonian paths between nodes 1 and 31 and nodes 1 and 15, respectively. Moreover, we can also check that TQ_5 is 3-hamiltonian. For example, if nodes 0, 8, and 16 are faulty, then "1-3-2-6-7-5-4-20-21-23-22-18-19-17-25-24-30-26-27-31-28-12-1 3-15-14-10-11-9-1" or "1-3-2-6-7-5-4-12-13-15-14-10-11-9-25-24-30-26-27-31-29-28-20-21-23-22-18-19-17-1" is a fault-free hamiltonian cycle in TQ_5.

To prove Theorem 2.1, we shall make use of the structure of $TQ_n \times K_2$. Let TQ_n^0 and TQ_n^1 be the two sides of $TQ_n \times K_2$; each side TQ_n^i (isomorphic to TQ_n) has 2^n nodes, $i=0, 1$. And TQ_n^0 and TQ_n^1 are connected by matching corresponding nodes. Recall that, for each node u of TQ_n, u^0 and u^1 are used to denote corresponding nodes of TQ_n^0 and TQ_n^1. These notations are used extensively throughout Theorem 2.1.

THEOREM 2.1. *Let n be a fixed odd integer and $n \geqslant 3$. If TQ_n is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected, then $TQ_n \times K_2$ is $(n-1)$-hamiltonian and $(n-2)$-hamiltonian connected.*

Proof. Let E_c be the set of crossing edges; that is, $E_c = \{(u^0, u^1) \mid \forall u \in TQ_n\}$. Let F be a faulty set, $F_0 = F \cap TQ_n^0$, $F_1 = F \cap TQ_n^1$, and $F_c = F \cap E_c$. And the cardinalities of F_0, F_1, F_c are f_0, f_1, f_c, respectively. In the following, we shall use the notation $HP^i(u^i, v^i)$ ($P^i(u^i, v^i)$respectively) to denote a hamiltonian path (a path, respectively) in the graph $TQ_n^i - F_i$ joining u^i and v^i for $i=0, 1$, and HC^i to denote a hamiltonian cycle in $TQ_n^i - F_i$ for $i=0, 1$.

First, we prove that $TQ_n \times K_2$ is $(n-1)$-hamiltonian. We will prove that $(TQ_n \times K_2) - F$ has a hamiltonian cycle if $F \subset (V(TQ_n \times K_2) \cup E(TQ_n \times K_2))$ and $|F| = n-1$ with the following two cases.

Case 1.1. $f_i = n-1$ for some $i=0, 1$. (All faults are on one side. See Fig. 4a).

Without loss of generality, we assume that $f_0 = n-1$. Since TQ_n^0 is $(n-2)$-hamiltonian, there exists a hamiltonian path $HP^0(u^0, v^0)$ joining some two vertices u^0, v^0 in $TQ_n^0 - F$. Since TQ_n^1 is hamiltonian connected, there exists another hamiltonian path $HP^1(u^1, v^1)$ in TQ_n^1, where u^i and v^1 are the corresponding nodes of u^0 and v^0, respectively. Then, $\langle u^0 \to HP^0(u^0, v^0) \to v^0 \to v^1 \to HP^1(v^1, u^1) \to u^1 \to u^0 \rangle$ forms a hamiltonian cycle in $(TQ_n \times K_2) - F$.

Case 1.2. $f_i \leqslant n-2$ for $i=0, 1$. (All faults are scattered over E_c, TQ_n^0, or TQ_n^1. See Fig. 4b.)

Since $2^n \geqslant n$, there exists an $u^0 \in V(TQ_n^0)$ such that u^0, (u^0, u^1), $u^1 \notin F$. Since TQ_n^0 (TQ_n^1, respectively) is $(n-2)$-hamiltonian, there exist at least $n-f_0$ ($n-f_1$, respectively) edges incident to u^o (u^1, respectively) in which each edge is on some hamiltonian cycles in $TQ_n^0 - F_0$ ($TQ_n^1 - F_1$, respectively). Now, we have at least $n-f_0$ ($n-f_1$, respectively) hamiltonian cycles and each hamiltonian cycle passes through an edge incident to u^0 (u^1) in $TQ_n^0 - F_0$ ($TQ_n^1 - F_1$, respectively). Because $f_0 + f_1 + f_c = n-1$, $(n-f_0) + (n-f_1) = 2n - f_0 - f_1 = n + (n - f_0 - f_1) = n + 1 + f_c > n$. By the

