Minimum Redundancy for ISI Free FIR Filterbank Transceivers

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***Abstract*—There has been great interest in the design of filterbank transceivers. Usually, with proper time domain equalization, the channel is modeled as an FIR filter. It is known that for FIR channels, the introduction of certain redundancy allows the receiver to cancel intersymbol interference (ISI) completely, and channel equalization is performed implicitly using FIR transceivers. This scheme allows us to trade bandwidth for ISI cancellation. In this paper, we will derive the minimum redundancy required for the existence of FIR transceivers for a given channel. We will see that the minimum redundancy is directly related to the zeros of the channel and to the Smith form of an appropriately defined channel matrix.**

I. Introduction

THE connection between an M-band filterbank and an M-band DMT (discrete multitone) or filterbank transceiver is well known [1]–[3]. When the analysis and synthesis banks of a perfect reconstruction filterbank are interchanged, the new structure becomes a filterbank transceiver (see Fig. 1). The system in this case has interpolation ratio $N = M$, and it is called *minimally interpolated*. When the channel $P(z)$ is a delay, i.e., ideal, the minimally interpolated M-band filterbank transceiver is ISI free if the corresponding filterbank has perfect reconstruction [1]. The ISI-free property means that there is no intraband or interband ISI. The discrete wavelet multitone (DWMT) system [4] is obtained by interchanging perfect reconstruction analysis and synthesis banks. However, when the channel is not ideal, the perfect reconstruction property of the filterbank no longer translates to ISI-free property of filterbank transceivers. The resulting ISI can seriously degrade the system performance [5], [6]. Additional interband and intraband equalization can be used to reduce ISI [4], [7].

When the interpolation ratio $N > M$, the filterbank transceiver is called *overinterpolated*; on average, every N output samples of the transmitter contain $K = N - M$ redundant samples. The cyclic prefix in DFT-based transceiver system [8], [9] and zero padding in vector coding transceiver system [10] are examples of such redundant samples. Using an overinterpolated filterbank transceiver, it is possible to cancel ISI completely with appropriate redundancy K. In a typical system model, the channel is an FIR filter $P(z)$ of order L upon time domain equalization. In the DFT-based transceiver system [9] or the vector coding system [10], zero ISI is achieved using redundancy $K = L$.

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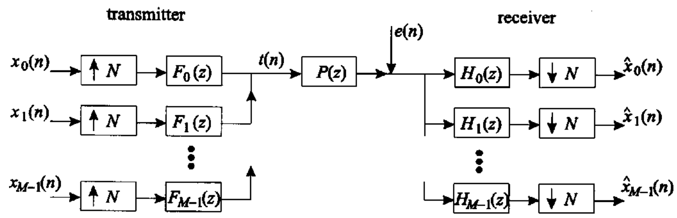


Fig. 1. M-band DMT transceiver over a channel $P(z)$ with additive noise $e(n)$.

As reducing redundancy leads to better bandwidth efficiency, designs with smaller redundancy has been of great research interest. Advances to the more general FIR overinterpolated system have been made in [11] and [12] for ISI cancellation using precoding. It has been shown that FIR transceivers exist for redundancy $K < L$ under very general conditions. In particular, for a given number of bands M and interpolation ratio N, the condition for the existence of FIR transceivers can be given in terms of the zeros of the channel $P(z)$. Let S be the set that contains the zeros of $P(z)$: $S = \{\alpha_1, \alpha_2, \ldots, \alpha_L\}$, with $P(\alpha_\ell) = 0$. The necessary and sufficient condition for the existence of FIR transceiver is [12]

$$\bigcap_{0\le\ell_1<\ell_2<\cdots<\ell_M\le N-1} \left(S_{\ell_1} \bigcup S_{\ell_2} \bigcup \cdots \bigcup S_{\ell_M}\right) = \phi \quad (1)$$

where

$$S_{\ell_k} = \{e^{-j2\pi\ell_k/N}\alpha_1, e^{-j2\pi\ell_k/N}\alpha_2, \ldots, e^{-j2\pi\ell_k/N}\alpha_L\}.$$

In [13], time-varying systems are employed for designing FIR transceivers. Suppose the channel is of order L with distinct roots and that the interpolation ratio N and number of bands M satisfy $N, M > L$. It is shown in [13] that we can always find a channel-independent time-varying transmitter such that FIR time-varying receivers exist. In particular, redundancy of one can be used as long as $N - 1 > L$ and the time-varying receiving filters are sufficiently long. In addition, in [13], one necessary condition for the existence of LTI transceivers is presented. Assume again that $N, M > L$ and that zeros of the channel are distinct; in addition, assume that the transmitter consists of a constant matrix. For the case where the channel has zeros of the form $re^{2\pi\ell_j/N}$, where ℓ_j for $j = 1, 2, \ldots \rho$

are integers, it is shown that the FIR receiver does not exist if $N - M < \rho$. However, for a given interpolation ratio, there has been no method that computes explicitly the minimum redundancy or, equivalently, the maximum number of bands that can be used for ISI-free transmission.

In this paper, we will derive a new necessary and sufficient condition for the existence of FIR transceivers. Given the zeros of the channel $P(z)$ and interpolation ratio N, we will be able to determine exactly the minimum redundancy for which FIR transceivers with the ISI-free property exist. Furthermore, solutions of FIR ISI-free transceivers with minimum redundancy will be provided. There are cases where minimum redundancy $= L$. The condition for such cases will be given. Moreover, we will consider minimum redundancy for the class of block-based DMT systems, in which case, the transmitter and receiver of the filterbank transceiver are characterized by constant matrices. The block-based system is the most widely used of all transceivers [9], [10], [13]–[15]; the DFT based system and the vector coding systems are both examples of block-based transceivers. We will show that the minimum redundancy for the block-based transceivers is given by $\lceil L/2 \rceil$. Furthermore, when ISI-free block-based transceivers with minimum redundancy $K = \lceil L/2 \rceil$ exist, the solutions of the transceivers will be parameterized.

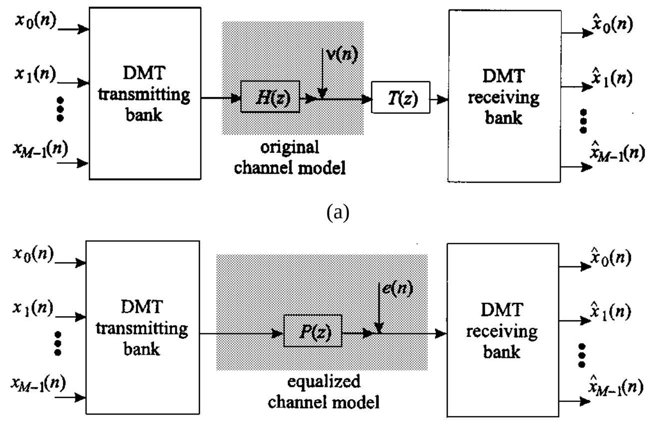
The sections are organized as follows: In Section II, we introduce the polyphase representation of the transmitter and receiver, which will be the framework throughout this paper. In the polyphase framework, the channel is formulated as a pseudo circulant matrix [16]. The minimally interpolated filterbank transceivers will be considered in Section III. In Section IV, minimum redundancy for the existence of FIR transceivers is presented. The block-based system is considered in Section V. Some properties of pseudo circulant matrices that are useful for our discussion are given in the Appendix.

A. Notations and Preliminaries

- Boldfaced lowercase letters are used to represent vectors, and boldfaced uppercase letters are reserved for matrices. The notations $\mathbf{A}^T$ and $\mathbf{A}^\dagger$ represent the transpose of $\mathbf{A}$ and transpose-conjugate of $\mathbf{A}$.
- For an $N \times M$ transfer matrix $\mathbf{A}(z)$, the notation $\widetilde{\mathbf{A}}(z)$ denotes $\mathbf{A}^\dagger(1/z^*)$. For transfer matrices with real coefficients, $\tilde{\mathbf{A}}(z) = \mathbf{A}^T(z^{-1})$.
- The notation $\mathbf{I}_N$ is used to represent the $N \times N$ identity matrix. The subscript is omitted whenever the size is clear from the context.
- *Unimodular Matrices:* An $N \times N$ matrix $\mathbf{A}(z)$ is called unimodular if $\det \mathbf{A}(z) = c$, which is a nonzero constant [17]. A causal unimodular FIR matrix $\mathbf{A}(z)$ has the property that $\mathbf{A}^{-1}(z)$ is also causal and FIR.

B. Channel Models

Fig. 2(a) shows the block diagram of a filterbank transceiver. The discrete time channel is modeled as an LTI filter $h(n)$ with additive noise $\nu(n)$, as shown in Fig. 2(a). A time domain equalizer (TEQ) $T(z)$ precedes the receiver. Typically, the filter $H(z)$ can be further modeled as a rational transfer function $H(z) = P(z)/B(z)$. The equalizer $T(z)$ is usually designed to cancel the poles of $H(z)$, and the resulting overall transfer function becomes the FIR filter $P(z)$, as shown in Fig. 2(b). Suppose $P(z)$ is of order L and that



(a)
(b)

Fig. 2. (a) Block diagram of the DMT transceiver, including a discrete-time channel model and an equalizer $T(z)$. (b) Block diagram of the DMT transceiver with an equalized channel model.

$$P(z) = p_0 + p_1 z^{-1} + \cdots + p_L z^{-L}.$$

The equalized impulse response of the channel is thus shortened to L. Each input sample of the channel will be spread to a duration of length $L+1$ as a result. The noise $e(n)$ shown in Fig. 2(b) is obtained by feeding the original noise $\nu(n)$ to the equalizer $T(z)$. The equalized channel model in Fig. 2(b) will be used throughout this paper; the channel refers to the equalized channel $P(z)$; and the channel noise refers to the equalized noise $e(n)$ in this paper.

