Stability-robustness Analysis for Linear Systems with State-space Models

by HORNG-GIOU CHEN *and* KUANG-WEI HAN

Institute of Electronics, National Chiao-Tung University, Hsinchu, Taiwan, R.O.C.

ABSTRACT: *The stability-robustness analysis for linear systems with state-space models is considered. The fundamental problem of linear control systems subject to unstructured perturbations is addressed, and the results are extended to the consideration of linear control systems subject to linear structured perturbations. A modified time-domain Lyapunov-based method of stability-robustness analysis is proposed where iterative interpolations of quadratic Lyapunov functions are considered. By use of the proposed method, less conservative allowable perturbation bounds are obtained, and the resulting Lyapunov matrices possess structural informations closely related to the perturbation-susceptible characteristics of the nominal system state matrix. The robustness behaviour of a vertical takeoff and landing (VTOL) aircraft control system, designed by use of the LQR state-feedback method, is illustrated.*

Notation

We denote the identity matrix by $\mathbf{I}$ and the all zero matrix by $\mathbf{0}$. Given a vector $x \in \mathbb{R}^n$, we take $\|x\| = (x'x)^{1/2}$. This induces the matrix norm $\|M\| = \sigma_{\max}(M)$ where $\sigma_{\max}(\cdot)$ denotes the operation of taking the largest singular value. The following matrix operations and matrix relations are denoted:

$[\cdot]^{1/2}$	square-root of positive-semidefinite matrix
$[\cdot]_s$	symmetric portion of square matrix
$[\cdot]_m$	matrix formed by replacing each entry of a matrix by its modulus value
$[\cdot]_{ps}$	positive-semidefinite matrix formed by replacing each eigenvalue of a symmetric matrix by its modulus value
$\mathbf{P} > 0$	square symmetric matrix $\mathbf{P}$ being positive-definite
$\mathbf{P} \geqslant 0$	square symmetric matrix $\mathbf{P}$ being positive-semidefinite
$\mathbf{P} > \mathbf{Q}$	square symmetric matrices $\mathbf{P}$ and $\mathbf{Q}$ that satisfy $\mathbf{P}-\mathbf{Q} > 0$
$\mathbf{P} \geqslant \mathbf{Q}$	square symmetric matrices $\mathbf{P}$ and $\mathbf{Q}$ that satisfy $\mathbf{P}-\mathbf{Q} \geqslant 0$

I. Introduction

In the analysis and design of robust control systems, the fundamental problems are that the assumed mathematical model for the systems are always inexact, and that the parameters of the systems may deviate away from their nominal values. Thus, it is desirable to be able to determine: (i) to what extent a nominal system remains stable when subject to a certain class of perturbations, and (ii) in what way a nominal controller compensator can be adjusted to rectify the perturbation-susceptible characteristics of the control system. This is called the *quantitative stability-robustness problem* (**1**–**17**).

The published literature on the quantitative stability-robustness analysis of linear systems can be categorized into two perspectives: (i) the *frequency-domain* analysis (**1**–**7**) which is based on the transfer-function representation of a system, and (ii) the *time-domain* analysis (**8**–**16**) which is based on a state-space representation of a system. The main approach in *frequency-domain* analysis is to extend the classical single-input single-output stability margins to multiple-input multiple-output systems by use of the singular-value decomposition method. In particular, singular-value decomposition of the return-difference transfer matrix of a stable feedback control system has been considered, and the tolerable gain and phase changes of an *unstructured* perturbation in frequency domain has been determined by Mukhopadhyay and Newsom (**5**). On the other hand, the *time-domain* approach is more amenable to the consideration of the *structured* perturbations in the form of parameter variations and nonlinearities (**17**). This paper treats the stability-robustness analysis in the time domain.

Starting with Patel *et al.* (**8**), considerable effort has been given to the reduction of conservatism in time-domain quantitative measures of robustness (**9**–**16**). In these contributions, the structural information of the perturbation is algebraically manipulated with the fundamental stability-robustness conditions which were derived for the case of unstructured perturbations. The fundamental stability-robustness conditions are the *Lyapunov-based* result of Patel and Toda (**9**) and the *rootlocus-based* result of Qiu and Davison (**14**), which are the two main techniques of the time-domain robustness analysis. Presently, the rootlocus-based approach (**14**–**16**) is known to produce less conservative measures of robustness where linear perturbations are considered. However, the capability of the Lyapunov-based approach (**9**–**13**) in dealing with nonlinear time-varying perturbations should not be overlooked. It has been argued that, the robustness measures indirectly derived from the quadratic Lyapunov functions are usually conservative (**15**). Nevertheless, the Lyapunov-based methods possess the exclusive feature of accompanying the robustness measure with a quadratic Lyapunov function. Thus, research on the Lyapunov-based stability-robustness analysis is conducted in this paper.

In Section II, we present the formulation of the problems concerning the use of quadratic Lyapunov functions for the stability-robustness analysis of linear state-space models with the associated *unstructured* and *structured* perturbations. It is shown that, the unstructured robustness-measure problem is fundamental to the stability-robustness analysis of a structurally perturbed system. Once an unstructured robustness-measure problem is solved, a pair of Lyapunov matrices is obtained, then the structured perturbation bounds can be derived by algebraically manipulating the Lyapunov matrices with the structured perturbations. A distinct feature of our approach is that, iterative interpolations of quadratic Lyapunov functions are considered to produce less conservative unstructured robustness measures. The extension of the structured perturbations is similar to the results developed by Zhou and Khargonekar (**12**), but less conservative robustness measures are achieved for control systems subject to structured perturbations.

In Section III, we show how to apply the interpolations of quadratic Lyapunov functions to achieve less conservative unstructured robustness measures. It is a mathematical fact that, the Lyapunov-based unstructured robustness measure is obtained by examining the perturbation-susceptible structure of the resulting Lyapunov matrices. A correct suggestion is that, the resulting Lyapunov matrices possess meaningful information concerning the structural content of perturbations to which the nominal system matrix is particularly susceptible. Thus, instead of producing the unstructured robustness measure alone, the perturbation-susceptible structure of the resulting Lyapunov matrices is explicitly derived. By use of the proposed method, improvements in robustness measures and the properties of the explicitly derived perturbation-susceptible structure are illustrated in Section IV by two examples.

Section V illustrates the application of the proposed robustness analysis to a vertical takeoff and landing (VTOL) aircraft control system which has been designed by use of the LQR state feedback method in both (**13**) and (**18**). It is shown that, in reaction to the perturbation-susceptible structure of the VTOL control system, the LQR state-feedback design can be modified to produce more robust results.

II. Problem Formulation and the Main Results

Consider the following dynamical system with perturbations:

$$\mathrm{d}x(t)/\mathrm{d}t = \mathbf{A}x(t)+f(x(t),t), \tag{1}$$

where $\mathbf{A}\in\mathbb{R}^{n\times n}$ is the stable nominal system matrix, and $f(x(t),t)$ is a vector perturbing function with $f(0,t) = 0$ for all time t. The main problem is to study the stability of the perturbed system described by Eq. (1) for various kinds of perturbations. We begin with a review of some results on robust stability due to Patel and Toda (**9**) and Zhou and Khargonekar (**12**).

Result 1. *Robustness bounds for unstructured perturbations* (**9**)

Let the perturbing vector function described in Eq. (1) be unstructured perturbations, i.e. an exact expression of $f(x(t),t)$ cannot be written explicitly, and a measure of perturbing magnitude is given by

$$\mu_f \equiv \mathrm{Max}\,\{\|f(x,t)\|/\|x\|\}, \tag{2}$$

where the Max operates over all $(x,t)\in\mathbb{R}^{n+1}$ with nonzero x. It is shown in (**9**) that the unstructurally perturbed system described by Eqs (1) and (2) is stable if

$$\mu_f < 1/\sigma_{\max}(\mathbf{P}_I) \equiv \mu_I, \tag{3}$$

where $\mathbf{P}_I$ is the unique matrix that satisfies the Lyapunov equation

$$\mathbf{A}'\mathbf{P}_I+\mathbf{P}_I\mathbf{A} = -2\mathbf{I}. \tag{4}$$

We note that, in the Lyapunov-based analysis of allowable perturbation bounds given in (**8**–**13**), the Lyapunov-matrix pair $\{\mathbf{P}_I,\mathbf{I}\}$ in the Lyapunov equation (4) has been adopted for whatever structured perturbations are considered. Thus, the Lyapunov-matrix pair $\{\mathbf{P}_I,\mathbf{I}\}$ constitutes the Lyapunov-based fundamental stability-robustness condition. Followed from the Lyapunov-matrix pair $\{\mathbf{P}_I,\mathbf{I}\}$, the Lyapunov-based robustness bounds for structurally perturbed linear dynamical systems are generalized in the work of Zhou and Khargonekar (**12**).