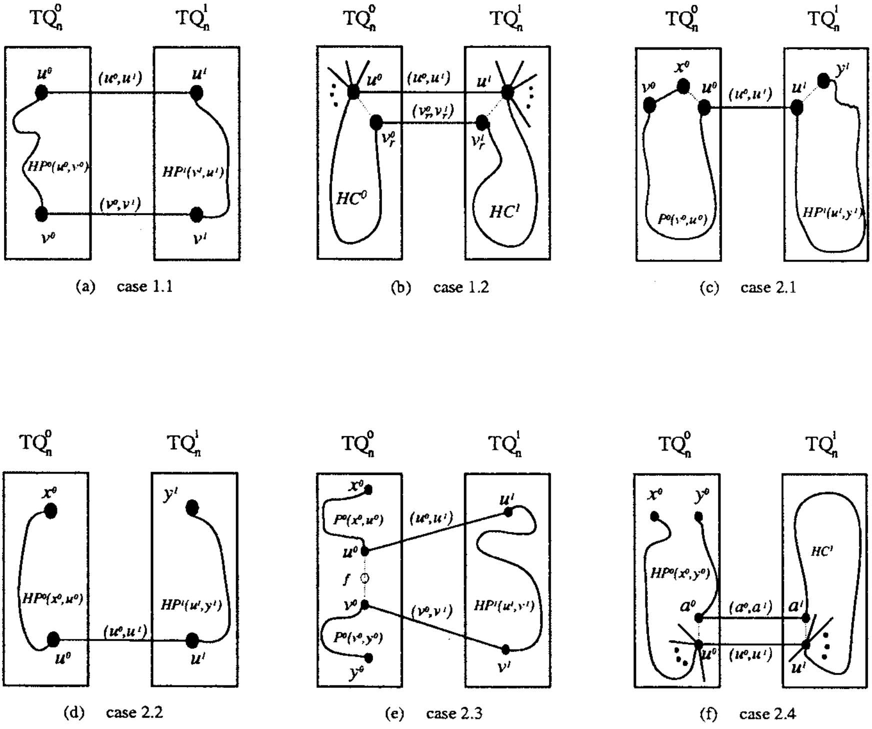

FIG. 4. Illustration for Theorem 2.1.

pigeonhole principle, u^0 (u^1, respectively) has a neighboring node v_r^0 (v_r^1, respectively) such that v_r^0 and v_r^1 are corresponding nodes, v_r^0, (v_r^0, v_r^1), $v_r^1 \notin F$, and (u^0, v_r^0) ((u^1, v_r^1), respectively) are on some hamiltonian cycles HC^0 (HC^1, respectively) in $TQ_n^0 - F_0$ ($TQ_n^1 - F_1$, respectively). Therefore, $HC^0 \cup HC^1 \cup \{(u^0, u^1)\}$, $\{(v^0, v_r^0)\} - \{(u^0, v_r^0), (u^1, v_r^1)\}$ forms a hamiltonian cycle in $(TQ_n \times K_2) - F$.

In the following, we prove that $TQ_n \times K_2$ is $(n-2)$-hamiltonian connected. We will prove that there exists a fault-free hamiltonian path between every pair of vertices x^i and $y^j \in V((TQ_n \times K_2) - F)$, where $F \subset (V(TQ_n \times K_2) \cup E(TQ_n \times K_2))$ and $|F| = n-2$, for $i, j \in \{0, 1\}$. We prove this part by the following cases.

Case 2.1. $i \neq j$, and $f_k = n-2$ for some $k = 0, 1$. (x^i and y^j are on different sides, and all faults are on one side. See Fig. 4c.)

Without loss of generality, we assume that $i = 0$, $j = 1$, and $f_0 = n-2$. Since TQ_n^0 is $(n-2)$-hamiltonian,there exists a hamiltonian cycle HC^0 in $TQ_n^0 - F$. Let $HC^0 = \langle x^0 \to u^0 \to P^0(u^0, v^0) \to v^0 \to x^0 \rangle$ where u^0, v^0 are the vertices incident to x^0 and $P^0(u^0, v^0)$ is a path between u^0, v^o. Because TQ_n^1 is hamiltonian connected, there exists a hamiltonian path between every pair of vertices of TQ_n^1. If $u^1 \neq y^1$, where u^1 is the corresponding node of u^0, then $\langle x^0 \to v^0 \to P^0(v^0, u^0) \to u^0 \to u^1 \to HP^1(u^1, y^1) \to y^1 \rangle$ forms a hamiltonian path joining x^o and y^1 in $TQ_n \times K_2 - F$. Otherwise, $v^1 \neq y^1$ and then $\langle x^0 \to u^0 \to P^0(u^0, v^0) \to v^0 \to v^1 \to HP^1(v^1, y^1) \to y^1 \rangle$ forms a hamiltonian path joining x^0 and y^1 in $TQ_n \times K_2 - F$.