II. Polyphase Representation of Filterbank Transceivers

Consider Fig. 1, where an M-band filterbank transceiver is shown. The channel is represented by an FIR filter $P(z)$ with additive noise $e(n)$, as explained in Section I-B. The filters $F_k(z)$ and $H_k(z)$ are called transmitting and receiving filters, respectively. It is not necessary for the interpolation ratio N to be the same as the number of bands M. Two cases will be studied: i) When $N = M$, we say the system is minimally interpolated; ii) when $N > M$, we say it is overinterpolated, and redundancy is introduced in this case. The case of $N < M$ is of no interest in our application because in this case, the input data $x_k(n)$ can never be fully recovered, no matter what the channel is.

Using polyphase decomposition, we can decompose the kth transmitting filter $F_k(z)$ with respect to the integer N [17]

$$F_k(z) = \sum_{n=0}^{N-1} G_{n,k}(z^N) z^{-n}. \tag{2}$$

Writing the polyphase representation for all the M transmitting filters, we have (3), shown at the bottom of the next page, where the $N \times M$ matrix $\mathbf{G}(z)$ is the polyphase matrix of the transmitter. Using the noble identity [17], we can interchange the expander and $\mathbf{G}(z^N)$. The transmitter can be implemented using

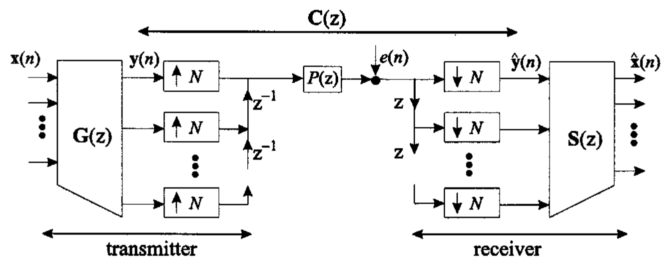


Fig. 3. Polyphase representation of the transmitter and receiver in a DMT transceiver.

its polyphase matrix, as shown in Fig. 3. In a similar manner, we can decompose the receiving filters as

$$H_k(z) = \sum_{n=0}^{N-1} S_{k,n}(z^N) z^n. \quad (4)$$

Then, by invoking the noble identity, the receiver can be redrawn as Fig. 3. The receiving filters $H_k(z)$ are related to the $M \times N$ polyphase matrix $\mathbf{S}(z)$ of the receiver as (5), shown at the bottom of the page.

1) Decomposition of the Channel: Using polyphase representation, we can decompose the channel as

$$C(z) = P_0(z^N) + P_1(z^N)z^{-1} + \cdots + P_{N-1}(z^N)z^{-N+1}. \quad (6)$$

In order to further simplify Fig. 3, we need to apply an identity from the multirate theory. It is shown in [17] that the multirate system in Fig. 4 is, in fact, equivalent to an LTI system with transfer function $A(z)$, which is given by

$$A(z) = \begin{cases} P_{i-j}(z), & \text{for } i \geq j \\ z^{-1}P_{N+i-j}(z), & \text{for } i < j \end{cases}$$

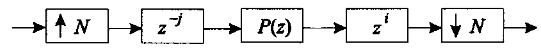


Fig. 4. Polyphase identity.

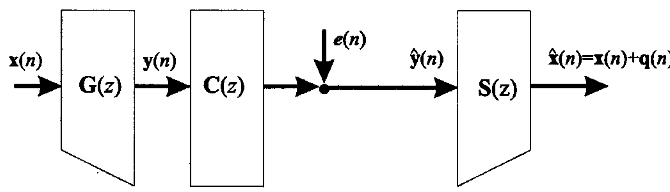


Fig. 5. Polyphase representation of a DMT transceiver.

where $P_k(z)$ is defined in (6). We see that the $N \times N$ system from $\mathbf{y}(n)$ to $\hat{\mathbf{y}}(n)$ in Fig. 3 is, in fact, an LTI system with transfer matrix $\mathbf{C}(z)$ given by

$$\mathbf{C}(z) = \begin{pmatrix} P_0(z) & z^{-1}P_{N-1}(z) & z^{-1}P_{N-2}(z) & \cdots & z^{-1}P_1(z) \\ P_1(z) & P_0(z) & z^{-1}P_{N-1}(z) & \cdots & z^{-1}P_2(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{N-1}(z) & P_{N-2}(z) & P_{N-3}(z) & \cdots & P_0(z) \end{pmatrix}. \quad (7)$$

Matrices in the above form are known as pseudo circulant matrices [16], [17]. A first detailed study of pseudocirculant matrices was made in [16]. Many useful properties, as well as applications of pseudocirculant matrices in QMF banks and block filtering, are given therein. Properties of $\mathbf{C}(z)$ that will be used in later discussions are given in the Appendix. With the channel matrix $\mathbf{C}(z)$, we can redraw Fig. 3 as Fig. 5. As we will see later, the polyphase representation in Fig. 5 will facilitate a systematic study of filterbank transceivers. Many useful theoretical and practical results can be drawn from such a representation.

2) Zero ISI Condition: From the polyphase decomposition in Fig. 5, we see that even though multirate building blocks are

$$[F_0(z) \quad F_1(z) \quad \cdots \quad F_{M-1}(z)] = [1 \quad z^{-1} \quad \cdots \quad z^{-N+1}] \underbrace{\begin{pmatrix} G_{0,0}(z^N) & G_{0,1}(z^N) & \cdots & G_{0,M-1}(z^N) \\ G_{1,0}(z^N) & G_{1,1}(z^N) & \cdots & G_{1,M-1}(z^N) \\ \vdots & \vdots & \ddots & \vdots \\ G_{N-1,0}(z^N) & G_{N-1,1}(z^N) & \cdots & G_{N-1,M-1}(z^N) \end{pmatrix}}_{\mathbf{G}(z^N)} \quad (3)$$

$$\begin{pmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{pmatrix} = \underbrace{\begin{pmatrix} S_{0,0}(z^N) & S_{0,1}(z^N) & \cdots & S_{0,N-1}(z^N) \\ S_{1,0}(z^N) & S_{1,1}(z^N) & \cdots & S_{1,N-1}(z^N) \\ \vdots & \vdots & \ddots & \vdots \\ S_{M-1,0}(z^N) & S_{M-1,1}(z^N) & \cdots & S_{M-1,N-1}(z^N) \end{pmatrix}}_{\mathbf{S}(z^N)} \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{N-1} \end{pmatrix} \quad (5)$$

used in a filterbank transceiver, it is, in fact, an LTI M-input M-output system. The transfer matrix $\mathbf{T}(z)$ of the overall system can be expressed as

$$\mathbf{T}(z) = \mathbf{S}(z)\mathbf{C}(z)\mathbf{G}(z). \tag{8}$$

The overall system is free from interband ISI if $\mathbf{T}(z)$ is a diagonal matrix. It is free from intraband ISI when the diagonal elements of $\mathbf{T}(z)$ are merely delays. If it is free from both interband and intraband ISI, we say that the filterbank transceiver is ISI free; in the absence of channel noise, the outputs of an ISI-free filterbank transceiver are identical to the inputs except delays and scalars. Without much loss of generality, we can use the ISI-free condition

$$\mathbf{S}(z)\mathbf{C}(z)\mathbf{G}(z) = \mathbf{I}_M. \tag{9}$$

III. FILTERBANK TRANSCEIVERS WITH NO REDUNDANCY

The filterbank transceiver in Fig. 1 is called minimally interpolated if the interpolation ratio $N = M$ and if there is no redundancy. In this section, we consider the solutions for such systems [18]. A number of properties of such systems will be derived. In particular, we will show that *no practical orthogonal system can yield inter-band ISI free solution* unless the channel $P(z)$ is a pure delay. Using FIR nonorthogonal transceivers, we can achieve only zero interband ISI but not zero ISI. Moreover, for nonminimum phase channel, there does not exist an ISI-free transceiver that is *causal and stable*.

1) Orthogonal Transmitters and Receivers: Suppose that the transceiver is orthogonal, that is

$$\mathbf{G}^{\dagger}(e^{j\omega})\mathbf{G}(e^{j\omega}) = \mathbf{I}, \quad \text{for all } \omega \tag{10}$$

and $\mathbf{S}(e^{j\omega}) = \mathbf{G}^{\dagger}(e^{j\omega})$. In the z-transform domain, this becomes $\mathbf{S}(z) = \tilde{\mathbf{G}}(z)$, and orthogonality translates to $\tilde{\mathbf{G}}(z)\mathbf{G}(z) = \mathbf{I}$ for all z. From (9), the condition for zero interband ISI becomes

$$\tilde{\mathbf{G}}(z)\mathbf{C}(z)\mathbf{G}(z) = \mathbf{\Lambda}(z), \tag{11}$$

where $\mathbf{\Lambda}(z)$ is a diagonal matrix. Premultiplying the above equation by $\mathbf{G}(z)$, we have

$$\mathbf{C}(z)\mathbf{G}(z) = \mathbf{G}(z)\mathbf{\Lambda}(z). \tag{12}$$

That is to say, the diagonal entries of $\mathbf{\Lambda}(z)$ contain the eigenvalues of $\mathbf{C}(z)$, and the columns of $\mathbf{G}(z)$ are the eigenvectors of $\mathbf{C}(z)$. However, from the property of pseudocirculant matrices given in (32), we see that the eigenvectors of the pseudo circulant matrix $\mathbf{C}(z)$ are the column vectors of $\mathbf{D}(z^{1/M})\mathbf{W}$, which consist of a fraction of a delay [17] and cannot be realized as rational transfer functions. Therefore, zero interband ISI property of orthogonal filterbank transceivers over nonideal channels cannot be achieved with finite cost. However, interband ISI-free property is possible if the transmitter and receiver are not constrained to be orthogonal, as we will see next.