Result 2. *Robustness bounds for linear structured perturbations* (**12**)

Let the perturbing vector function described in Eq. (1) be linear structured perturbations, i.e.

$$f(x(t),t) = \mathbf{E}(t)x(t) = \sum_{i=1}^{m} k_i(t)\mathbf{E}_i x(t), \tag{5}$$

where $\mathbf{E}_i \in \mathbb{R}^{n\times n}$ are constant matrices, $k_i(t)$ are uncertain time-varying parameters, and the magnitudes of $k_i(t)$ are assumed to vary in the intervals around zero, i.e. $k_i(t) \in [-\varepsilon_i, \varepsilon_i]$. It is shown in (**12**) that the structurally perturbed system described by Eqs (1) and (5) is stable if,

(i) for $m \geqslant 2$,

$$|k_j(t)| < 1/\sigma_{\max}\left(\sum_{i=1}^{m} [\mathbf{P}_i]_m\right) \equiv \delta_l, \quad j = 1,2,\ldots,m; \tag{6}$$

or (ii) for $m = 1$,

$$|k_1(t)| < 1/\sigma_{\max}(\mathbf{P}_1), \tag{7}$$

where matrices $\mathbf{P}_i$ are defined by algebraic manipulations of the perturbation matrices $\mathbf{E}_i$ given in Eq. (5) with the Lyapunov-matrix pair $\{\mathbf{P}_l, \mathbf{I}\}$ given in the Lyapunov equation (4), i.e.

$$\mathbf{P}_i = [\mathbf{P}_l\mathbf{E}_i]_s, \quad i = 1,2,\ldots,m. \tag{8}$$

Instead of confining ourselves to work within the fundamental stability-robustness condition specified by the Lyapunov-matrix pair $\{\mathbf{P}_l, \mathbf{I}\}$ in the Lyapunov equation (4), our main result is started with the advent of a modified fundamental stability-robustness condition specified by the Lyapunov-matrix pair $\{\mathbf{P}, \mathbf{Q}\}$ such that the following Lyapunov equation is fulfilled:

$$\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A} = -2\mathbf{Q}, \tag{9}$$

where matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is symmetric positive-definite, and matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ is the symmetric positive-definite solution of the Lyapunov equation. Given Lyapunov-matrix pair $\{\mathbf{P}, \mathbf{Q}\}$ that fulfills the Lyapunov equation (9), the *unstructured* quantitative measure of robustness is given in the following theorem.

Theorem I

Given Lyapunov matrices $\mathbf{P}$ and $\mathbf{Q}$ that fulfill the Lyapunov equation (9), the unstructurally perturbed system described by Eqs (1) and (2) is stable if,

$$\mu_f \leqslant \mathrm{Min}\,\{x'\mathbf{Q}x/\|\mathbf{P}x\|\} \equiv \mu_{\mathrm{L}}, \tag{10}$$

where the Min operates over all $x \in \mathbb{R}^n$ with $\|x\| = 1$. The unstructured robustness bound μ_{L} is also given by

$$\mu_L \equiv 1/\mathrm{Max}\,\{\|\mathbf{Q}^{-1/2}y\|\ \|\mathbf{PQ}^{-1/2}y\|\}, \tag{11}$$

where the Max operates over all $y \in \mathbb{R}^n$ with $\|y\| = 1$.

Proof: Since $\mathbf{V}(x) = x'\mathbf{P}x$ is a Lyapunov function of the stable nominal system matrix $\mathbf{A}$, a sufficient condition for the stability of the perturbed system (1) is

$$(\mathbf{A}x+f)'\mathbf{P}x+x'\mathbf{P}(\mathbf{A}x+f) \leqslant 0, \tag{12}$$

for all $x \in \mathbb{R}^n$ with $\|x\| = 1$. Following the Lyapunov equation (9), we have

$$f'\mathbf{P}x \leqslant x'\mathbf{Q}x, \tag{13}$$

which is sufficiently justified by

$$\|f\|\ \|\mathbf{P}x\| \leqslant x'\mathbf{Q}x. \tag{14}$$

Given the magnitude of perturbation μ_f defined in Eq. (2), we have

$$\mu_f \leqslant x'\mathbf{Q}x/\{\|\mathbf{P}x\|\ \|x\|\}, \tag{15}$$

and the allowable upper bound on the magnitude of perturbation is given by Eq. (10).

By making the replacement of

$$x = \mathbf{Q}^{-1/2}y/\|\mathbf{Q}^{-1/2}y\|, \tag{16}$$

the relation given in (15) becomes

$$\mu_f \leqslant 1/\{\|\mathbf{Q}^{-1/2}y\|\ \|\mathbf{PQ}^{-1/2}y\|\}, \tag{17}$$

and the allowable upper bound on the magnitude of perturbation is given by Eq. (11). ■

Remark (1)

Given matrices $\mathbf{P}$ and $\mathbf{Q}$ that fulfill the Lyapunov equation (9), the unstructured robustness bound μ_L defined in Eq. (11) can be determined numerically. Thus, the unstructured quantitative stability-robustness problem amounts to the judicious choice of matrix $\mathbf{Q}$ in the Lyapunov equation (9) such that less conservative unstructured robustness bound (μ_L) is achieved. This issue is treated in Section III and the Appendix.

On the other hand, given Lyapunov-matrix pair $\{\mathbf{P}, \mathbf{Q}\}$ that fulfills the Lyapunov equation (9), the quantitative measure of robustness for *structurally perturbed* linear dynamical systems is given in the following theorem.

Theorem II

Given matrices $\mathbf{P}$ and $\mathbf{Q}$ that fulfill the Lyapunov equation (9), the structurally perturbed system described by (1) and (5) is stable if

$$|k_j(t)| < 1/\sigma_{\max}\left(\sum_{i=1}^{m} [\mathbf{P}_i]_{ps}\right) \equiv \delta_L, \quad j = 1, 2, \ldots, m; \tag{18}$$

where matrices $\mathbf{P}_i$ are defined by algebraic manipulations of the perturbation matrices $\mathbf{E}_i$ given in Eq. (5) with the Lyapunov-matrix pair $\{\mathbf{P},\mathbf{Q}\}$ given in the Lyapunov equation (9), i.e.

$$\mathbf{P}_i = \mathbf{Q}^{-1/2}[\mathbf{P}\mathbf{E}_i]_s\mathbf{Q}^{-1/2}, \quad i = 1,2,\ldots,m. \tag{19}$$

Proof: We will show that $\mathbf{V}(x,t) = x'\mathbf{P}x$ is a Lyapunov function of the structurally perturbed system under the condition given in Eq. (18).

A simple computation shows that

$$\mathrm{d}\mathbf{V}/\mathrm{d}t = 2x'\mathbf{Q}^{1/2}\left(\sum_{i=1}^{m} k_i(t)\mathbf{P}_i - \mathbf{I}\right)\mathbf{Q}^{1/2}x. \tag{20}$$

It is clear that $\mathrm{d}\mathbf{V}/\mathrm{d}t < 0$ if

$$\sigma_{\max}\left(\sum_{i=1}^{m} k_i(t)\mathbf{P}_i\right) < 1. \tag{21}$$

Note that, for all $i = 1,2,\ldots,m$ and for all $\varepsilon \in [-1,1]$, we have

$$[\mathbf{P}_i]_{ps} - \varepsilon\mathbf{P}_i \geqslant 0. \tag{22}$$

Thus,

$$\sum_{i=1}^{m} |k_i(t)|\,[\mathbf{P}_i]_{ps} \geqslant \sum_{i=1}^{m} k_i(t)\mathbf{P}_i, \tag{23}$$

and

$$\operatorname*{Max}_j |k_j(t)|\,\sigma_{\max}\left(\sum_{i=1}^{m} [\mathbf{P}_i]_{ps}\right) \geqslant \sigma_{\max}\left(\sum_{i=1}^{m} k_i(t)\mathbf{P}_i\right). \tag{24}$$

Hence, Eq. (18) implies (21). ■

III. Derivation of the Robustness Related Perturbation Structure

Let the perturbation-susceptible structure of the matrices **P** and **Q** in the Lyapunov equation (9) be taken into consideration in the analysis of stability robustness, the *robustness-related perturbation structure* of a nominal system matrix **A** is defined in the following definition.

Definition (1)

Consider a stable nominal system matrix **A**. Let matrices **P** and **Q** fulfill the Lyapunov equation (9) and bring forth the unstructured robustness bound μ_L defined in Eq. (11). The robustness-related perturbation structure of the nominal system matrix **A** is the unity-rank matrix $\mu_L vw'$, where v and w are unit vectors within $\mathbb{R}^n$, such that the Lyapunov function $x'\mathbf{P}x$ of the nominal system matrix **A** fails being a Lyapunov function of the perturbed system matrix $\mathbf{A} + \mu_L vw'$.