Case 2.2. $i \neq j$ and both $f_0, f_1 \leqslant n-3$. (x^i and y^j are on different sides, and all faults are scattered over E_c, TQ_n^0, or TQ_n^1. See Fig. 4d.)

Without loss of generality, we assume that $i=0$ and $j=1$. Because $2^n \geqslant n+1$ for $n \geqslant 3$, there exists vertices u^0, $u^1 \notin (F \cup \{x^0, y^1\})$ and $(u^0, u^1) \notin F$. Since both TQ_n^0 and TQ_n^1 are $(n-3)$-hamiltonian connected, $n-3 \geqslant f_0$ and $n-3 \geqslant f_1$, the graphs $TQ_n^0 - F_0$ and $TQ_n^1 - F_1$ are hamiltonian connected. Thus there exist hamiltonian paths $HP^0(x^0, u^0)$ and $HP^1(u^1, y^1)$ in $TQ_n^0 - F_0$ and $TQ_n^1 - F_1$, respectively. Therefore, $\langle x^0 \to HP^0(x^0, u^0) \to u^0 \to u^1 \to HP^1(u^1, y^1) \to y^1 \rangle$ forms a hamiltonian path joining x^0 and y^1 in $TQ_n \times K_2 - F$.

Case 2.3. $i=j$ and $f_i = n-2$. (x^i, y^j, and all faults are on the same side. See Fig. 4c.)

Without loss of generality, we assume that $i=j=0$. Let f be a fault of F. Since TQ_n^0 is $(n-3)$-hamiltonian connected, $TQ_n^0 - (F - \{f\})$ contains a hamiltonian path $HP^0(x^0, y^0)$. Thus $TQ_n^0 - F$ contains two node-disjoint paths $P^0(x^0, u^0)$ and $P^0(v^0, y^0)$ where $P^0(x^0, u^0) \cup P^0(v^0, y^0) = HP^0(x^0, y^0) - \{f\}$. Because TQ_n^1 is $(n-3)$-hamiltonian connected and $n-3 \geqslant 0$, there exists a hamiltonian path $HP^1(u^1, v^1)$ in $TQ_n^1 - F_1$. Therefore, $\langle x^0 \to P^0(x^0, u^0) \to u^0 \to u^1 \to HP^1(u^1, v^1) \to v^1 \to v^0 \to P^0(v^0, y^0) \to y^0 \rangle$ forms a hamiltonian path in $(TQ_n \times K_2) - F$.

Case 2.4. $i=j$ and $f_i \leqslant n-3$. (x^i and y^i are on the same side, but not all faults are on the same side with x^i and y^i. See Fig. 4f.)

This case can be proved in a similar way to Case 1.2. Without loss of generality, we assume that $i=j=0$. Because $2^n \geqslant n+1$, there exists u^0, $u^1 \notin (F \cup \{x^0, y^0\})$ such that $(u^0, u^1) \notin F$. Since TQ_n^0 is $(n-3)$-hamiltonian connected and $n-3 \geqslant f_0$, there exist $n-1-f_0$ edges incident to u^0 in which each edge is on some hamiltonian path $HP^0(x^0, y^0)$ in $TQ_n^0 - F_0$. On the other hand, because TQ_n^1 is $(n-2)$-hamiltonian and $n-2 \geqslant f_1$, there exist $n-f_1$ edges incident to u^1 in which each edge is on some hamiltonian cycle in $TQ_n^1 - F_1$. Since $f_0 + f_1 + f_c = n-2$, there exist $n-1-f_0+n-f_1-n = 1+f_c$ vertices, denoted by a_i^0 for $1 \leqslant i \leqslant 1+f_c$, such that (u^0, a_i^0) is on some hamiltonian path in $TQ_n^0 - F_0$ and (u^1, a_i^1) is on some hamiltonian cycle in $TQ_n^1 - F_1$. Hence, there exists an edge $(a^0, a^1) \notin F$ such that (u^0, a^0) is on some hamiltonian path $HP^0(x^0, y^0)$ in $TQ_n^0 - F_0$ and (u^1, a^1) is on some hamiltonian cycle HC^1 in $TQ_n^1 - F_1$. Therefore, $(HP^0(x^0, y^0) \cup HC^1 \cup \{(u^0, u^1), (a^0, a^1)\}) - \{(u^0, a^0), (u^1, a^1)\}$ forms a hamiltonian path joining x^0 and y^0 in $(TQ_n \times K_2) - F$. This theorem is proved. ▮