2) FIR Nonorthogonal Transceiver: The matrix $\mathbf{C}(z)$ is a causal FIR matrix and can be decomposed using the Smith form decomposition described in the Appendix

$$\mathbf{C}(z) = \mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{V}(z) \tag{13}$$

where $\mathbf{U}(z)$ and $\mathbf{V}(z)$ are causal FIR unimodular matrices, and $\mathbf{\Gamma}(z)$ is a causal FIR diagonal matrix. Note that $\mathbf{U}^{-1}(z)$ and $\mathbf{V}^{-1}(z)$ are also causal FIR unimodular matrices as $\det \mathbf{U}(z)$ and $\det \mathbf{V}(z)$ are constants. Therefore, if we choose the transmitter and receiver as

$$\mathbf{G}(z) = \mathbf{V}^{-1}(z) \quad \text{and} \quad \mathbf{S}(z) = \mathbf{U}^{-1}(z)$$

then $\mathbf{T}(z) = \mathbf{\Gamma}(z)$. Thus, using FIR nonorthogonal filterbank transceivers, we can achieve interband ISI free. Although interband ISI is canceled, intraband ISI cannot be removed completely; in the minimally interpolated case, there is no FIR transceiver that can achieves zero ISI for nonideal channels. To see this, we can consider the determinant of the overall system $\det(\mathbf{T}(z)) = \det(\mathbf{S}(z)\mathbf{G}(z))\det(\mathbf{C}(z))$. As $P(z)$ is FIR, $\det(\mathbf{C}(z))$ is FIR, and $\det(\mathbf{C}(z))$ is a delay if and only if $P(z)$ is a delay. If the transceiver is FIR, then $\det(\mathbf{S}(z)\mathbf{G}(z))$ is FIR, and it follows that $\det(\mathbf{T}(z))$ is also FIR. When there is zero ISI, $\det(\mathbf{T}(z))$ is a delay, i.e., $\det(\mathbf{S}(z)\mathbf{G}(z))\det(\mathbf{C}(z))$ is a delay. Therefore, $\det(\mathbf{S}(z)\mathbf{G}(z))$ cannot be FIR unless $\det\mathbf{C}(z)$ is a delay. Therefore, it is not possible to achieve zero ISI using FIR transmitters and receivers for a nonideal channel.

3) IIR Transceivers: If we are allowed to use IIR filters, one possible ISI-free solution is $\mathbf{G}(z) = \mathbf{V}^{-1}(z)$ and $\mathbf{S}(z) = \mathbf{\Gamma}^{-1}(z)\mathbf{U}^{-1}(z)$. Caution must be taken in doing so. The term $\mathbf{\Gamma}^{-1}(z)$ may not be stable. In fact, if the channel $P(z)$ does not have minimum phase, there exists no ISI-free transceiver that is both *causal* and *stable*.

Lemma 3.1: There exists a causal and stable ISI-free minimally interpolated filterbank transceiver if and only if $P(z)$ is a minimum-phase filter. Furthermore, FIR transceivers with the ISI-free property can be obtained only if $P(z)$ is a delay.

Proof—Sufficiency of Minimum Phase $P(z)$: As $\mathbf{U}(z)$ and $\mathbf{V}(z)$ are unimodular matrices, we have $\det \mathbf{C}(z) = p_0^N \det \mathbf{\Gamma}(z) = p_0^N \Pi_{k=0}^{M-1} \gamma_k(z)$. Using the second property of pseudo circulant matrices derived in the Appendix, we know that $\det \mathbf{C}(z)$ has zeros at α_ℓ^N if $P(z)$ has zeros at α_ℓ for $\ell = 1, 2, \ldots, L$. It follows that the zeros of $\gamma_k(z)$ are α_ℓ^N. If $P(z)$ has minimum phase, the zeros satisfy $|\alpha_\ell| < 1$. The zeros α_ℓ^N of $\gamma_k(z)$ are also inside the unit circle. For a causal and stable transceiver solution, we can choose

$$\mathbf{G}(z) = \mathbf{V}^{-1}(z), \quad \text{and } \mathbf{S}(z) = \mathbf{\Gamma}^{-1}(z)\mathbf{U}^{-1}(z).$$

Whenever there exists a causal and stable transceiver pair $(\mathbf{G}(z), \mathbf{S}(z))$, we can use

$$(\mathbf{G}(z)\mathbf{\Theta}(z), \mathbf{\Theta}^{-1}(z)\mathbf{S}(z))$$

to obtain a new causal and stable transceiver pair, where $\mathbf{\Theta}(z)$ is any causal and stable transfer matrix with a causal and stable inverse.

Necessity of Minimum Phase $P(z)$. When the filterbank transceiver is ISI free, we have

$$\det(\mathbf{S}(z)\mathbf{C}(z)\mathbf{G}(z)) = cz^{-n_0}$$

for some constant c and integer n_0. As $\det \mathbf{C}(z)$ contains the factor $(1 - \alpha_\ell^N z^{-1})$, therefore, either $\det \mathbf{S}(z)$ or $\det \mathbf{G}(z)$ contains the factor $1/(1 - \alpha_\ell^N z^{-1})$. If $P(z)$ does not have minimum phase, then $|\alpha_\ell| > 1$, and thus, $|\alpha_\ell^N| > 1$ for some ℓ. Therefore, $\mathbf{S}(z)$ and $\mathbf{G}(z)$ cannot both be stable. In other words, if $P(z)$

does not have minimum phase, there exists no causal and stable ISI-free transceiver. △

For the minimally interpolated filterbank transceiver, the existence of a causal and stable transceiver depends on the minimum-phase property of the given channel $P(z)$. The stability problem also explains why nonminimum phase channels are difficult to equalize for minimally interpolated filterbank transceivers. However, as we will see later, this is not the case if a certain redundancy is allowed, i.e., the filterbank transceiver is overinterpolated. In fact, if the added redundancy is large enough, there always exist FIR ISI-free filterbank transceivers.

Remarks: For single input single output (SISO) system, it is well known that the inverse of an FIR system is always IIR. The IIR inverse is i) causal and stable if the original system has minimum phase and ii) stable and possibly noncausal if the original system has no zero on the unit circle. The result in Lemma 3.1 for the minimally interpolated systems can be viewed as a generalization of the SISO case.

IV. FIR Transceivers With Minimum Redundancy

We say a filterbank transceiver is overinterpolated if the interpolation ratio N is greater than the number of bands M. In this case, there are more samples at the output of the transmitter than the input. There are $K=N-M$ redundant samples in every N samples of the transmitter output. By introducing proper redundancy to the transmitter output, the channel can be equalized perfectly to achieve ISI-free property using FIR transceivers. For example, in the DFT-based system, redundancy is introduced by adding cyclic prefix. The transmitting and receiving filters are FIR of length N and M, respectively. In this section, we will consider general FIR transceivers. For a given interpolation ratio N, we will derive the minimum redundancy for the existence of FIR transceivers. The minimum redundancy can be determined from the location of the zeros of the channel $P(z)$. It can also be related to the Smith form decomposition of the channel matrix $\mathbf{C}(z)$.

With the number of bands M and interpolation ratio N, the transmitter $\mathbf{G}(z)$ and $\mathbf{S}(z)$ are, respectively, of dimension $N\times M$ and $M\times N$. The channel matrix $\mathbf{C}(z)$ is of dimension $N\times N$.

Definition 1: For a given N and channel matrix $\mathbf{C}(z)$ with Smith form $\mathbf{\Gamma}(z)$, the notation $\rho(N)$ denotes the number of nonunity terms in the diagonal of the Smith form.

The number $\rho(N)$ depends only on the given channel and the interpolation ratio N. We can express $\mathbf{\Gamma}(z)$ as

$$\mathbf{\Gamma}(z)=\text{diag}\begin{pmatrix}1 & 1 & \cdots & 1 & \gamma_{N-\rho(N)}(z) & \cdots & \gamma_{N-1}(z)\end{pmatrix} \quad (14)$$

where $\rho(N)$ is an integer satisfying $\rho(N)\leq\min(L,N)$. The following lemma gives the smallest rank of $\mathbf{C}(z)$, which in terms will give the condition for the existence of FIR transceivers.

Lemma 4.1: The smallest rank of $\mathbf{C}(z)$ is $N-\rho(N)$, where $\rho(N)$ is the number of nonidentity terms on the diagonal of the Smith form $\mathbf{\Gamma}(z)$, as given in (14).

Proof: The determinants of the two unimodular matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ in the Smith form decomposition are nonzero constants. The rank of $\mathbf{C}(z)$ is the same as the rank of the Smith form $\mathbf{\Gamma}(z)$. We can consider the rank of $\mathbf{\Gamma}(z)$. Observing (14), we can see that the smallest rank of $\mathbf{\Gamma}(z)$ is $N-\rho(N)$. This happens when $z=z_0$ is a zero of $\gamma_{N-\rho(N)}(z)$. △

Theorem 4.1: Consider the filterbank transceiver in Fig. 1. Let $\mathbf{C}(z)$ have the Smith-form decomposition $\mathbf{C}(z)=\mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{V}(z)$, and let the Smith-form $\mathbf{\Gamma}(z)$ be as given in (14). Then, there exist FIR $\mathbf{G}(z)$ and $\mathbf{S}(z)$ such that the transceiver is ISI free if and only if the redundancy $K\geq\rho(N)$, where $\rho(N)$ is as given in Definition 1. When an FIR transceiver exists, the solution is not unique. One choice of ISI-free FIR transceiver is

$$\mathbf{G}(z)=\mathbf{V}^{-1}(z)\begin{pmatrix}\mathbf{I}_{N-\rho(N)}\\ \mathbf{0}\end{pmatrix}$$
$$\mathbf{S}(z)=\begin{pmatrix}\mathbf{I}_{N-\rho(N)} & \mathbf{0}\end{pmatrix}\mathbf{U}^{-1}(z). \quad (15)$$

The minimum redundancy for FIR transceiver solutions is $\rho(N)$.