Given matrices **P** and **Q** that fulfill the Lyapunov equation (9), the existence of the robustness-related perturbation structure $\mu_L vw'$ can be proved by the following theorem.

Theorem III

Let μ_L be the unstructured robustness bound derived by use of Theorem I with matrices **P** and **Q** that fulfill the Lyapunov equation (9). There exist unit vectors v and w within $\mathbb{R}^n$ such that, treating the unity-rank matrix $\mu_L vw'$ as a perturbation to the nominal system matrix **A**, the function $x'\mathbf{P}x$ fails being a Lyapunov function of the perturbed system matrix $\mathbf{A}+\mu_L vw'$.

Proof: The unstructured robustness bound μ_L is given in Eq. (11) for a specific $y \in \mathbb{R}^n$ with $\|y\| = 1$, i.e.

$$\mu_L = 1/\{\|\mathbf{Q}^{-1/2}y\| \, \|\mathbf{P}\mathbf{Q}^{-1/2}y\|\}, \tag{25}$$

and $\mathbf{V}(x) = x'\mathbf{P}x$ is a Lyapunov function of the stable nominal system matrix **A**.

Let unit vectors v and w be chosen as:

$$v = \mathbf{P}\mathbf{Q}^{-1/2}y/\|\mathbf{P}\mathbf{Q}^{-1/2}y\|, \tag{26}$$

and

$$w = \mathbf{Q}^{-1/2}y/\|\mathbf{Q}^{-1/2}y\|. \tag{27}$$

Employing the quadratic function $\mathbf{V}(x)$ on the perturbed system matrix $\mathbf{A}+\mu_L vw'$, we have

$$\mathrm{d}\mathbf{V}(x)/\mathrm{d}t = 2x'(\mathbf{A}'\mathbf{P}+\mu_L wv'\mathbf{P})x. \tag{28}$$

Following the Lyapunov equation (9), we have

$$\mathrm{d}\mathbf{V}(x)/\mathrm{d}t = 2x'(-\mathbf{Q}+\mu_L wv'\mathbf{P})x. \tag{29}$$

We will show that, given relations (25), (26) and (27), the choice of $x = w$ nullifies the right-hand part of Eq. (29) which causes $\mathbf{V}(x)$ to fail being a Lyapunov function of the perturbed system matrix.

Following Eqs (25) and (26), we have the following two relations:

$$w'\mathbf{Q}w = \|y\|^2/\|\mathbf{Q}^{-1/2}y\|^2, \tag{30}$$

and

$$v'\mathbf{P}w = \|\mathbf{P}\mathbf{Q}^{-1/2}y\|/\|\mathbf{Q}^{-1/2}y\|. \tag{31}$$

Thus,

$$w'(-\mathbf{Q}+\mu_L wv'\mathbf{P})w = -1/\|\mathbf{Q}^{-1/2}y\|^2+\mu_L\|\mathbf{P}\mathbf{Q}^{-1/2}y\|/\|\mathbf{Q}^{-1/2}y\| = 0. \tag{32}$$

■

Remark (2)

It is shown implicitly in Definition (1) and Theorem III that, given matrices **P** and **Q** that fulfill the Lyapunov equation (9), the unstructured robustness bound (μ_L) is derived by examining the perturbation-susceptible structure of the Lyapunov matrices. The suggestion is that the Lyapunov matrices possess meaningful information concerning the structural content of perturbations to which the nominal system matrix **A** is particularly susceptible. Presumptively, the less conservative way the unstructured robustness-measure problem is solved, the closer the robustness-related perturbation structure is related to the perturbation-susceptible structure of the nominal system matrix **A**. This relation is illustrated by examples given in Section IV.

Remark (3)

Iterative procedures have been devised such that the proper sequential choice of matrix **Q** in the Lyapunov equation (9) is made and a less conservative unstructured robustness bound (μ_L) is obtained along the process. The proposed procedure is developed with the discovery of some interpolating properties of Lyapunov equations. Theoretical developments and two algorithms for the generation of the Lyapunov matrices **P** and **Q** are given in the Appendix.

We note that, following Definition (1) and Theorem III, the robustness-related perturbation structure of the nominal system matrix **A** is derived by replacing the perturbation $f(x(t), t)$ denoted in the perturbed system (1) by a linear unity-rank perturbation, i.e.

$$f(x(t), t) = \mu_L v w' x(t), \tag{33}$$

where v and w are unit vectors within $\mathbb{R}^n$.

Let $\mathbf{V}(x) = x'\mathbf{P}x$ remain a Lyapunov function for the system (1) perturbed by Eq. (33), then

$$(\mathbf{A}+\mu_L v w')'\mathbf{P}+\mathbf{P}(\mathbf{A}+\mu_L v w') \leqslant 0. \tag{34}$$

Following the Lyapunov equation (9), relation (34) becomes

$$w v'\mathbf{P}+\mathbf{P}v w' \leqslant 2\mu_L^{-1}\mathbf{Q}, \tag{35}$$

or

$$\mathbf{Q}^{-1/2}(w v'\mathbf{P}+\mathbf{P}v w')\mathbf{Q}^{-1/2} \leqslant 2\mu_L^{-1}\mathbf{I}. \tag{36}$$

Thus, there are two relations to be satisfied for all $x \in \mathbb{R}^n$ with $\|x\| = 1$, i.e.

$$x'\mathbf{Q}^{-1/2}\mathbf{P}v w'\mathbf{Q}^{-1/2}x \leqslant \mu_L^{-1}, \tag{37}$$

and

$$w'\mathbf{Q}^{-1/2}x x'\mathbf{Q}^{-1/2}\mathbf{P}v \leqslant \mu_L^{-1}, \tag{38}$$

which constitute the procedure for deriving the robustness-related perturbation structure $\mu_L v w'$.

Algorithm I

Given matrices **P** and **Q** that fulfill the Lyapunov equation (9), we have:

Step 1. Initially, let x be the first singular vector of the matrix $\mathbf{Q}^{-1/2}\mathbf{P}^2\mathbf{Q}^{-1/2}$.

Step 2. Given x, let w and v be correspondingly the left first and right first singular vectors of the matrix $\mathbf{Q}^{-1/2}x x'\mathbf{Q}^{-1/2}\mathbf{P}$. Equivalently, we have

$$w = \mathbf{Q}^{-1/2}x/\|\mathbf{Q}^{-1/2}x\|, \tag{39}$$

$$v = \mathbf{P}\mathbf{Q}^{-1/2}x/\|\mathbf{P}\mathbf{Q}^{-1/2}x\|, \tag{40}$$

and an intermediate value of the unstructured robustness bound μ_L is given by

$$\mu_L = \{\|\mathbf{Q}^{-1/2}x\| \, \|\mathbf{PQ}^{-1/2}x\|\}^{-1}. \tag{41}$$

Step 3. Following v and w given in Eqs (39) and (40), let new x be selected as the first singular vector of the matrix $\mathbf{Q}^{-1/2}[\mathbf{P}vw']_s\mathbf{Q}^{-1/2}$, and repeat Step 2 until a convergent condition is detected.

Algorithm I converges to the condition of $x \in \mathbb{R}^n$ being the first singular vector of the matrix $[\mathbf{Q}^{-1/2}\mathbf{P}^2\mathbf{Q}^{-1/2}xx'\mathbf{Q}^{-1}]_s$. The successful convergence of Algorithm I is assured by examining a matrix relation given in the following theorem.