From Lemma 1.1, $TQ_{n+2} = (TQ_n \times K_2) \oplus_M (TQ_n \times K_2)$ for some perfect matching M. Let G be the graph $TQ_n \times K_2$. The graph $G \oplus_M G$ has two copies of G, denoted by G^0 and G^1. So $G^0 \oplus_M G^1 = G \oplus_M G$. Moreover, the graph G^0 (G^1, respectively) itself has two copies of TQ_n, denoted by TQ_n^{00} and TQ_n^{10} (TQ_n^{01} and TQ_n^{11}, respectively).

Remarks about the notations used below are required. In Theorem 2.1, we consider the graph $TQ_n \times K_2 = TQ_n^0 \oplus_M TQ_n^1$, where TQ_n^0 and TQ_n^1 are connected by matching corresponding nodes. That is, $M = \{(u^0, u^1) \mid \forall u \in TQ_n\}$, where we use u^0

and u^1 to denote corresponding nodes of TQ_n^0 and TQ_n^1. In the following theorem, we consider the graph $TQ_{n+2} = G^0 \oplus_M G^1$, where $G = TQ_n \times K_2$. The matching M, however, does not connect corresponding nodes of G^0 and G^1; it does connect the nodes of G^0 and G^1 in pair and such a pair of nodes are called *matching nodes.* Instead of using superscript, e.g., u^0 and u^1, we shall use small letters with subscript 0 (subscript 1, respectively) to denote the nodes of G^0 (G^1, respectively), e.g., x_0 and u_0, etc. (x_1 and u_1, etc., respectively). The same letters with different subscripts 0 and 1 are used to denote matching nodes; e.g., the matching node of u_0 is u_1. Again, these notations are used extensively throughout the following theorem.

THEOREM 2.2. *Let n be a fixed odd integer for* $n \geqslant 3$. *If* TQ_n *is* $(n-2)$-*hamiltonian and* $(n-3)$-*hamiltonian connected, then* $G^0 \oplus_M G^1$ *is n-hamiltonian and* $(n-1)$-*hamiltonian connected, where* $G^0 = G^1 = G = TQ_n \times K_2$.