Proof:

Sufficiency: Consider that the choice of FIR transmitter $\mathbf{G}(z)$ is given in (15). Then

$$\mathbf{C}(z)\mathbf{G}(z)=\mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{V}(z)\mathbf{V}^{-1}(z)\begin{pmatrix}\mathbf{I}_{N-\rho(N)}\\ \mathbf{0}\end{pmatrix}=\mathbf{U}(z)\begin{pmatrix}\mathbf{I}_{N-\rho(N)}\\ \mathbf{0}\end{pmatrix}.$$

Therefore, if we choose the receiver as in (15), the transceiver is FIR and ISI free. The unimodular matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ are not unique in the Smith-form decomposition; therefore, $\mathbf{G}(z)$ and $\mathbf{S}(z)$ are not unique. In particular, if $(\mathbf{G}(z),\mathbf{S}(z))$ is a pair of FIR ISI-free solutions, then $(\mathbf{U}'(z)\mathbf{G}(z),\mathbf{S}(z)\mathbf{U}'^{-1}(z))$ is also a pair of FIR ISI-free solutions for any choice of unimodular matrix $\mathbf{U}'(z)$.

Necessity: Suppose that the added redundancy $K<\rho(N)$ and that there exist FIR $\mathbf{G}(z)$ and $\mathbf{S}(z)$ such that the system is ISI free, i.e., $\mathbf{S}(z)\mathbf{C}(z)\mathbf{G}(z)=\mathbf{I}_M$. Using $\mathbf{C}(z)=\mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{V}(z)$, we have

$$(\mathbf{S}(z)\mathbf{U}(z))\mathbf{\Gamma}(z)(\mathbf{V}(z)\mathbf{G}(z))=\mathbf{I}_M. \quad (16)$$

As $\gamma_k(z)$ divides $\gamma_{k+1}(z)$, the zeros of $\gamma_k(z)$ are also zeros of $\gamma_{k+1}(z)$. The last $\rho(N)$ nonidentity terms have a common factor $\gamma_{N-\rho(N)}(z)$. If $\gamma_{N-\rho(N)}(z_0)=0$, then

$$\gamma_{N-\rho(N)}(z_0)=\gamma_{N-\rho(N)+1}(z_0)=\cdots=\gamma_{N-1}(z_0)=0.$$

It follows that $\mathbf{\Gamma}(z_0)$ has rank $N-\rho(N)$, and the left-hand side of (16) at $z=z_0$ has at most rank $N-\rho(N)$. However, the rank of the right-hand side of (16) is always equal to $M=N-K$, which is greater than $N-\rho(N)$ when $K<\rho(N)$. Therefore, we have a contradiction in this case. △

The necessary and sufficient condition given in the above theorem can be replaced as

$$\text{rank}[\mathbf{C}(z)]\geq M, \quad \text{for all } z. \quad (17)$$

We can see this by using the result in Lemma 4.1 that the smallest rank of $\mathbf{C}(z)$ is $N-\rho(N)$. As $K=N-M$, the condition $K\geq\rho(N)$ holds if and only if $N-\rho(N)\geq M$, i.e., $\min_z\text{rank}(\mathbf{C}(z))\geq M$. Therefore, we have (17). It turns out that the smallest rank of $\mathbf{C}(z)$ can be determined by the number of *congruous zeros* of the channel $P(z)$ to be defined below.

Definition 2—Congruous Zeros: A set of zeros $\{\alpha_{k_1}, \alpha_{k_2}, \ldots \alpha_{k_q}\}$ of $P(z)$ are congruous with respect to N if

i) $\alpha_{k_1}, \alpha_{k_2}, \ldots \alpha_{k_q}$ are distinct;
ii) $\alpha_{k_1}^N = \alpha_{k_2}^N = \cdots = \alpha_{k_q}^N$.

Definition 3: The notation $\mu(N)$ denotes the cardinal of the largest set of congruous zeros with respect to N. If there are no congruous zeros, we define $\mu(N) = 1$.

The zeros that are congruous are distinct, but their magnitudes are the same, and their angles differ by an integer multiple of $2\pi/N$. That is

$$|\alpha_{k_1}| = |\alpha_{k_2}| = \cdots = |\alpha_{k_q}|$$

and

$$\alpha_{k_m}/\alpha_{k_n} = e^{j\ell_{m,n}2\pi/N}, \quad \text{where } 1 \leq \ell_{m,n} \leq N-1.$$

The number $\mu(N)$ represents the largest number of distinct zeros that have the same magnitude, and their differences in angles are integer multiples of $2\pi/N$.

Lemma 4.2: For the $N \times N$ channel matrix $\mathbf{C}(z)$ given in (7), we have $\rho(N) = \mu(N)$, where $\rho(N)$ and $\mu(N)$ are as in Definition 1 and 3.

Proof: Consider the decomposition in (32). Because $\mathbf{W}$ and $\mathbf{D}(z)$ are unitary matrices, the rank of $\mathbf{C}(z)$ is the same as the rank of $\mathbf{\Sigma}(z)$. Recall that $\mathbf{\Sigma}(z) = \mathrm{diag}(P(z)\ \ P(zW^{-1})\ \ \cdots\ \ P(zW^{-(N-1)}))$. The number of terms on the diagonal of $\mathbf{\Sigma}(z)$ that have common zeros determines the smallest rank of $\mathbf{\Sigma}(z)$.

Observe that the zeros of $P(zW^{-k})$ are those of $P(z)$ rotated by $2k\pi/N$. If $P(z)$ and $P(zW^{-k})$ have a common zero α, then both α and $\alpha e^{j2k\pi/N}$ are zeros of $P(z)$; the two zeros α and $\alpha e^{j2k\pi/N}$ are congruous. The largest number of terms on the diagonal of $\mathbf{\Sigma}(z)$ that have common zeros is the same as the largest number of congruous zeros $\mu(N)$. Therefore, the smallest rank of $\mathbf{\Sigma}(z)$ is $N - \mu(N)$. By Lemma 4.1, we know the smallest rank of $\mathbf{C}(z)$ is $N - \rho(N)$. Therefore, we have $\rho(N) = \mu(N)$. Δ

By combining Theorem 4.1 and Lemma 4.2, we can relate the existence of FIR transceivers to the zeros of the channel $P(z)$.

Theorem 4.2: Consider the filterbank transceiver in Fig. 1 with interpolation ratio N, number of bands M, and redundancy $K = N - M$. Then, FIR transceivers exist if and only if $K \geq \mu(N)$, where $\mu(N)$ is the largest number of congruous zeros, as given in Definition 3.

Example 1: Consider the second-order channel $P(z) = 1 + 2z^{-1} + z^{-2}$. The channel has double zeros at $z = -1$. The number of zeros on the unit circle is 2, but the number of distinct zeros is one. In this case, $\rho(N) = 1$ for all N. For instance, $M = 1$, and $K = 1$; we have $N = 2$. The polyphases of $P(z)$ with respect to $N = 2$ are $P_0(z) = 1 + z^{-1}$ and $P_1(z) = 2$. The channel matrix $\mathbf{C}(z)$ is given by

$$\mathbf{C}(z) = \begin{pmatrix} 1+z^{-1} & 2z^{-1} \\ 2 & 1+z^{-1} \end{pmatrix}.$$

The Smith form $\mathbf{\Gamma}(z)$ of $\mathbf{C}(z)$ is

$$\mathbf{\Gamma}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1-2z^{-1}+z^{-2} \end{pmatrix}.$$

One set of choices of $\mathbf{U}(z)$ and $\mathbf{V}(z)$ is

$$\mathbf{U}(z) = \begin{pmatrix} 1+z^{-1} & 1 \\ 2 & 0 \end{pmatrix} \quad \text{and}$$
$$\mathbf{V}(z) = \begin{pmatrix} 1 & 0.5(1+z^{-1}) \\ 0 & -0.5 \end{pmatrix}.$$

We can choose $\mathbf{G}(z)$ and $\mathbf{S}(z)$ according to (15):

$$\mathbf{G}(z) = \mathbf{V}^{-1}(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\mathbf{S}(z) = (1 \quad 0)\mathbf{U}^{-1}(z) = (0 \quad 0.5).$$

Example 2: Consider the channel $P(z) = 1 + z^{-3}$ with $L = 3$. The channel has zeros at $z = e^{j\pi/3}$, $e^{-j\pi/3}$, and -1. Let $M = 1$ and $K = 1$. In this case, $N = 2$. The polyphases of $P(z)$ with respect to $N = 2$ are $P_0(z) = 1$ and $P_1(z) = z^{-1}$. The channel matrix is given by

$$\mathbf{C}(z) = \begin{pmatrix} 1 & z^{-2} \\ z^{-1} & 1 \end{pmatrix}.$$

As there are no congruous zeros with respect to $N = 2$, we have $\rho(2) = 1$. The nonunity diagonal element is

$$\gamma_1(z) = (1 - e^{j2\pi/3}z^{-1})(1 - e^{-j2\pi/3}z^{-1}) \times (1 - (-1)^2 z^{-1}) = 1 - z^{-3}.$$

The Smith form of $\mathbf{C}(z)$ is

$$\mathbf{\Gamma} = \begin{pmatrix} 1 & 0 \\ 0 & 1-z^{-3} \end{pmatrix}.$$

Therefore, the choice $K = 1$ ensures the existence of FIR transceivers. In this example, we also see that the interpolation ratio N can be smaller than the channel order L. Now, suppose we increase redundancy to $K = 2$, and N becomes 3. The polyphases of $P(z)$ with respect to $N = 3$ are $P_0(z) = 1+z^{-1}$, $P_1(z) = 0$, and $P_2(z) = 0$. The channel matrix $\mathbf{C}(z)$ is given by

$$\mathbf{C}(z) = \begin{pmatrix} 1+z^{-1} & 0 & 0 \\ 0 & 1+z^{-1} & 0 \\ 0 & 0 & 1+z^{-1} \end{pmatrix}.$$

It is already in Smith form; the Smith form $\mathbf{\Gamma}(z) = \mathbf{C}(z)$. The number of nonidentity terms $\rho(3)$ on the diagonal of $\mathbf{\Gamma}(z)$ is 3. This result is also consistent with the fact that the three zeros $z = e^{j\pi/3}$, $e^{-j\pi/3}$, and -1 are congruous with respect to $N = 3$. FIR transceiver solutions do not exist in this case.