Theorem IV

Consider the symmetric positive-definite matrix given by

$$\mathbf{M} = \mathbf{Q}^{-1/2}\mathbf{P}^2\mathbf{Q}^{-1/2}/(x'\mathbf{Q}^{-1/2}\mathbf{P}^2\mathbf{Q}^{-1/2}x) + \mathbf{Q}^{-1}/(x'\mathbf{Q}^{-1}x), \tag{42}$$

where vector $x \in \mathbb{R}^n$ with $\|x\| = 1$ and positive-definite matrices $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n\times n}$ are given. If $\sigma_{\max}(\mathbf{M}) = 2$, then, for all $z \in \mathbb{R}^n$ with $\|z\| = 1$,

$$\|\mathbf{PQ}^{-1/2}x\| \, \|\mathbf{Q}^{-1/2}x\| \geqslant \|\mathbf{PQ}^{-1/2}z\| \, \|\mathbf{Q}^{-1/2}z\|. \tag{43}$$

Proof: Since $x'\mathbf{M}x = 2$ and $\sigma_{\max}(\mathbf{M}) = 2$, we have

$$2 \geqslant z'\mathbf{M}z = \|\mathbf{PQ}^{-1/2}z\|^2/\|\mathbf{PQ}^{-1/2}x\|^2 + \|\mathbf{Q}^{-1/2}z\|^2/\|\mathbf{Q}^{-1/2}x\|^2. \tag{44}$$

Thus,

$$\{\|\mathbf{PQ}^{-1/2}z\|/\|\mathbf{PQ}^{-1/2}x\| - \|\mathbf{Q}^{-1/2}z\|/\|\mathbf{Q}^{-1/2}x\|\}^2 \leqslant 2 - 2(\|\mathbf{PQ}^{-1/2}z\|/\|\mathbf{PQ}^{-1/2}x\|)(\|\mathbf{Q}^{-1/2}z\|/\|\mathbf{Q}^{-1/2}x\|), \tag{45}$$

which makes

$$(\|\mathbf{PQ}^{-1/2}z\| \, \|Q^{-1/2}z\|)/(\|\mathbf{PQ}^{-1/2}x\| \, \|\mathbf{Q}^{-1/2}x\|) \leqslant 1. \tag{46}$$

■

IV. Properties of the Robustness Related Perturbation Structure

Employing Algorithm A1 (or A2) of the Appendix, matrices **P** and **Q** that fulfill the Lyapunov equation (9) are chosen to produce less conservative unstructured robustness bound μ_L defined in Eq. (11). By use of Algorithm I, the perturbation-susceptible structure of the resulting Lyapunov matrices **P** and **Q** is explicitly derived, while the unstructured robustness bound (μ_L) is achieved. Since the quantitative robustness-bound measure problem for unstructurally perturbed systems is formulated with the least knowledge concerning the structural content of the perturbations, the resulting Lyapunov matrices possess structural properties closely related to the perturbation-susceptible characteristics of the nominal system matrix. This relation is illustrated by the use of Example 1.

On the other hand, given the Lyapunov matrices **P** and **Q**, Theorem II is employed to derive the robust stability bounds (δ_L) for structurally perturbed systems. As long as the robustness-related perturbation structure $\mu_L v w'$ is closely related to the genuine perturbation-susceptible structure of the nominal system matrix **A**, the result of producing less conservative unstructured robustness bounds (μ_L) attributes to the generation of less conservative structured robustness bounds (δ_L). By use of the proposed method, Example 2 is given to demonstrate the improvement in the analysis of quantitative robustness measures for perturbed systems described by Eq. (1).

Example 1

Consider a perturbed system described by Eq. (1) with the nominal system matrix given by

$$\mathbf{A} = \begin{bmatrix} -1 & 10 \\ 0 & -1 \end{bmatrix}. \tag{47}$$

The perturbation-susceptible structure of matrix **A** can be obtained by observation, which is approximately given by

$$\mu_{L0} v_0 w_0' = 0.1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}'. \tag{48}$$

Employing Algorithm A1 of the Appendix, the iterative process produces a sequence of Lyapunov-matrix pairs $\{\mathbf{P}_i, \mathbf{Q}_i\}$ that fulfill the Lyapunov equation (9). Let each pair of the matrices $\mathbf{P}_i$ and $\mathbf{Q}_i$ be used in Algorithm I, a sequence of the robustness-related perturbation structure $\{\mu_{Li} v_i w_i'\}$ [defined in Definition (1); also referred to as the perturbation-susceptible structure of the Lyapunov matrices $\mathbf{P}_i$ and $\mathbf{Q}_i$] is obtained. Additionally, a matchness index $\chi \equiv v_0' v_i w_i' w_0$ is computed to indicate the closeness of the robustness-related perturbation structure $\{\mu_{Li} v_i w_i'\}$ to the genuine perturbation-susceptible structure given in Eq. (48). The results are summarized in Table I.

Note that, by reducing the conservatism in the unstructured robustness bounds

TABLE I
Unstructured robustness measures and values of matchness index for Example 1

Iterations	μ_{Li}	χ
Initial $\mathbf{Q} = \mathbf{I}$	0.0194	0.0981
1st	0.0347	0.6342
2nd	0.0492	0.8733
3rd	0.0616	0.9411
4th	0.0714	0.9659
5th	0.0779	0.9760
6th	0.0813	0.9798
7th	0.0823	0.9807
8th	0.0824	0.9808

(μ_L), the robustness-related perturbation structure $\mu_L vw'$ consistently approaches the genuine perturbation-susceptible characteristics of the stable nominal matrix (i.e. the matchness index χ approaches 1). By use of the proposed method, the process converges to the robustness-related perturbation structure given by

$$\mu_L vw' = 0.0824\begin{bmatrix} 0.1381 \\ 0.9904 \end{bmatrix}\begin{bmatrix} 0.9903 \\ 0.1392 \end{bmatrix}'. \tag{49}$$

Note also that Algorithm A2 of the Appendix converges to the same result in one iteration.

Example 2

Consider a perturbed system described by Eq. (1) with the nominal system matrix given by

$$\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}. \tag{50}$$

Employing Algorithm A2 of the Appendix, the iterate process converges to a Lyapunov-matrix pair $\{\mathbf{P}, \mathbf{Q}\}$, where

$$\mathbf{Q} = \begin{bmatrix} 5.2361 & 2.6180 \\ 2.6180 & 2.6180 \end{bmatrix}, \tag{51}$$

and

$$\mathbf{P} = \begin{bmatrix} 2.1817 & 1.3090 \\ 1.3090 & 3.0544 \end{bmatrix}. \tag{52}$$

(A) *Unstructurally perturbed case.* Let matrices **P** and **Q** in Eqs (51) and (52) be used in Algorithm I, the unstructured robustness bound (μ_L) is obtained. The results are summarized in Table II. Obviously, the proposed robustness bound (μ_L) is less conservative than the bound (μ_l) given by Result 1 from Ref. (**9**).

(B) *Structurally perturbed case.* For systems (1) perturbed by a single unity-rank linear perturbation, it is advisable to always conduct the stability-robustness analysis using the rootlocus-based techniques (**14–16**) which give exact bounds of stability robustness. In this example, by use of the Lyapunov-based techniques, structured robustness measures are derived and compared for those rank-2 linear

TABLE II
Unstructured robustness measures for Example 2

Result 1 μ_l	Theorem I μ_L
0.3820	0.4842

TABLE III
Structured robustness bounds for Example 2

E_i $(i=1,2)$	Result 2 δ_l	Theorem II δ_L
$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$	0.4805	0.7868
$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$	1.0000	1.1790
$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$	0.3820	0.4973
$\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$	0.5000	0.7073
$\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$	0.3028	0.3574
$\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$	0.3246	0.4288

perturbations where two entries of the nominal system matrix **A** in Eq. (50) are independently subject to variations.

Let matrices **P** and **Q** in Eqs (51) and (52) be used in Theorem II, the structured robustness bound (δ_L) is obtained. The results are summarized in Table III. Obviously, the proposed robustness bounds (δ_L) are less conservative than the bounds (δ_l) given by Result 2 from Ref. (**12**).

V. Robustness Behaviour of a Vertical Takeoff and Landing Aircraft

Consider the control system of a vertical takeoff and landing (VTOL) aircraft given in (**13**) and (**18**). The linearized model of the VTOL aircraft in the vertical plane is described by

$$\mathrm{d}x(t)/\mathrm{d}t = (\mathbf{F}+\Delta\mathbf{F})x(t)+(\mathbf{G}+\Delta\mathbf{G})u(t). \tag{53}$$

The components of the state vector $x \in \mathbb{R}^4$ and the control vector $u \in \mathbb{R}^2$ are given by

x_1 horizontal velocity (knots)
x_2 vertical velocity (knots)
x_3 pitch rate (deg s^{-1})
x_4 pitch angle (deg)
u_1 "collective" pitch control
u_2 "longitudinal cyclic" pitch control.

Essentially, control is achieved by varying the angle of attack with respect to air of the rotor blades. The collective control u_1 is mainly used for controlling the motion of the aircraft vertically up and down. The longitudinal cyclic control u_2 is basically used to control the horizontal velocity of the aircraft. For typical loading and flight conditions of the VTOL aircraft at a speed of 135 knots, the matrices **F** and **G** are given by

$$\mathbf{F} = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{54}$$

and

$$\mathbf{G} = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5992 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}. \tag{55}$$

A standard Riccati equation is employed in both (**13**) and (**18**) such that the nominal state-feedback control, that stabilizes the nominal closed-loop system, is given by

$$u(t) = \mathbf{K}x(t) = -(r\mathbf{R})^{-1}\mathbf{G}'\mathbf{S}x(t), \tag{56}$$

where the matrix **S** satisfies the algebraic equation

$$\mathbf{F}'\mathbf{S} + \mathbf{S}\mathbf{F} + \mathbf{H} = \mathbf{S}\mathbf{G}(r\mathbf{R})^{-1}\mathbf{G}'\mathbf{S}, \tag{57}$$

and the scalar variable r serves as the design variable while weighting matrices **H** and **R** are given.