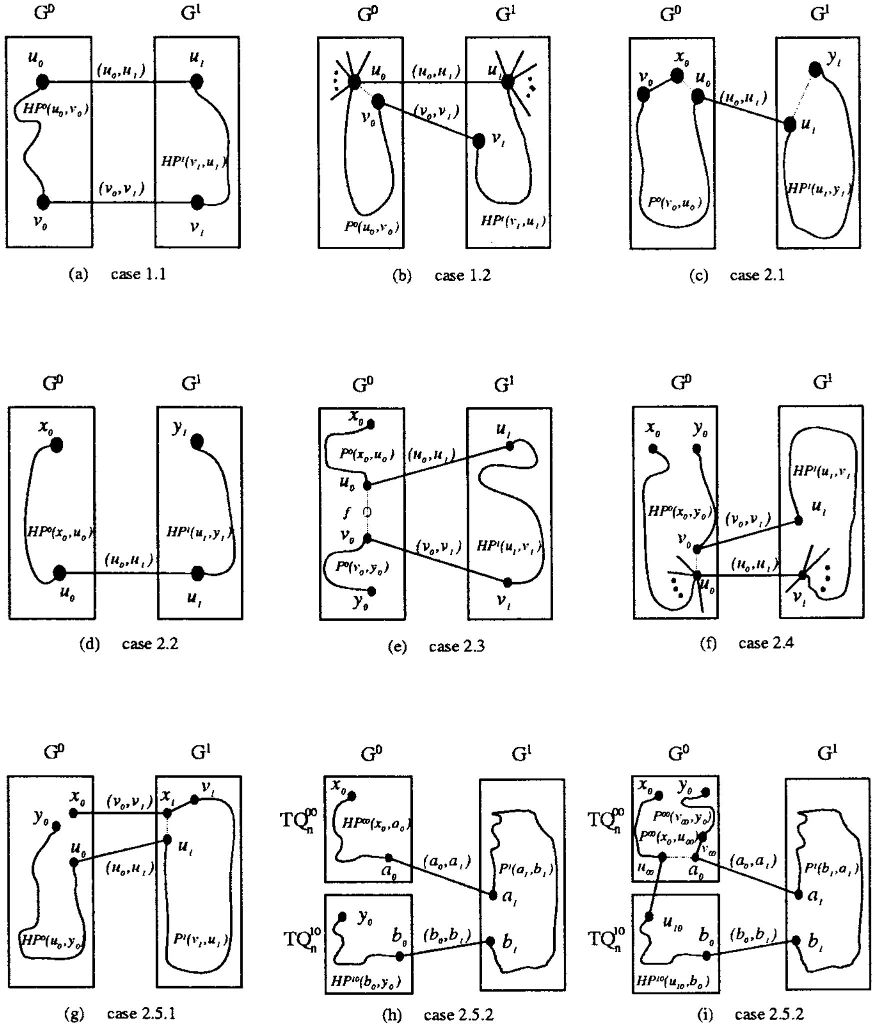

FIG. 5. Illustration for Theorem 2.2.

Proof. Applying Theorem 2.1, we know that $G = TQ_n \times K_2$ is $(n-1)$-hamiltonian and $(n-2)$-hamiltonian connected. Let E_c be the set of crossing edges; that is, $E_c = \{(u_0, u_1) \mid (u_0, u_1) \in M\}$. Let F be a faulty set, $F_0 = F \cap G^0$, $F_1 = F \cap G^1$, and $F_c = F \cap E_c$. And the cardinalities of F_0, F_1, F_c are f_0, f_1, f_c, respectively. In the following, we shall use the notation $HP^i(u_i, v_i)$ ($P^i(u_i, v_i)$, respectively) to denote a hamiltonian path (a path, respectively) in the graph $G^i - F_i$ joining u_i and v_i for $i = 0, 1$, and HC^i to denote a hamiltonian cycle in $G^i - F_i$ for $i = 0, 1$.

In order to prove that $G^0 \oplus_M G^1$ is n-hamiltonian, we will prove that $(G^0 \oplus_M G^1) - F$ has a hamiltonian cycle, if $F \subset (V(G^0 \oplus_M G^1) \cup E(G^0 \oplus_M G^1))$ and $|F| = n$ with the following two cases.

Case 1.1. $f_i = n$ for some $i = 0, 1$. (All faults are on one side. See Fig. 5a.)

Without loss of generality, we assume that $f_0 = n$. Since $G^0 = TQ_n \times K_2$ is $(n-1)$-hamiltonian, there exists a hamiltonian path $HP^0(u_0, v_0)$ joining some two vertices u_0, v_0 in $G^0 - F$. Since $G^1 = TQ_n \times K_2$ is hamiltonian connected, there exists another hamiltonian path $HP^1(u_1, v_1)$, where u_1 and v_1 are the matching nodes of u_0 and v_0. Then, $\langle u_0 \to HP^0(u_0, v_0) \to v_0 \to v_1 \to HP^1(v_1, u_1) \to u_1 \to u_0 \rangle$ forms a hamiltonian cycle in $(G_0 \oplus_M G^1) - F$.

Case 1.2. $f_i \leqslant n-1$ for both $i = 0, 1$. (All faults are scattered over E_c, G^0, or G^1. See Fig. 5b.)

Without loss of generality, we assume that $f_0 \geqslant f_1$. Because $f_0 + f_1 \leqslant n$ and $f_1 \leqslant f_0 \leqslant n-1$ for $n \geqslant 3$, therefore $f_1 \leqslant n-2$. Since G^0 is $(n-1)$-hamiltonian and $n-1 \geqslant f_0$, there exists a hamiltonian cycle HC^0 in $G^0 - F_0$. And G^1 is $(n-2)$-hamiltonian connected and $n-2 \geqslant f_1$, $G^1 - F_1$ is a hamiltonian-connected graph. Since $2^{n+1} > 2n+1$ for $n \geqslant 3$, there exist two vertices u_0, v_0 such that edge (u_0, v_0) is on HC^0 and $u_0, v_0, u_1, v_1, (u_0, u_1), (v_0, v_1) \notin F$. Thus, there exists a hamiltonian path $HP^1(v_1, u_1)$. Let $HC^0 = \langle v_0 \to u_0 \to P^0(u_0, v_0) \to v_0 \rangle$. Then, $\langle u_0 \to P^0(u_0, v_0) \to v_0 \to v_1 \to HP^1(v_1, u_1) \to u_1 \to u_0 \rangle$ forms a hamiltonian cycle in $(G^0 \oplus_M G^1) - F$.