Example 3: Consider the second-order channel $P(z) = 1 + 2\sin\epsilon z^{-1} + z^{-2}$. The zeros are $e^{j(\pi/2+\epsilon)}$ and $e^{-j(\pi/2+\epsilon)}$. Let $N = 2$. For small ϵ, the zeros are almost congruous with respect to $N = 2$. That is, when $N = 2$, the two zeros of $\det \mathbf{C}(z)$ are distinct but clustered. The channel matrix is given by

$$\mathbf{C}(z) = \begin{pmatrix} 1+z^{-1} & 2\sin\epsilon z^{-1} \\ 2\sin\epsilon & 1+z^{-1} \end{pmatrix}.$$

When $\sin\epsilon = 0$, the Smith form of $\mathbf{C}(z)$ is the same as $\mathbf{C}(z)$, and FIR transceivers do not exist. When $\sin\epsilon \neq 0$, it has the following Smith-form decomposition:

$$\mathbf{C}(z) = \begin{pmatrix} 1+z^{-1} & -1/2\sin\epsilon \\ 2\sin\epsilon & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+2\cos 2\epsilon z^{-1}+z^{-2} \end{pmatrix} \times \begin{pmatrix} 1 & 1+\frac{1}{2\sin\epsilon}(1+z^{-1}) \\ 0 & 1 \end{pmatrix}.$$

If $\sin\epsilon \approx 0$, we have $\mathbf{C}(-1) \approx \mathbf{0}$. However, rank$(\mathbf{\Gamma}(-1)) \geq 1$, as long as $\sin\epsilon \neq 0$. We can see that the unimodular matrices $\mathbf{U}(-1)$ and $\mathbf{V}(-1)$ reduce the rank of $\mathbf{\Gamma}(-1)$. Therefore, $\mathbf{U}(-1)$ or $\mathbf{V}(-1)$ is an ill-conditioned matrix, although they are unimodular and have constant determinants. To be more specific, one can compute the condition number. For an $N \times N$ matrix $\mathbf{A}$, the condition number is defined as $||\mathbf{A}||||\mathbf{A}^{-1}||$, where $||\cdot||$ denotes matrix norm. Let us use the matrix norm defined as the maximum of the absolute column sum, i.e., $||\mathbf{A}|| = \max_j \sum_{i=0}^{N-1} |[\mathbf{A}]_{i,j}|$. We can verify that the condition number of $\mathbf{V}(-1)$ is one, whereas for small $\sin\epsilon$, the condition number of $\mathbf{U}(-1)$ is $|1/(2\sin\epsilon)|^2$. The condition number of $\mathbf{U}(-1)$ goes to infinity as $\sin\epsilon$ approaches zero.

Remarks:

1) The number $\rho(N)$ is the largest number of *distinct* zeros of $P(z)$ that have the same magnitude but differs in angles by integer multiples of $2\pi/N$. Zeros of multiplicity greater than one count as one. This is demonstrated in Example 1. The channel $P(z)$ has double roots at $z=-1$, but $\rho(2)=1$.
2) When $\rho(N)=1$, we only need to use redundancy $K=1$, which is the lowest redundancy possible for any nonideal channel when the transceiver is FIR. The case $\rho(N)=1$ occurs when $\det\mathbf{C}(z)$ has distinct zeros. We know that $\det\mathbf{C}(z)$ has roots at α_ℓ^N, where α_ℓ for $\ell = 1,2,\ldots,L$ are the L roots of the channel $P(z)$. The roots of $\det\mathbf{C}(z)$ are distinct if and only if α_ℓ have the property that

$$\frac{\alpha_k}{\alpha_\ell} \neq e^{j2m\pi/N}, \quad \text{where } m \text{ is any integer in the range } 1 \leq m \leq N-1. \quad (18)$$

That means that if two roots α_k and α_ℓ are of the same magnitude, their phase difference can not be a multiple of $2\pi/N$. This condition is similar to that given in [11]. However, $\det\mathbf{C}(z)$ having distinct zeros is not necessary for $\rho(N)=1$ as $P(z)$ can have multiple zeros. For practical channels, the probability that the roots of $P(z)$ satisfy (18) is almost one. Therefore, redundancy of $K=1$ is sufficient for the existence of FIR ISI-free transceivers for most practical cases. However, when $\det\mathbf{C}(z)$ has distinct but clustered zeros, the condition number of the transmitter or receiver are very large, as demonstrated in Example 3.
3) *When is $\rho(N)$ equal to L?* The minimum redundancy required for FIR transceivers falls into the range $1 \leq \rho(N) \leq L$. The minimum redundancy $\rho(N)=L$ if and only if all the zeros of $P(z)$ are congruous. This happens if and only if $P(z)$ has distinct zeros and these zeros lie on the same circle with angles difference that are integer multiples of $2\pi/N$,
4) When $P(z)$ has distinct zeros, the number $\rho(N)$ can be determined from the multiplicities of the zeros of det$(\mathbf{C}(z))$. Suppose det$(\mathbf{C}(z))$ has q distinct roots $\beta_1,\beta_2,\ldots,\beta_q$ with multiplicities, respectively, $\rho_1,\rho_2,\ldots,\rho_q$. Then, it can be verified that $\rho(N)$ is equal to the maximum of the multiplicities, i.e.,

$$\rho(N) = \max\{\rho_1,\rho_2,\ldots,\rho_q\}.$$

5) Suppose solutions of FIR transceivers exist for a given K. FIR solutions do not necessarily exist if we increase redundancy from K to $K+1$ and keep M fixed. The channel in Example 2 demonstrates that when $M=1$, $K=1$, and $N=2$, we have $\rho(2)=1$; FIR solutions exist for the case. However, when we increase K to 2, keeping $M=1$, i.e., $N=3$, we have $\rho(3)=3$; there are no FIR solutions in this case.
6) Since the order of channel is finite, we can always find N such that $\mu(N)=1$, and redundancy $K=1$ can be used.
7) For a given M and N, the condition in (1) provides a test for the existence of FIR transceivers [11]. If the condition is not satisfied, FIR transceivers do not exist for the given pair of (N,M). It does not provide a permissible pair of solution. On the other hand, for a given N, Theorem 4.1 gives the minimum redundancy or the maximum M that ensures the existence of FIR transceivers.

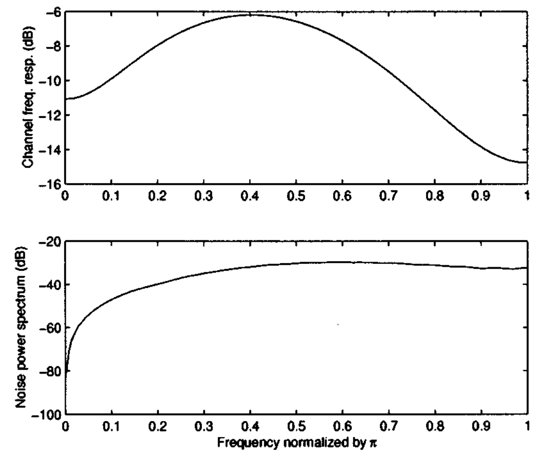


Fig. 6. (Top) Magnitude response of the channel $P(z)$. (Bottom) Power spectrum of the additive noise $e(n)$.

Example 4: Consider the channel $P(z)$ and power spectrum of the colored noise $\mathbf{e}(n)$ shown in Fig. 6. The coefficient of the channel $p(n)$ is $(0.1659\ 0.3045\ -0.1159\ -0.0733\ -0.0015)$. The channel $P(z)$ has order $=4$. The channel and channel noise are drawn from an ADSL environment. For $N=5$, the minimum redundancy is one, and we choose $K=1$ and $M=4$. The FIR transmitter and receiver is as given in (15). The inputs are BPSK symbols, rendering a bit rate of 0.8 bits/sample. The transmission power is the variance of the signal $t(n)$, as indicated in Fig. 1. The plot of bit error rate versus transmission power is given in Fig. 7. For comparison, we also plot the bit error rate performance of DFT-based system with the same bit

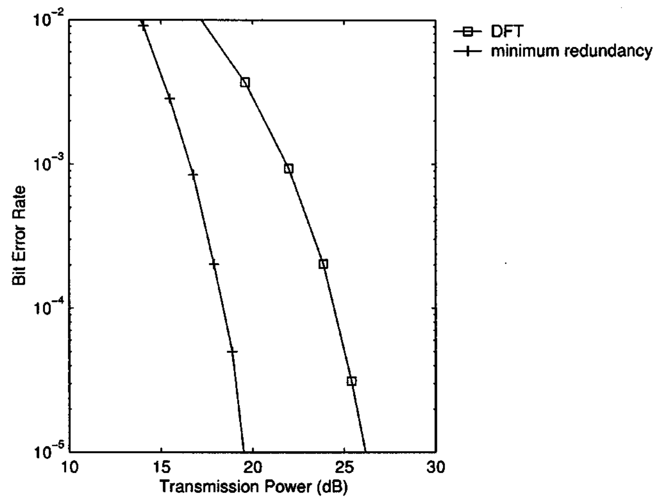


Fig. 7. Bit error rate for the DMT system with minimum redundancy and DFT-based system with the usual redundancy.

rate and relative redundancy, i.e., same K/N or same K/M. We choose $M = 16$, $K = 4$. The system with minimum redundancy requires a much less transmission power for the same bit error rate.