It is known that, even for optimal control problems where the system parameters are completely known, the choice of the weighting matrices **H** and **R** in the algebraic Riccati equation (57) is not an easy one. A commonly followed procedure, that relates these weighting matrices to the subjective criterion of the pilot, is to make matrices **H** and **R** diagonal with the elements of the matrices inversely proportional to the square of the maximum allowable variations of the state variables and control variables respectively. Thus, as proposed by Narendra and Tripathi (**18**), we choose

$$\mathbf{H} = \begin{bmatrix} 1/25 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{58}$$

and

$$\mathbf{R} = \begin{bmatrix} 1/25 & 0 \\ 0 & 1/9 \end{bmatrix}. \tag{59}$$

Applying the proposed method of robustness analysis, the robustness behaviour of the VTOL aircraft control system is given in the following.

(A) *Unstructurally perturbed case.* If the LQR state feedback is employed, then the nominal closed-loop system matrix is given for each value of r as

$$\mathbf{A}(r) = \mathbf{F}+\mathbf{GK} = \mathbf{F}-\mathbf{G}(r\mathbf{R})^{-1}\mathbf{G}'\mathbf{S}. \tag{60}$$

Employing Algorithm I and Algorithm A2 of the Appendix on the nominal system matrix $\mathbf{A}(r)$, unstructured robustness bounds (μ_L) are obtained and the results are summarized as shown in Fig. 1. Obviously, the proposed robustness bounds (μ_L) are less conservative than the bounds (μ_I) given by Result 1 from Ref. (**9**).

We note that the unstructured robustness bound (μ_L) is optimized for control-weighting parameter r at near 1.34. The attainable unstructured robustness bound is 0.294. Note also that $r = 1$ was arbitrarily selected by Narendra and Tripathi (**18**) without the robustness analysis shown in Fig. 1.

(B) *Structurally perturbed case.* As the airspeed changes, significant variations take place in the elements $\mathbf{F}_{32}$, $\mathbf{F}_{34}$ and $\mathbf{G}_{21}$ of the nominal state matrix $\mathbf{F}$ and the input matrix $\mathbf{G}$ given in Eqs (54) and (55), respectively. As given in (**18**) that, for range of airspeed from 60 to 170 knots, in-phase variations of elements $\mathbf{F}_{32}$, $\mathbf{F}_{34}$ and $\mathbf{G}_{21}$ are observed such that

$$|\Delta\mathbf{F}_{32}| = 0.302; \quad |\Delta\mathbf{F}_{34}| = 1.300; \quad |\Delta\mathbf{G}_{21}| = 2.567. \tag{61}$$

The perturbed closed-loop system is given for each value of r as

$$\mathrm{d}x(t)/\mathrm{d}t = [\mathbf{A}(r)+\mathbf{E}(r)]x(t), \tag{62}$$

where $\mathbf{A}(r)$ is given in Eq. (60), and

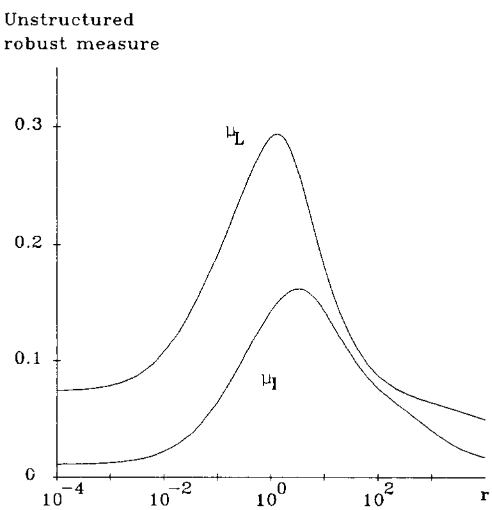

FIG. 1. Unstructured robustness bounds for VTOL aircraft control system designed by use of LQR method with weighting matrices given in Eqs (58) and (59).

$$\mathbf{E}(r) = \Delta\mathbf{F} - \Delta\mathbf{G}(r\mathbf{R})^{-1}\mathbf{G}'\mathbf{S}. \tag{63}$$

For in-phase variations given in Eq. (61), the perturbation matrix can be expressed as

$$\mathbf{E}(r) = k_1(t)\mathbf{E}_1 = k_1(t)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.302 & 0 & 1.300 \\ 0 & 0 & 0 & 0 \end{bmatrix} - k_1(t)\begin{bmatrix} 0 & 0 \\ 2.567 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}(r\mathbf{R})^{-1}\mathbf{G}'\mathbf{S}. \tag{64}$$

Thus, the matrix pair $\{\mathbf{A}(r), \mathbf{E}(r)\}$ denoted in Eq. (62) constitutes the problem of robustness analysis for structured perturbations.

Employing Theorem II and Algorithm A2 of the Appendix on the structurally perturbed problem specified by the matrix pair $\{\mathbf{A}(r), \mathbf{E}(r)\}$ denoted in Eq. (62), the robustness bounds (δ_L) are obtained and the results are summarized in Fig. 2. Obviously, the proposed robustness bounds (δ_L) are less conservative than the bounds (δ_I) given by Result 2 from Ref. (**12**).

We note that the structured robustness bound (δ_L) is optimized for control-weighting parameter r at near 160. The attainable structured robustness bound is 0.939. Note also that in this example the value of structured robustness bound (δ_L) greater than 1 is required to assure the stability of the closed-loop control system described by Eq. (62). Thus, for the design of the LQR state feedback, the weighting matrices [given in Eqs (58) and (59)] proposed by Narendra and Tripathi are not adequate (**18**).

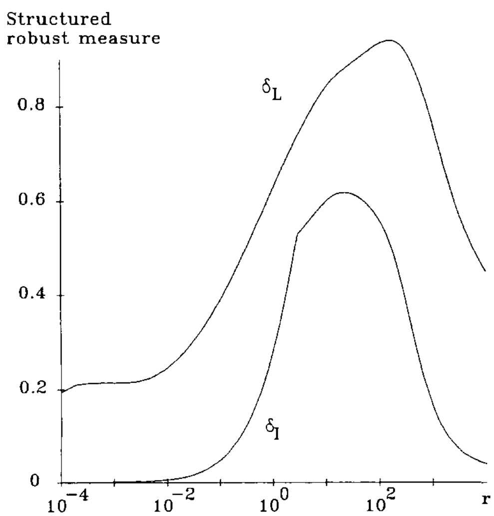

FIG. 2. Structured robustness bounds for VTOL aircraft control system designed by use of LQR method with weighting matrices given in Eqs (58) and (59).

(C) *Robustness-related perturbation structure of the VTOL control system.* Given weighting matrices specified in Eqs (58) and (59) with weighting parameter $r = 1$, the closed-loop nominal system matrix is obtained from (60), i.e.

$$\mathbf{A} = \begin{bmatrix} -0.4421 & -0.3055 & 0.1475 & -0.1275 \\ -0.8779 & -14.2802 & 0.4727 & 0.5195 \\ 3.4356 & 12.3970 & -1.8964 & -4.0941 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{65}$$

Applying Algorithm I and Algorithm A2 of the Appendix, the robustness-related perturbation structure of the matrix **A** given in Eq. (65) is derived, i.e.

$$\mu_L v w' = 0.2915 \begin{bmatrix} 0.8715 \\ 0.2326 \\ 0.2791 \\ -0.3295 \end{bmatrix} \begin{bmatrix} 0.5552 \\ 0.0019 \\ 0.8316 \\ -0.0172 \end{bmatrix}', \tag{66}$$

which is proposed to represent the perturbation-susceptible structure of the closed-loop system matrix **A**.

Since the stability of the matrix **A** given in Eq. (65) is particularly susceptible to the robustness-related perturbation structure given in Eq. (66), then the deficiency of the VTOL aircraft control design can be analysed by examining impulse responses of a system described by:

$$\mathrm{d}x(t)/\mathrm{d}t = (\mathbf{A}+\mathbf{E})x(t) + v\delta(t), \tag{67}$$

and

$$y(t) = w'x(t), \tag{68}$$

where **A** is the matrix given in Eq. (65), v and w are the column matrices that constitute the perturbation-susceptible structure given in (66), $\delta(t)$ is the Dirac impulse function, and **E** is the perturbation matrix obtained from (5.12). By use of computer simulation, Figs 3–5 illustrate the impulse responses of the system described by Eqs (67) and (68), i.e. the horizontal velocity x_1, the vertical velocity x_2 and the composite output signal y. Examining the nominal case (1.35 Mach), it is observed that both x_1 and x_2 are well-regulated responses, while y exhibits underdamped characteristics.