Then, we prove that $(G^0 \oplus_M G^1)$ is $(n-1)$-hamiltonian connected. In other words, we will prove that there exists a fault-free hamiltonian path between every pair of vertices x_i and $y_j \in (V(G^0 \oplus_M G^1) - F)$, where $F \subset (V(G^0 \oplus_M G^1) \cup E(G^0 \oplus_M G^1))$ and $|F| = n-1$ for $i, j \in \{0, 1\}$. We prove this part by the following cases.

Case 2.1. $i \neq j$ and $f_k = n-1$ for some $k = 0, 1$. (x_i and y_j are on different sides, and all faults are on one side. See Fig. 5c.)

Without loss of generality, we assume that $i = 0$, $j = 1$, and $f_0 = n-1$. Since G_0 is $(n-1)$-hamiltonian, there exists a hamiltonian cycle HC^0 in $G^0 - F$. Let $HC^0 = \langle x_0 \to u_0 \to P^0(u_0, v_0) \to v_0 \to x_0 \rangle$, in which u_0, v_0 are the vertices incident to x_0 and $P^0(u_0, v_0)$ is a path between u_0, v_0. Because G^1 is hamiltonian connected, there exists a hamiltonian path between every pair of vertices of G^1. If $u_1 \neq y_1$ where u_1 is the matching nodes of u_0, then $\langle x_0 \to v_0 \to P^0(v_0, u_0) \to u_0 \to u_1 \to HP^1(u_1, y_1) \to y_1 \rangle$ forms a hamiltonian path joining x_0 and y_1 in $(G^0 \oplus_M G^1) - F$. Otherwise, $v_1 \neq y_1$ where v_1 is in the matching nodes of v_0, and then $\langle x_0 \to u_0 \to P^0(u_0, v_0) \to v_0 \to v_1 \to HP^1(v_1, y_1) \to y_1 \rangle$ forms a hamiltonian path joining x_0 and y_1 in $(G^0 \oplus_M G^1) - F$.

Case 2.2. $i \neq j$ and both $f_0, f_1 \leqslant n-2$. (x_i and y_j are on different sides, and all faults are scattered over E_c, G^0, or G^1. See Fig. 5d.)

Without loss of generality, we assume that $i=0$ and $j=1$. Because $2^{n+1} \geqslant n+2$ for $n \geqslant 3$, there exist two vertices $u_0, u_1 \notin (F \cup \{x_0, y_1\})$ and $(u_0, u_1) \notin F$. Since both G^0 and G^1 are $(n-2)$-hamiltonian connected, and $n-2 \geqslant f_0$ and $n-2 \geqslant f_1$, the graphs $G^0 - F_0$ and $G^1 - F_1$ are hamiltonian connected. Thus there exist hamiltonian paths $HP^0(x_0, u_0)$ and $HP^1(u_1, y_1)$ in G^0 and G^1, respectively. Therefore, $\langle x_0 \rightarrow HP^0(x_0, u_0) \rightarrow u_0 \rightarrow u_1 \rightarrow HP^1(u_1, y_1) \rightarrow y_1 \rangle$ forms a hamiltonian path joining x_0 and y_1 in $(G^0 \oplus_M G^1) - F$.

Case 2.3. $i=j$ and $f_i = n-1$. (x_i, y_j, and all faults are on the same side. See Fig. 5e.)

Without loss of generality, we assume that $i=j=0$. Let w be a fault of F. Since G^0 is $(n-2)$-hamiltonian connected, $G^0 - (F - \{w\})$ contains a hamiltonian path $HP^0(x_0, y_0)$. Thus $G^0 - F$ contains two node-disjoint paths $P^0(x_0, u_0)$ and $P^0(v_0, y_0)$ where $P^0(x_0, u_0) \cup P^0(v_0, y_0) = HP^0(x_0, y_0) - \{w\}$. Because G^1 is $(n-2)$-hamiltonian connected and $n-2 \geqslant 0$, there exists a hamiltonian path $HP^1(u_1, v_1)$ in G^1. Therefore, $\langle x_0 \rightarrow P^o(x_0, u_0) \rightarrow u_0 \rightarrow u_1 \rightarrow HP^1(u_1, v_1) \rightarrow v_1 \rightarrow v_0 \rightarrow P^0(v_0, y_0) \rightarrow y_0 \rangle$ forms a hamiltonian path of $(G^0 \oplus_M G^1) - F$.

Case 2.4. $i=j$ and both $f_0, f_1 \leqslant n-2$. (x_i and y_j are on the same side, and all faults are scattered over E_c, G^0, or G^1. See Fig. 5f.)