Remarks: In the above example, the minimum redundancy is one, whereas the usual redundancy is 4. In most cases, the minimum redundancy is less than the usual redundancy L. At the same relative redundancy, the system with minimum redundancy has a smaller M, i.e., a shorter block length. For the same bit error rate, the system with minimum redundancy enjoys a smaller transmission power. However, as the system is not DFT based, the transceiver solution has more channel-dependent elements in the design and implementation phases.

V. Block-Based Transceivers

The M-band filterbank transceiver shown in Fig. 3 is called block based if the transmitter and the receiver are constant matrices, i.e., $\mathbf{G}(z) = \mathbf{G}$ and $\mathbf{S}(z) = \mathbf{S}$. The encoding at the transmitter side and the decoding at the receiver end can be performed blockwise. Typically, in block-based DMT (BDMT) systems, the redundancy K is chosen to be the order of the channel L for ISI cancellation. In this section, we will consider BDMT transceiver with redundancy $K \leq L$. Moreover, we will derive minimum redundancy for BDMT systems. When ISI-free solutions of the BDMT system with minimum redundancy exist, complete parameterization of the transmitter and receiver will be given.

A. Block-Based Transceivers With Reduced Redundancy

The block-based DMT (BDMT) system can be seen as a special case of FIR transceivers, where the transmitting filters and receiving filters have length $\leq$ the interpolation ratio N. The BDMT transceivers have been studied by a number of researchers [10], [13]–[15]. For a given FIR channel $P(z)$ with order L, redundancy of length $K = L$ is sufficient for the existence of BDMT transceivers.

1) Two Widely Used BDMT Transceivers: Most of the BDMT transceivers fall into the categories of trailing-zero transmitters and leading-zero receivers. In the DFT-based DMT systems [9], redundancy is in the form of cyclic prefix of length L. The prefix is discarded at the receiving end; the receiver is of the form (leading zeros)

$$\mathbf{S}_{LZ} = (\mathbf{0} \quad \mathbf{S}') \tag{19}$$

where $\mathbf{S}'$ is of dimensions $M \times M$. Another commonly used form of redundancy is zero padding. Zero padding of length L are used in [10], [13], [15]. In this case, the transmitter $\mathbf{G}$ is of the form (trailing zeros)

$$\mathbf{G}_{TZ} = \begin{pmatrix} \mathbf{G}' \\ \mathbf{0} \end{pmatrix} \tag{20}$$

where $\mathbf{G}'$ is of dimensions $M \times M$.

2) Useful Special Case of BDMT Transceivers With Reduced Redundancy: Let us consider a subclass of BDMT system with reduced redundancy. Suppose the transmitter is in the form of trailing zeros (20). Assume that the redundancy is $L/2 \leq K \leq L$ and that the receiver is in the form of leading zeros $\mathbf{S} = (\mathbf{0}\ \mathbf{S}')$, where $\mathbf{S}'$ is an $M \times (M + 2K - L)$ matrix. Unlike the conventional leading-zero receiver, it has only the first $L - K$ columns equal to zeros. In this case, the ISI-free condition in (8) becomes

$$\mathbf{SC}(z)\mathbf{G} = \mathbf{S}'\mathbf{B}\mathbf{G}' = \mathbf{I} \tag{21}$$

where $\mathbf{B}$ is the bottom left $(M + 2K - L) \times L$ submatrix of $\mathbf{C}(z)$. The matrix $\mathbf{B}$ is Toeplitz, given by

$$\mathbf{B} = \begin{pmatrix} p_{L-K} & \cdots & p_0 & 0 & \cdots & 0 \\ \vdots & \ddots & & & & \\ p_K & & & & & \\ \vdots & \ddots & & \ddots & & p_0 \\ p_L & & & & & \vdots \\ 0 & & & \ddots & & p_{L-K} \\ \vdots & & & & & \vdots \\ 0 & & & 0 & p_L \; \cdots & p_K \end{pmatrix}_{(M+2K-L)\times M.} \tag{22}$$

The necessary and sufficient condition for the existence of the ISI-free transceiver is that the matrix $\mathbf{B}$ has a left inverse. When $K = L/2$ (L even case), $\mathbf{B}$ is M by M, and the inverse is unique when it exists. If $K > L/2$, the left inverse of B, when it exists, is not unique. For a given $\mathbf{G}'$, we can choose $\mathbf{S}'$ as

$$\mathbf{S}' = \mathbf{G}'^{-1}\mathbf{Q} \tag{23}$$

where $\mathbf{Q}$ is any left inverse of $\mathbf{B}$. For most of the practical channels $P(z)$ in our experiments, the matrix $\mathbf{B}$ has a left inverse.

Example 5—Comparison of ISI-Free DCT Transceivers With Different Redundancy: Consider the channel $P(z)$ and noise power spectrum (Fig. 6) used in Example 4. Let us consider block-based DCT transceivers with two different cases of redundancy: reduced redundancy $K = 3$ and conventional length of redundancy $K = L = 4$. The transmitter used in this example is in trailing zero form (20), and $\mathbf{G}'$ is an $M \times M$ DCT matrix. From (21), we know, for an ISI-free solution, that we can choose $\mathbf{S} = (\mathbf{0}\ \mathbf{G}'^{-1}\mathbf{Q})$, where $\mathbf{Q} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}$ is an left inverse of

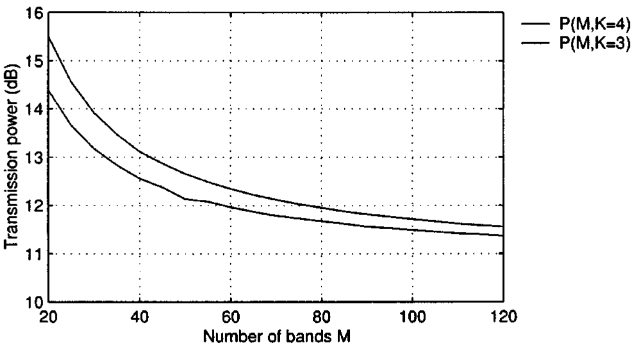


Fig. 8. Transmission power of the DCT transceiver with reduced redundancy $K = 3$ and with conventional length of redundancy $K = 4$ for transmission bit rate $R_b = 3$ bits/sample and symbol error rate $= 10^{-6}$.

$\mathbf{B}$. The bits are allocated optimally as in [15]. For a fixed probability of error P_e and transmission bit rate R_b, the required transmission power $\mathcal{P}(M, K)$ is a function of the number of bands M and redundancy K. With transmission rate $R_b = 3$ bits/sample and symbol error rate $= 10^{-6}$, Fig. 8 shows the required transmission power $\mathcal{P}(M, K = 3)$ and $\mathcal{P}(M, K = 4)$ for different values of M. We can see that the required transmission power decreases as the number of bands M increases. For the same M, the DMT system with reduced redundancy $K = 3$ requires less power than the one with the usual redundancy $K = 4$. On the other hand, we can also compare these two systems with the same relative redundancy K/N or same K/M. For the same relative redundancy, the DMT system with $K = L$ has a larger M, and the performance is comparable. For example, the required power $\mathcal{P}(M = 60, K = 3)$ and $\mathcal{P}(M = 80, K = 4)$ are about the same. The DMT system with reduced redundancy can achieve the same performance with a smaller number of bands.

3) Minimum Redundancy of BDMT Transceivers: In what follows, we will consider more general BDMT systems that are not restricted to the leading-zeros form in (19) or trailing-zero form in (20). The transmitter $\mathbf{G}$ is a general $N \times M$ matrix, and the receiver $\mathbf{S}$ is a general $M \times N$ matrix. Assuming $N > L$, the channel matrix is causal, FIR, of first order, as in (34). The overall transfer function $\mathbf{T}(z)$ is also causal, of first order, $\mathbf{T}(z) = \mathbf{T}_0 + z^{-1}\mathbf{T}_1$, where $\mathbf{T}_0 = \mathbf{S}\mathbf{C}_0\mathbf{G}$, and $\mathbf{T}_1 = \mathbf{S}\mathbf{C}_1\mathbf{G}$. The BDMT is ISI-free if

$$\begin{aligned} &\mathbf{S}\mathbf{C}_0\mathbf{G} = \mathbf{I}, \quad \text{(condition i)} \quad \text{and} \\ &\mathbf{S}\mathbf{C}_1\mathbf{G} = \mathbf{0}, \quad \text{(condition ii)} \end{aligned} \tag{24}$$

where $\mathbf{C}_0$ and $\mathbf{C}_1$ are as defined in (34). When the second condition holds, the system has zero interblock interference (IBI). This condition is necessary for blockwise encoding and decoding. In [15], it is shown that an IBI-free condition can be achieved with redundancy $K = \lceil L/2 \rceil$. The transceiver considered therein has a transmitter in the form of trailing zeros (20) and a receiver in the form of leading zeros (19). The following lemma will show that the redundancy $K = \lceil L/2 \rceil$ is also the minimum redundancy for an IBI-free property.

Lemma 5.1: Consider the DMT transceiver in Fig. 3 with interpolation ratio N and redundancy K. Suppose it is block based with $\mathbf{G}(z) = \mathbf{G}$ and $\mathbf{S}(z) = \mathbf{S}$. The DMT system is IBI free, i.e., $\mathbf{S}\mathbf{C}_1\mathbf{G} = \mathbf{0}$ only if redundancy K satisfies $2K \geq L$.