Summarizing the examinations in impulse responses, we have:

(i) the LQR state feedback does establish a well-regulated response in accordance with the state weighting matrix **H** given in Eq. (58) where x_1 and x_2 are proportionally weighted to satisfy the subjective criterion of the pilot;
(ii) the VTOL aircraft control design is deficient in robustness since an underdamped perturbation-susceptible composite output signal y is observed.

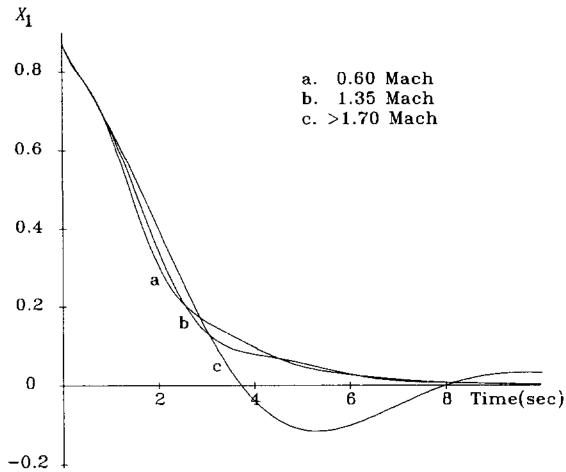

FIG. 3. Impulse response of the horizontal velocity x_1 for the composite system described by Eqs (67) and (68). The composite system is derived from the control system designed by use of LQR method with weighting matrices given in Eqs (58) and (59).

(D) *To robustify the LQR control design.* To robustify the LQR control design of the VTOL aircraft given by Narendra and Tripathi (**18**), it is suggested that the composite output signal y be weighted in the state weighting matrix instead of x_1. Compromising the state weightings on x_1 and on y, we consider the state weighting matrix given by

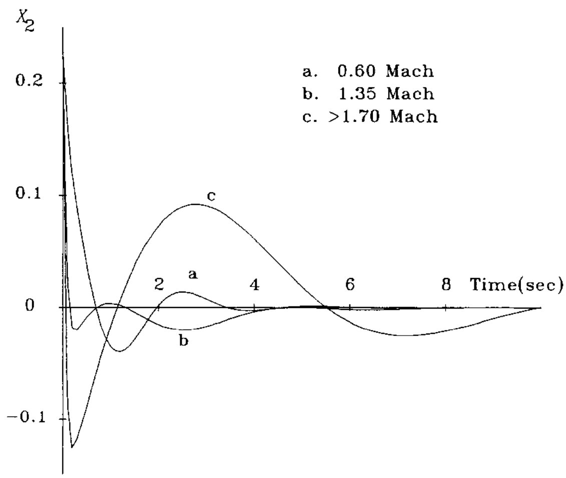

FIG. 4. Impulse response of the vertical velocity x_2 for the composite system described by Eqs (67) and (68). The composite system is derived from the control system designed by use of LQR method with weighting matrices given in Eqs (58) and (59).

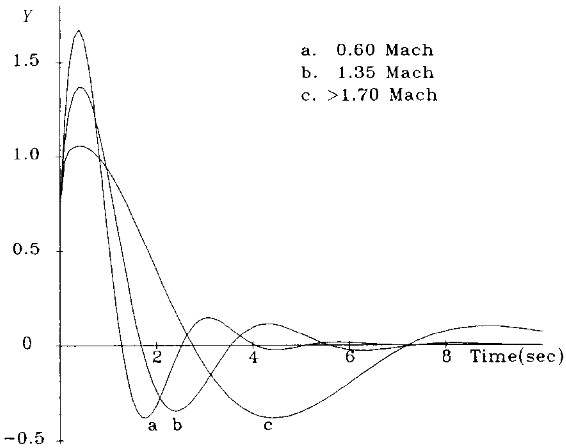

FIG. 5. Impulse response of the composite output signal y for the composite system described by Eqs (67) and (68). The composite system is derived from the control system designed by use of LQR method with weighting matrices given in Eqs (58) and (59).

$$\mathbf{H} = \begin{bmatrix} h_{x1} & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + h_y \begin{bmatrix} 0.5552 \\ 0 \\ 0.8316 \\ 0 \end{bmatrix} \begin{bmatrix} 0.5552 \\ 0 \\ 0.8316 \\ 0 \end{bmatrix}', \tag{69}$$

where

$$h_{x1} + h_y (0.5552)^2 = 1/25. \tag{70}$$

Applying the proposed method of robustness analysis, for various values of the compromising parameter h_{x1}, the robustness bounds of the modified VTOL aircraft control system are obtained as shown in Figs 6 and 7.

Figure 6 displays the unstructured robustness bounds (μ_L), where $h_{x1} = 0.04$ represents the original design. Note that a slight change in the compromising parameter (h_{x1}) may give considerable improvement in robustness bounds. Figure 7 displays the structured robustness bounds (δ_L). Note that by compromising the state weightings such that $h_{x1} \to 0$, the attainable robustness bound is improved.

Thus, selecting $h_{x1} = 0.02$ and $r = 0.1$, the modified LQR state-feedback design is derived by assigning a state-weighting matrix as

$$\mathbf{H} = \begin{bmatrix} 0.04 & 0 & 0.03 & 0 \\ 0 & 0.25 & 0 & 0 \\ 0.03 & 0 & 0.0449 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{71}$$

The stabilized closed-loop nominal system matrix is given by

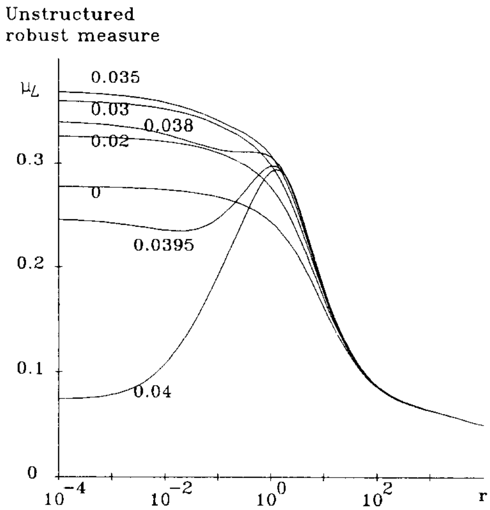

FIG. 6. Unstructured robustness bounds for VTOL aircraft control system designed by use of LQR method with state-weighting matrix given by Eqs (69) and (70) for some values of compromising parameter h_{x1}.

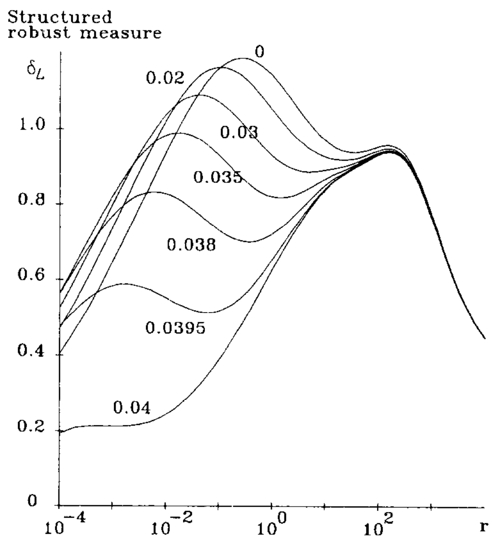

FIG. 7. Structured robustness bounds for VTOL aircraft control system designed by use of LQR method with state-weighting matrix given by Eqs (69) and (70) for some values of compromising parameter h_{x1}.

$$\mathbf{A} = \begin{bmatrix} -1.3872 & -0.5166 & 1.2706 & 1.4103 \\ -4.5836 & -43.7041 & 5.6180 & 7.9124 \\ 12.3583 & 35.6323 & -13.0510 & -19.6257 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{72}$$

which possesses the robustness-related perturbation structure given by

$$\mu_L v w' = 0.3105 \begin{bmatrix} 0.9808 \\ 0.1107 \\ 0.1511 \\ -0.0534 \end{bmatrix} \begin{bmatrix} 0.8633 \\ 0.0217 \\ 0.3512 \\ 0.3618 \end{bmatrix}', \tag{73}$$

and the structured robustness bound is given by

$$\delta_L = 1.1601 > 1. \tag{74}$$

Since the stability of the matrix **A** given in (72) is particularly susceptible to the robustness-related perturbation structure given in (73), whether or not the modified LQR control design is better in robustness can be analysed by examining impulse responses of a system described by (67) and (68). Similarly, by use of computer simulation, we have Figs 8–10 to illustrate the impulse responses of x_1, x_2 and y. It can be seen that, for all cases, x_1, x_2 and y are well-regulated responses which evidence to justify the robustified LQR control design.