Without loss of generality, we may assume that $i=j=0$. Since G^0 is $(n-2)$-hamiltonian connected and $n-2 \geqslant f_0$, there exists a hamiltonian path $HP^0(x_0, y_0)$. Because $2^{n+1} \geqslant 2n$ for $n \geqslant 3$, there exists an edge (u_0, v_0) on the path $HP^0(x_0, y_0)$ such that u_1, v_1, (u_0, u_1), and (v_0, v_1) are not in F. Since G^1 is $(n-2)$-hamiltonian connected and $n-2 \geqslant f_1$, there exists a hamiltonian path $HP^1(u_1, v_1)$ in G^1. Thus, $(HP^0(x_0, y_0) \cup \{(u_0, u_1), (v_0, v_1)\} \cup HP^1(u_1, v_1)) - \{(u_0, v_0)\}$ forms a hamiltonian path joining x_0 and y_0 in $(G^0 \oplus_M G^1) - F$.

Case 2.5. $i=j$ and $f_k = n-1$ for $k \neq i$. (x_i and y_j are on the same side, but all faults are on the other side.)

Without loss of generality, we may assume that $i=j=0$ and $f_1 = n-1$. We will prove this case by the following subcases.

Subcase 2.5.1. $x_1 \notin F$ or $y_1 \notin F$, where x_1 and y_1 are the matching nodes of x_0 and y_0, respectively. Without loss of generality, we may assume that $x_1 \notin F$. (See Fig. 5g.)

Since G^1 is $(n-1)$-hamiltonian, there exists a hamiltonian cycle $HC^1 = \langle x_1 \rightarrow u_1 \rightarrow P^1(u_1, v_1) \rightarrow v_1 \rightarrow x_1 \rangle$. Because G^0 is $(n-2)$-hamiltonian and $n-2 \geqslant 1$, $G^0 - \{x_0\}$ is a hamiltonian-connected graph. Let $HP^0(z_0, y_0)$ denote a hamiltonian path joining z_0 and y_0 in $G^0 - \{x_0\}$ for every node z_0 in $G^0 - \{x_0\}$. If $u_0 \neq y_0$, where u_0 is the matching nodes of u_1, then $\langle x_0 \rightarrow x_1 \rightarrow v_1 \rightarrow P^1(v_1, u_1) \rightarrow u_1 \rightarrow u_0 \rightarrow HP^0(u_0, y_0) \rightarrow y_0 \rangle$ forms a hamiltonian path in $(G^0 \oplus_M G^1) - F$. Otherwise, $v_0 \neq y_0$, and then $\langle x_0 \rightarrow x_1 \rightarrow u_1 \rightarrow P^1(u_1, v_1) \rightarrow v_1 \rightarrow v_0 \rightarrow HP^0(v_0, y_0) \rightarrow y_0 \rangle$ forms a hamiltonian path in $(G^0 \oplus_M G^1) - F$.

Subcase 2.5.2. $x_1 \in F$ and $y_1 \in F$. The discussion of this case is a little complicated. Since G^1 is $(n-1)$-hamiltonian and $f_1 = n-1$, there exists a hamiltonian cycle HC^1 in $G^1 - F_1$. Moreover, there are two consecutive nodes a_1 and b_1 on this cycle HC^1, such that their matching nodes a_0 and b_0 are on different sides of $G^0 = TQ_n \times K_2$, say $a_0 \in TQ_n^{00}$ and $b_0 \in TQ_n^{10}$, where TQ_n^{00} and TQ_n^{10} are the two sides of G^0. Let $HC^1 = \langle a_1 \to P^1(a_1, b_1) \to b_1 \to a_1 \rangle$.

Consider the case that x_0 and y_0 are on different sides of $G^0 = TQ_n \times K_2$. Without loss of generality, we may assume that $x_0 \in TQ_n^{00}$ and $y_0 \in TQ_n^{10}$ (See Fig. 5h.) Since TQ_n is hamiltonian connected, there exist hamiltonian paths $HP^{00}(x_0, a_0)$ and $HP^{10}(y_0, b_0)$ in TQ_n^{00} and TQ_n^{10}, respectively. Thus $\langle x_0 \to HP^{00}(x_0, a_0) \to a_0 \to a_1 \to P^1(a_1, b_1) \to b_1 \to b_0 \to HP^{10}(b_0, y_0) \to y_0 \rangle$ forms a hamiltonian path joining x_0 and y_0 in $(G^0 \oplus_M G^1) - F$.

Next, consider that x_0 and y_0 are on the same side of $G^0 = TQ_n \times K_2$. Without loss of generality, we may assume that $x_0, y_0 \in TQ_n^{00}$. (See Fig. 5i.)