Proof: The matrix $\mathbf{C}_1$ is Toeplitz, and it has rank L as p_L is assumed to be nonzero. In addition, $\mathbf{G}$ is full rank of dimensions $N \times (N - K)$; the nullity or the dimension of the null space of $\mathbf{G}^T$ is K. We have

$$\text{rank}(\mathbf{C}_1\mathbf{G}) \geq L - K. \tag{25}$$

The equality holds if and only if the null space of $\mathbf{G}^T$ is contained in the row space of $\mathbf{C}_1$. Similarly, the nullity of $\mathbf{S}$ is K; we have

$$\text{rank}(\mathbf{S}\mathbf{C}_1\mathbf{G}) \geq \text{rank}(\mathbf{C}_1\mathbf{G}) - K \geq L - 2K. \tag{26}$$

The first inequality becomes an equality if and only if the null space of $\mathbf{S}$ is in the range space of $\mathbf{C}_1\mathbf{G}$. The second inequality is due to (25). When the system is IBI free, $\text{rank}(\mathbf{S}\mathbf{C}_1\mathbf{G}) = 0$, and from (26), we can see that this is true only if $K \geq L/2$. $\Delta\Delta\Delta$

Remarks: For a given N, we can compute the minimum redundancy $\mu(N)$ for the existence of FIR transceivers, as in Section IV. When $\mu(N) > L/2$, FIR solutions do not exist, let alone block-based solutions. The condition in Lemma 5.1 gives only the necessary condition for the existence of IBI-free block-based transceivers. It does not guarantee existence. The problem of finding the minimum redundancy sufficient for the existence of IBI-free block-based DMT transceivers is still open.

B. Parameterization of Block-Based DMT Systems With Minimum Redundancy

When ISI-free block-based DMT systems with minimum redundancy exist, we can parameterize the solutions. We will assume that L is even and that $K = L/2$. Let the top right $L \times L$ submatrix of $\mathbf{C}_1$ be $\mathbf{C}_\Delta$; then

$$\mathbf{C}_1 = \begin{pmatrix} \mathbf{0} & \mathbf{C}_\Delta \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

The matrix $\mathbf{C}_\Delta$ is nonsingular as $p_L \neq 0$. Let

$$\mathbf{S} = (\mathbf{S}_0 \quad \mathbf{S}_1), \quad \mathbf{G} = \begin{pmatrix} \mathbf{G}_0 \\ \mathbf{G}_1 \end{pmatrix}$$

where $\mathbf{S}_0$ and $\mathbf{S}_1$ are of dimensions $M \times L$ and $M \times (M - L/2)$, respectively, and $\mathbf{G}_0$ and $\mathbf{G}_1$ are of dimensions $(M - L/2) \times M$ and $L \times M$, respectively. Then, condition ii) of (24) becomes

$$\mathbf{S}\mathbf{C}_1\mathbf{G} = \mathbf{S}_0\mathbf{C}_\Delta\mathbf{G}_1 = \mathbf{0}.$$

Lemma 5.2: Consider the block-based DMT transceiver with redundancy $K = L/2$, where L is even. a) The DMT system is ISI free only if $\text{rank}(\mathbf{S}_0) = \text{rank}(\mathbf{G}_1) = L/2$. b) The transceivers satisfying these rank conditions in a) are of the form

$$\begin{aligned} \mathbf{S} &= \mathbf{S}_M \begin{pmatrix} \mathbf{\Phi}_S & \mathbf{I}_M \\ \mathbf{0} & \mathbf{I}_M \end{pmatrix} \begin{pmatrix} \mathbf{P}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{M-L/2} \end{pmatrix} \\ \mathbf{G} &= \begin{pmatrix} \mathbf{I}_{M-L/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_G \end{pmatrix} \begin{pmatrix} \mathbf{I}_M \\ \mathbf{0} \quad \mathbf{\Phi}_G \end{pmatrix} \mathbf{G}_M \end{aligned} \tag{27}$$

where $\mathbf{\Phi}_S$ and $\mathbf{\Phi}_G$ are $L/2$ by $L/2$ arbitrary matrices, and $\mathbf{P}_S$ and $\mathbf{P}_G$ are $L \times L$ permutation matrices.

Proof:

a) The column space of $\mathbf{S}_0^T$ is orthogonal to that of $\mathbf{C}_\Delta\mathbf{G}_1$. As $\mathbf{C}_\Delta$ has full rank, the rank of $\mathbf{C}_\Delta\mathbf{G}_1$ is the same as the rank of $\mathbf{G}_1$. The condition $\mathbf{S}_0\mathbf{C}_\Delta\mathbf{G}_1 = \mathbf{0}$ implies that rank$(\mathbf{S}_0)+$rank$(\mathbf{G}_1) \le L$. On the other hand, notice that condition i) of (24) requires that $\mathbf{G}$ and $\mathbf{S}$ be full rank. The matrix $\mathbf{G}_0$ is $(M-L/2)$ by M and has rank at most $M-L/2$. It follows that

$$M = \text{rank}(\mathbf{G}) \le \text{rank}(\mathbf{G}_0) + \text{rank}(\mathbf{G}_1) \\ \le M - L/2 + \text{rank}(\mathbf{G}_1).$$

This means that rank$(\mathbf{G}_1) \ge L/2$. Similarly, for $\mathbf{S}$ to be full rank, it is necessary to have rank$(\mathbf{S}_0) \ge L/2$. Combining these with the condition rank$(\mathbf{S}_0) +$ rank$(\mathbf{G}_1) \le L$, we can conclude rank$(\mathbf{S}_0) =$ rank$(\mathbf{G}_1) = L/2$.

b) We first consider the case where the first $L/2$ columns of $\mathbf{S}_0$ are linear combinations of the last $L/2$ columns, i.e., $\mathbf{S}_0 = \mathbf{S}'_0(\mathbf{\Phi}_s \;\; \mathbf{I}_{L/2})$, where $\mathbf{\Phi}_s$ is of dimensions $L/2$ by $L/2$, and $\mathbf{S}'_0$ is $M \times L/2$. Then, we have

$$\mathbf{S} = (\mathbf{S}_0 \quad \mathbf{S}_1) \\ = \underbrace{(\mathbf{S}'_0 \quad \mathbf{S}_1)}_{S_M} \begin{pmatrix} \mathbf{\Phi}_S & \mathbf{I}_{L/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{M-L/2} \end{pmatrix}$$

where the matrix $\mathbf{S}_M$ is $M \times M$. More general $\mathbf{S}$ can be obtained by introducing a permutation matrix $\mathbf{P}_S$ for $\mathbf{S}_0$, as given in (27). In a similar manner, we can obtain $\mathbf{G}$ as in (27).

$\Delta\Delta\Delta$

Note that the matrices $\mathbf{S}_M$ and $\mathbf{G}_M$ are M by M, and they are nonsingular because $\mathbf{S}$ and $\mathbf{G}$ are full rank. Using (27), condition ii) in (24) becomes

$$(\mathbf{\Phi}_S \quad \mathbf{I}_{L/2})\mathbf{P}_S\mathbf{C}_\Delta\mathbf{P}_G \begin{pmatrix} \mathbf{I}_{L/2} \\ \mathbf{\Phi}_G \end{pmatrix} = \mathbf{0}. \tag{28}$$

Let

$$\mathbf{P}_S\mathbf{C}_\Delta\mathbf{P}_G = \begin{pmatrix} \mathbf{\Delta}_{00} & \mathbf{\Delta}_{01} \\ \mathbf{\Delta}_{10} & \mathbf{\Delta}_{11} \end{pmatrix}.$$

Then, (28) can be rewritten as

$$\mathbf{\Phi}_S\mathbf{\Delta}_{00} + \mathbf{\Phi}_S\mathbf{\Delta}_{01}\mathbf{\Phi}_G + \mathbf{\Delta}_{10} + \mathbf{\Delta}_{11}\mathbf{\Phi}_G = \mathbf{0}. \tag{29}$$

Using $\mathbf{G}$ and $\mathbf{S}$ in (27), condition i) in (24) becomes

$$\mathbf{S}_M\mathbf{C}_M\mathbf{G}_M = \mathbf{I}, \quad \text{where} \\ \mathbf{C}_M = \begin{pmatrix} \mathbf{\Phi}_S & \mathbf{I}_M \\ \mathbf{0} & \end{pmatrix} \begin{pmatrix} \mathbf{P}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{M-L/2} \end{pmatrix} \mathbf{C}_0 \\ \times \begin{pmatrix} \mathbf{I}_{M-L/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_G \end{pmatrix} \begin{pmatrix} \mathbf{\Phi}_G & \mathbf{I}_M \\ \mathbf{0} & \end{pmatrix}. \tag{30}$$

Using (27), we have converted the two conditions in (24)–(30). From (29) and (30), we can solve for the receiver when the transmitter is given, and similarly, we can solve for the transmitter when the receiver is given. For example, suppose the transmitter is given, that is, $\mathbf{\Phi}_G$ and $\mathbf{G}_M$ are given. We can solve for $\mathbf{\Phi}_S$ in (29). In particular, if $\mathbf{\Delta}_{00} + \mathbf{\Delta}_{01}\mathbf{\Phi}_G$ is nonsingular, we have

$$\mathbf{\Phi}_S = -(\mathbf{\Delta}_{10} + \mathbf{\Delta}_{11}\mathbf{\Phi}_G)(\mathbf{\Delta}_{00} + \mathbf{\Delta}_{01}\mathbf{\Phi}_G)^{-1}. \tag{31}$$

Equation (30) can be satisfied if $\mathbf{C}_M$ is nonsingular. In this case, $\mathbf{S}_M = \mathbf{G}_M^{-1}\mathbf{C}_M^{-1}$.