VI. Conclusions

In this paper, a modified Lyapunov-based method for time-domain stability-robustness analysis has been proposed for linear systems with state-space models.

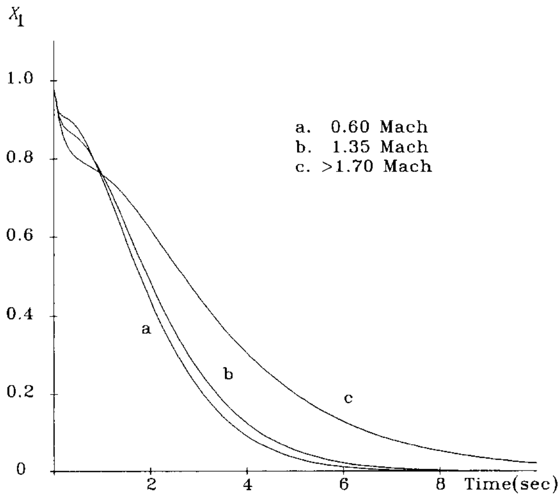

FIG. 8. Impulse response of the horizontal velocity x_1 for the composite system described by Eqs (67) and (68). The composite system is derived from the control system designed by use of LQR method with state-weighting matrix given by Eq. (71).

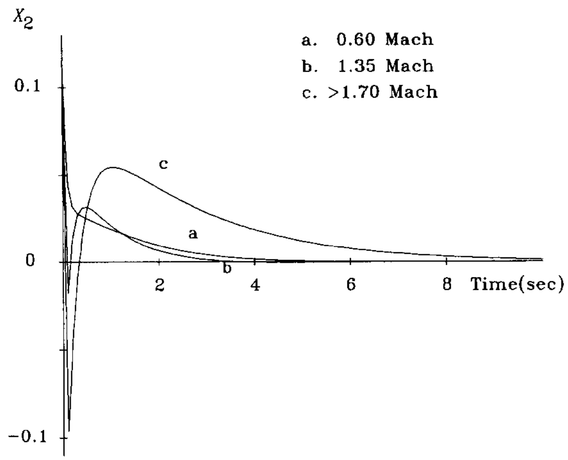

FIG. 9. Impulse response of the vertical velocity x_2 for the composite system described by Eqs (67) and (68). The composite system is derived from the control system designed by use of LQR method with state-weighting matrix given in Eq. (71).

Following the proposed algorithms, Lyapunov matrices are obtained from which a less conservative quantitative measure of robustness and a scalar internal feedback structure are derived. It has been shown that the derived perturbation-susceptible structure of the Lyapunov matrices is closely related to the perturbation-susceptible

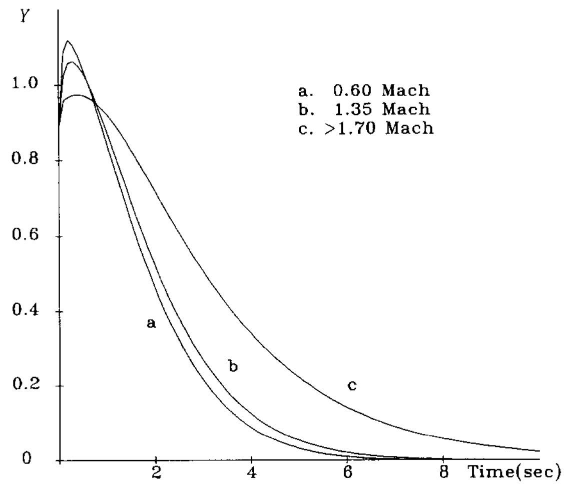

FIG. 10. Impulse response of the composite output signal y for the composite system described by Eqs (67) and (68). The composite system is derived from the control system designed by use of LQR method with state-weighting matrix given in Eq. (71).

property of the nominal system matrix. In addition, the robustification of a VTOL aircraft state-feedback control system by use of a modified state-weighting matrix has been illustrated.

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Appendix

Let $\mathbf{A}$ be a stable system matrix, so $\mathbf{A}'$ will be stable. The following Lyapunov equations produce two symmetric positive-definite matrices $\mathbf{P}_1$ and $\mathbf{P}_2$:

$$\mathbf{A}'\mathbf{P}_1+\mathbf{P}_1\mathbf{A}+2\mathbf{Q}_1=\mathbf{0}, \tag{A1}$$

$$\mathbf{P}_2\mathbf{A}'+\mathbf{A}\mathbf{P}_2+2\mathbf{Q}_2=\mathbf{0}, \tag{A2}$$

where matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are symmetric positive-definite. Simultaneously, we have the following alternative expressions of Eqs (A1) and (A2):

$$\mathbf{P}_1^{-1}\mathbf{A}'+\mathbf{A}\mathbf{P}_1^{-1}+2\mathbf{P}_1^{-1}\mathbf{Q}_1\mathbf{P}_1^{-1}=\mathbf{0}, \tag{A3}$$

$$\mathbf{A}'\mathbf{P}_2^{-1}+\mathbf{P}_2^{-1}\mathbf{A}+2\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}=\mathbf{0}. \tag{A4}$$

Lyapunov equations (A1) and (A4) provide us with two robustness-related perturbation structures. As it is defined in the main text, these perturbation-susceptible structures are $\mu_1 v_1 w_1'$ and $\mu_2 v_2 w_2'$ such that the following relations are fulfilled:

$$\mu_1^{-1}v_1w_1'=\mathbf{P}_1\mathbf{Q}_1^{-1/2}x_1x_1'\mathbf{Q}_1^{-1/2}, \tag{A5}$$

$$\mu_2^{-1}v_2w_2'=\mathbf{Q}_2^{-1/2}x_2x_2'\mathbf{Q}_2^{-1/2}\mathbf{P}_2, \tag{A6}$$

where v_1, w_1, x_1, v_2, w_2 and x_2 are unit vectors within $\mathbb{R}^n$, and

$$\mu_1^{-1}=\|\mathbf{P}_1\mathbf{Q}_1^{-1/2}x_1\|\,\|\mathbf{Q}_1^{-1/2}x_1\|\geqslant\|\mathbf{P}_1\mathbf{Q}_1^{-1/2}x\|\,\|\mathbf{Q}_1^{-1/2}x\|, \tag{A7}$$

$$\mu_2^{-1}=\|\mathbf{P}_2\mathbf{Q}_2^{-1/2}x_2\|\,\|\mathbf{Q}_2^{-1/2}x_2\|\geqslant\|\mathbf{P}_2\mathbf{Q}_2^{-1/2}x\|\,\|\mathbf{Q}_2^{-1/2}x\|, \tag{A8}$$

for all $x\in\mathbb{R}^n$ with $\|x\|=1$.

We note that (A1) and (A4) can be interpolated to make new Lyapunov equations, and that (A2) and (A3) can be interpolated in a dual manner.

Lemma A1

Given Lyapunov equations (A1) and (A2), the following interpolated Lyapunov equations are satisfied for all interpolating parameters a, b, c, $d>0$, i.e.

$$\mathbf{A}'\mathbf{X}_1+\mathbf{X}_1\mathbf{A}+2\mathbf{Y}_1=\mathbf{0}, \tag{A9}$$

$$\mathbf{X}_2\mathbf{A}'+\mathbf{A}\mathbf{X}_2+2\mathbf{Y}_2=\mathbf{0}, \tag{A10}$$

where

$$\mathbf{X}_1=a\mathbf{P}_1+b\mathbf{P}_2^{-1}, \tag{A11}$$

$$\mathbf{Y}_1=a\mathbf{Q}_1+b\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}, \tag{A12}$$

$$\mathbf{X}_2=c\mathbf{P}_2+d\mathbf{P}_1^{-1}, \tag{A13}$$

$$\mathbf{Y}_2=c\mathbf{Q}_2+d\mathbf{P}_1^{-1}\mathbf{Q}_1\mathbf{P}_1^{-1}. \tag{A14}$$

Proof: Equation (A9) is the direct interpolated result of Eqs (A1) and (A4). Equation (A10) is the direct interpolated result of Eqs (A2) and (A3) ∎

The interpolated Lyapunov equations (A9) and (A10) also provide us with two robustness-related perturbation structures. These perturbation-susceptible structures $v_1\bar{v}_1\bar{w}'_1$ and $v_2\bar{v}_2\bar{w}'_2$ such that the following relations are fulfilled:

$$v_1^{-1}\bar{v}_1\bar{w}'_1 = \mathbf{X}_1\mathbf{Y}_1^{-1/2}\bar{x}_1\bar{x}'_1\mathbf{Y}_1^{-1/2}, \tag{A15}$$

$$v_2^{-1}\bar{v}_2\bar{w}'_2 = \mathbf{Y}_2^{-1/2}\bar{x}_2\bar{x}'_2\mathbf{Y}_2^{-1/2}\mathbf{X}_2, \tag{A16}$$

where $\bar{v}_1$, $\bar{w}_1$, $\bar{x}_1$, $\bar{v}_2$, $\bar{w}_2$ and $\bar{x}_2$ are unit vectors within $\mathbb{R}^n$, and

$$v_1^{-1} = \|\mathbf{X}_1\mathbf{Y}_1^{-1/2}\bar{x}_1\|\ \|\mathbf{Y}_1^{-1/2}\bar{x}_1\| \geqslant \|\mathbf{X}_1\mathbf{Y}_1^{-1/2}x\|\ \|\mathbf{Y}_1^{-1/2}x\|, \tag{A17}$$

$$v_2^{-1} = \|\mathbf{X}_2\mathbf{Y}_2^{-1/2}\bar{x}_2\|\ \|\mathbf{Y}_2^{-1/2}\bar{x}_2\| \geqslant \|\mathbf{X}_2\mathbf{Y}_2^{-1/2}x\|\ \|\mathbf{Y}_2^{-1/2}x\|, \tag{A18}$$

for all $x \in \mathbb{R}^n$ with $\|x\| = 1$.