We need to define notations before further discussions. The graph $G^0 = TQ_n \times K_2$ has two sides, denoted by TQ_n^{00} and TQ_n^{10}. For each node u_{00} (u_{10}, respectively) in TQ_n^{00} (TQ_n^{10}, respectively), its matching node with respect to the two sides TQ_n^{00} and TQ_n^{10} is denoted by u_{10} (u_{00}, respectively).

Since TQ_n is $(n-3)$-hamiltonian connected and $n-3 \geqslant 0$, there exists a hamiltonian path $HP^{00}(x_0, y_0) = \langle x_0 \to P^{00}(x_0, u_{00}) \to u_{00} \to a_0 \to v_{00} \to P^{00}(v_{00}, y_0) \to y_0 \rangle$, where u_{00} and v_{00} are the two adjacent nodes of a_0 on this path. Let $HP^{10}(z_{10}, b_0)$ denote a hamiltonian path joining z_{10} and b_0 in TQ_n^{10}. If $u_{10} \neq b_0$, where u_{10} is the matching node of u_{00} with respect to the two sides TQ_n^{00} and TQ_n^{10}, then $\langle x_0 \to P^{00}(x_0, u_{00}) \to u_{00} \to u_{10} \to HP^{10}(u_{10}, b_0) \to b_0 \to b_1 \to P^1(b_1, a_1) \to a_1 \to a_0 \to v_{00} \to P^{00}(v_{00}, y_0) \to y_0 \rangle$ forms a hamiltonian path joining x_0 and y_0 in $(G^0 \oplus_M G^1) - F$. Otherwise, $v_{10} \neq b_0$ where v_{10} is the matching node of v_{00} with respect to the two sides TQ_n^{00} and TQ_n^{10}, and then $\langle x_0 \to P^{00}(x_0, u_{00}) \to u_{00} \to a_0 \to a_1 \to P^1(a_1, b_1) \to b_1 \to b_0 \to HP^{10}(b_0, v_{10}) \to v_{10} \to v_{00} \to P^{00}(v_{00}, y_0) \to y_0 \rangle$ forms a hamiltonian path joining x_0 and y_0 in $(G^0 \oplus_M G^1) - F$. This completes the induction proof. ▌

Now we are ready to prove our main theorem:

THEOREM 2.3. *The twisted n-cube TQ_n is $(n-2)$-hamiltonian and $(n-3)$-hamiltonian connected, for all odd integer $n \geqslant 3$.*

Proof. By Lemma 2.2, TQ_3 is 1-hamiltonian and hamiltonian connected. And $TQ_{n+2} = (TQ_n \times K_2) \oplus_M (TQ_n \times K_2)$ for some perfect matching M. By Theorems 2.1 and 2.2, and by a simple induction, this theorem follows. ▌

It is obvious that the fault-tolerant hamiltonicity $\mathcal{H}_f(G)$ (the fault-tolerant hamiltonian connectivity $\mathcal{H}_f^\kappa(G)$, respectively) of a graph G is no greater than $\delta(G)-2$ ($\delta(G)-3$, respectively), and TQ_n is a regular graph of degree n. From Theorem 2.3 above, we have the following result.

COROLLARY 2.1. *$\mathcal{H}_f(TQ_n) = n-2$ and $\mathcal{H}_f^\kappa(TQ_n) = n-3$, for all odd integer $n \geqslant 3$.*

3. CONCLUSIONS

In this paper, we consider a faulty twisted n-cube TQ_n with edge and/or node faults. We prove that TQ_n remains hamiltonian (hamiltonian connected, respectively), even if it has up to $n-2$ ($n-3$, respectively) edge and/or node faults. This result is optimum in the sense that the fault-tolerant hamiltonicity (the fault-tolerant hamiltonian connectivity, respectively) of TQ_n is at most $n-2$ ($n-3$, respectively). As far as the hypercube network Q_n is concerned, its vertex fault-tolerant hamiltonicity is 0 and edge fault-tolerant hamiltonicity is $n-2$, for $n \geqslant 2$. Recently, many topological properties of the twisted n-cube have been studied [1, 2, 6, 8, 9]. All these results indicate that the performance of TQ_n is better than that of the hypercube in many aspects. Therefore, the twisted n-cube is an attractive alternative to the hypercube network.

As noted in this paper, we observe that the fault-tolerant hamiltonicity and the fault-tolerant hamiltonian connectivity are essential parameters of an interconnection network [10]. It would be an interesting issue to study more on this subject.

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