The design procedure can be summarized as follows. Consider the case when the transmitter is given. Choose $\mathbf{G}_M$, $\mathbf{\Phi}_G$, and $\mathbf{P}_G$ for the transmitter in (27) and also choose $\mathbf{P}_S$ for the receiver. The matrix $\mathbf{G}_M$ is an arbitrary $M \times M$ nonsingular matrix, and $\mathbf{P}_G$ and $\mathbf{P}_S$ are arbitrary permutation matrices. We can solve for $\mathbf{\Phi}_S$ according to (31). Form the matrix $\mathbf{C}_M$ in (30), and compute $\mathbf{S}_M = \mathbf{G}_M^{-1}\mathbf{C}_M^{-1}$. For the case when the receiver is given, the design procedure is similar.

In the parameterization, no additional assumption has been made on the transmitter matrix and the receiver matrix, except that they achieve zero ISI. Therefore, whenever BDMT with redundancy $K = L/2$ exists, it can be parameterized as in this section. The parameterization is useful in cases where ISI-free BDMT solutions exist but there are no ISI-free solutions with trailing-zero and leading-zero constraints. One such example is given below.

Example 6: Consider the FIR channel $P(z) = (1 - z^{-2})^3$ with order $L = 6$. Let $M = 5$ and $K = 3$; then, we have $N = M + K = 8$. We can verify that in this case, the matrix $\mathbf{B}$ given in (22) is singular. There are no ISI-free solutions for BDMT with trailing-zero and leading-zero constraints. On the other hand, let us choose

$$\mathbf{\Phi}_G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can verify that the matrix $\mathbf{\Delta}_{00} + \mathbf{\Delta}_{01}\mathbf{\Phi}_G$ is nonsingular and that the matrix $\mathbf{C}_M$ in (30) is also nonsingular. We can obtain the solution of $\mathbf{\Phi}_S$ from (31) and $\mathbf{S}_M = \mathbf{G}_M^{-1}\mathbf{C}_M^{-1}$ for arbitrarily chosen nonsingular $\mathbf{G}_M$.

Remarks:

1) In the parameterization, no additional assumption has been made on the transmitter matrix and the receiver matrix, except that they achieve zero ISI. Therefore, whenever block-based DMT with redundancy $K = L/2$ exists, it can be parameterized as in this section.
2) The parameterization presented in this section is for even L. Using similar techniques, we can obtain solutions for odd L.

VI. Conclusion

In this paper we show that for a given interpolation ratio N, the minimum redundancy or the maximum number of bands usable for FIR transceivers can be determined exactly. It is directly related to the number of *congruous zeros* of the channel $P(z)$ defined in the paper. In particular, the minimum redundancy that ensures the existence of FIR ISI-free DMT systems is equal to the maximum number of congruous zeros with respect to N. This number, in almost all cases, is less than the usual redundancy used in most systems. However, like all non-DFT-based systems, transceiver design is more channel dependent. The transceiver solutions depend on the channel, and the performance depends on the accuracy of channel estimation. We also demonstrate, through examples, that minimum redundancy may lead to transceiver solutions that contain matrices with large condition numbers. This happens when the zeros are almost congruous, i.e., $\det \mathbf{C}(z)$ have distinct but clustered zeros.

In this paper, we have also shown that for the block-based DMT transceivers, the minimum redundancy is $\lceil L/2 \rceil$, where L is the order of the FIR channel. When a block-based DMT system with redundancy $\lceil L/2 \rceil$ has ISI-free solutions, the solutions are parameterized. The free parameters can be useful for optimizing the transceiver for minimizing output noise or minimizing transmission power for a given probability of error and transmission bit rate. However, the redundancy of length $\lceil L/2 \rceil$ does not guarantee the existence of ISI-free solutions. The determination of the minimum redundancy that guarantees the existence of ISI-free block-based DMT transceiver is still an open problem.

APPENDIX
PROPERTIES OF THE CHANNEL MATRIX

In this Appendix, we give a collection of the properties of the channel matrix in (7) that is useful for our discussion. Some of these properties are known and can be found in text books, e.g., [17]. Some have not been shown explicitly before and will be derived.

1) A pseudo circulant matrix $\mathbf{C}(z)$ of the form in (7) is shown in [16] to assume the decomposition

$$\mathbf{C}(z^N) = \mathbf{D}(z)\mathbf{W}\mathbf{\Sigma}(z)\mathbf{W}^\dagger \mathbf{D}(z^{-1}) \quad (32)$$

where

$$\mathbf{D}(z) = \text{diag}(1 \quad z^{-1} \quad \cdots \quad z^{-N+1})$$
$$\mathbf{\Sigma}(z) = \text{diag}(P(z) \quad P(zW^{-1}) \quad \cdots \quad P(zW^{-N+1})).$$

The matrix $\mathbf{W}$ is the $N \times N$ DFT matrix given by

$$[\mathbf{W}]_{kn} = \frac{1}{\sqrt{N}} W^{kn} \quad \text{with}$$
$$W = e^{-j2\pi/N} \quad \text{for } 0 \le k, \quad n \le N-1.$$

2) When the channel $P(z)$ is a causal FIR filter of order L, $\det \mathbf{C}(z)$ is also a causal FIR filter of order L. Furthermore, suppose $P(z)$ has a zero at α; then, $\det \mathbf{C}(z)$ has a zero at α^N.

Proof: Using (32), we can obtain

$$\det\mathbf{C}(z^N) = \det(\mathbf{\Sigma}(z)) = \prod_{k=0}^{N-1} P(zW^{-k}).$$

It follows that $\det \mathbf{C}(\alpha^N) = \prod_{k=0}^{N-1} P(\alpha W^{-k}) = 0$. As $P(z)$ is of order L, the product filter $\prod_{k=0}^{N-1} P(zW^{-k})$ is an FIR filter of order NL. We know that $\det \mathbf{C}(z)$ is an FIR filter as the polyphases of $P(z)$ are FIR. We can conclude that $\det \mathbf{C}(z)$ is of order L.

3) *Smith-Form Decomposition:* An $N \times N$ polynomial matrix $\mathbf{A}(z)$ in z^{-1} can be represented using the Smith-form decomposition [17]

$$\mathbf{A}(z) = \mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{V}(z) \quad (33)$$

where all three matrices in the decomposition are matrix polynomials in the variable z^{-1}. The matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ are unimodular matrices, the definition of which is given in Section I-A; $\mathbf{\Gamma}(z)$ is a diagonal matrix

$$\mathbf{\Gamma}(z) = \begin{pmatrix} \gamma_0(z) & 0 & \cdots & 0 \\ 0 & \gamma_1(z) & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \gamma_{N-1}(z) \end{pmatrix}.$$

Moreover, the unimodular matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ can be so chosen that the polynomials $\gamma_k(z)$ are monic (i.e., highest power has unity coefficient), and $\gamma_k(z)$ is a factor of $\gamma_{k+1}(z)$, i.e., $\gamma_k(z)$ divides $\gamma_{k+1}(z)$ for $0 \le k \le N-2$. The matrix $\mathbf{\Gamma}(z)$, which is called the Smith form of $\mathbf{A}(z)$, is unique. Although $\mathbf{\Gamma}(z)$ is unique, the unimodular matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ are not. As $\det \mathbf{U}(z)$ and $\det \mathbf{V}(z)$ are both constants, we have

$$\det \mathbf{A}(z) = c \det \mathbf{\Gamma}(z) = c\Pi_{k=0}^{N-1}\gamma_k(z)$$

where $c = \det\mathbf{U}(z)\det\mathbf{V}(z)$.

The Smith form of $\mathbf{C}(z)$: Let the Smith-form decomposition of $\mathbf{C}(z)$ be

$$\mathbf{C}(z) = \mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{V}(z).$$

Note that $\det \mathbf{C}(z) = c\Pi_{k=0}^{N-1}\gamma_k(z)$. The polynomials $\gamma_k(z)$, for $k = 0, 1, \ldots, N-1$, in the diagonal of $\mathbf{\Gamma}(z)$ have the property that $\gamma_k(z)$ divides $\gamma_{k+1}(z)$. On the other hand, from Property 2, we know that $\det \mathbf{C}(z)$ is an FIR filter with order L. This implies that there are at most L nonunity terms among $\{\gamma_k(z)\}$.

4) In many applications [8], [9], [13], the interpolation ratio N is chosen to be larger than the order L of $P(z)$. In this case, the N polyphases of $P(z)$ are constants, and the last $N-L-1$ polyphases are zero. The matrix $\mathbf{C}(z)$ is causal, and of order one

$$\mathbf{C}(z) = \mathbf{C}_0 + z^{-1}\mathbf{C}_1$$

where

$$\mathbf{C}_0 = \begin{pmatrix} p_0 & 0 & \cdots & \cdots & & 0 \\ p_1 & p_0 & & & & 0 \\ \vdots & & \ddots & & & \vdots \\ p_L & p_{L-1} & & & & \\ 0 & p_L & & & & \\ \vdots & \vdots & \ddots & & \ddots & \vdots \\ 0 & 0 & & & & p_0 \end{pmatrix} \quad \text{and}$$

$$\mathbf{C}_1 = \begin{pmatrix} 0 & \cdots & 0 & p_L & p_{L-1} & \cdots & p_1 \\ 0 & & 0 & 0 & p_L & \cdots & p_2 \\ \vdots & & \vdots & \vdots & & \ddots & \vdots \\ & & & & & & p_L \\ & & & & & & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (34)$$

The matrices $\mathbf{C}_0$ and $\mathbf{C}_1$ are both $N \times N$ and Toeplitz; $\mathbf{C}_0$ is lower triangular, and $\mathbf{C}_1$ is upper triangular.

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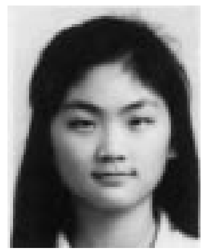

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