The following Lemmas will be useful when interpolating Lyapunov equations are considered for improving the unstructured robustness measures.

Lemma A2

Given Lyapunov equations (A1) and (A2) with their robustness-related perturbation structures $\mu_1 v_1 w'_1$ and $\mu_2 v_2 w'_2$, the interpolated Lyapunov equations (A9) and (A10) defined in Lemma A1 produce the robustness-related perturbation structures $v_1\bar{v}_1\bar{w}'_1$ and $v_2\bar{v}_2\bar{w}'_2$ such that, for all interpolating parameters a, b, c, $d > 0$,

$$\text{Min}\,\{v_1, v_2\} \geqslant \text{Min}\,\{\mu_1, \mu_2\}. \tag{A19}$$

Proof: We shall prove that $v_1 \geqslant \text{Min}\,\{\mu_1, \mu_2\}$, and the other relation showing that $v_2 \geqslant \text{Min}\,\{\mu_1, \mu_2\}$ can be proved in a similar way.

The robustness-related perturbation structure $v_1\bar{v}_1\bar{w}'_1$ fulfills the relation given in (A15) Since it is given in (A11) that

$$\mathbf{X}_1 = a\mathbf{P}_1 + b\mathbf{P}_2^{-1},$$

relation (A15) becomes

$$\begin{aligned} v_1^{-1}\bar{v}_1\bar{w}'_1 &= a\mathbf{P}_1\mathbf{Y}_1^{-1/2}\bar{x}_1\bar{x}'_1\mathbf{Y}_1^{-1/2} + b\mathbf{P}_2^{-1}\mathbf{Y}_1^{-1/2}\bar{x}_1\bar{x}'_1\mathbf{Y}_1^{-1/2}, \\ &= a\mathbf{P}_1\mathbf{Q}_1^{-1/2}\mathbf{Q}_1^{1/2}\mathbf{Y}_1^{-1/2}\bar{x}_1\bar{x}'_1\mathbf{Y}_1^{-1/2}\mathbf{Q}_1^{1/2}\mathbf{Q}_1^{-1/2} \\ &\quad + b\mathbf{Q}_2^{-1/2}\mathbf{Q}_2^{1/2}\mathbf{P}_2^{-1}\mathbf{Y}_1^{-1/2}\bar{x}_1\bar{x}'_1\mathbf{Y}_1^{-1/2}\mathbf{P}_2^{-1}\mathbf{Q}_2^{1/2}\mathbf{Q}_2^{-1/2}\mathbf{P}_2. \end{aligned} \tag{A20}$$

Defining vectors y_1 and z_1 as

$$y_1 = a^{1/2}\mathbf{Q}_1^{1/2}\mathbf{Y}_1^{-1/2}\bar{x}_1, \tag{A21}$$

$$z_1 = b^{1/2}\mathbf{Q}_2^{1/2}\mathbf{P}_2^{-1}\mathbf{Y}_1^{-1/2}\bar{x}_1, \tag{A22}$$

relation (A20) becomes

$$v_1^{-1}\bar{v}_1\bar{w}'_1 = \mathbf{P}_1\mathbf{Q}_1^{-1/2}y_1y'_1\mathbf{Q}_1^{-1/2} + \mathbf{Q}_2^{-1/2}z_1z'_1\mathbf{Q}_2^{-1/2}\mathbf{P}_2. \tag{A23}$$

Thus, by use of the properties given in (A7) and (A8), we have

$$\begin{aligned} v_1^{-1} &= \sigma_{\max}(\mathbf{P}_1\mathbf{Q}_1^{-1/2}y_1y'_1\mathbf{Q}_1^{-1/2} + \mathbf{Q}_2^{-1/2}z_1z'_1\mathbf{Q}_2^{-1/2}\mathbf{P}_2), \\ &\leqslant \|\mathbf{P}_1\mathbf{Q}_1^{-1/2}y_1\|\ \|\mathbf{Q}_1^{-1/2}y_1\| + \|\mathbf{Q}_2^{-1/2}z_1\|\ \|\mathbf{P}_2\mathbf{Q}_2^{-1/2}z_1\|, \\ &\leqslant \|y_1\|^2\|\mathbf{P}_1\mathbf{Q}_1^{-1/2}x_1\|\ \|\mathbf{Q}_1^{-1/2}x_1\| + \|z_1\|^2\|\mathbf{Q}_2^{-1/2}x_2\|\ \|\mathbf{P}_2\mathbf{Q}_2^{-1/2}x_2\|, \\ &= \|y_1\|^2\mu_1^{-1} + \|z_1\|^2\mu_2^{-2}. \end{aligned} \tag{A24}$$

On the other hand, it is given in (A12) that

$$\mathbf{Y}_1 = a\mathbf{Q}_1 + b\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}.$$

Followed from (A21) and (A22), it can be shown that

$$\|y_1\|^2 + \|z_1\|^2 = 1. \tag{A25}$$

Finally, the relation given in (A24) becomes

$$v_1^{-1} \leqslant \|y_1\|^2\mu_1^{-1} + (1 - \|y_1\|^2)\mu_2^{-1}, \tag{A26}$$

and

$$v_1 \geqslant \text{Min}\{\mu_1, \mu_2\}. \tag{A27}$$

■

Remark (A1)

Lemma A2 provides the fact that interpolating Lyapunov equations will not devastate the unstructured robustness bound, i.e. $\text{Min}\{v_1, v_2\} \geqslant \text{Min}\{\mu_1, \mu_2\}$. The Lyapunov matrices that bring forth a more conservative result are always improved by the Lyapunov matrices that provide a less conservative result.

Lemma A3

Given Lyapunov equations (A1) and (A2) with their robustness-related perturbation structures $\mu_1 v_1 w_1'$ and $\mu_2 v_2 w_2'$, the interpolated Lyapunov equations (A9) and (A10) defined in Lemma A1 produce the robustness-related perturbation structures $v_1\bar{v}_1\bar{w}_1'$ and $v_2\bar{v}_2\bar{w}_2'$ such that, if either

(1) $\mu_1 \geqslant \mu_2$ and

$$\mu_1 v_1'\mathbf{P}_2^{-1}w_1 < w_1'\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}w_1, \tag{A28}$$

or (2) $\mu_2 \geqslant \mu_1$ and

$$\mu_2 w_2'\mathbf{P}_1^{-1}v_2 < v_2'\mathbf{P}_1^{-1}\mathbf{Q}_1\mathbf{P}_1^{-1}v_2, \tag{A29}$$

then there are some interpolating parameters a, b, c, $d > 0$ that make

$$\text{Max}\{v_1, v_2\} \geqslant \text{Max}\{\mu_1, \mu_2\}. \tag{A30}$$

Proof: Lemma A2 implies the result for the case of $\mu_1 = \mu_2$. We shall prove that $v_1 \geqslant \mu_1$ while $\mu_1 > \mu_2$, and the remaining relation showing that $v_2 \geqslant \mu_2$ while $\mu_2 > \mu_1$ can be proved in a similar way. The proof is divided into two stages:

Stage 1. The interpolated unstructured robustness bound v_1 fulfills the relation given in (A17), i.e.

$$v_1^{-1} = \text{Max}\{\|\mathbf{X}_1\mathbf{Y}_1^{-1/2}x\|\ \|\mathbf{Y}_1^{-1/2}x\|\},$$

where the Max operates over all $x \in \mathbb{R}^n$ with $\|x\| = 1$. Since it is given in (A11) that

$$\mathbf{X}_1 = a\mathbf{P}_1 + b\mathbf{P}_2^{-1},$$

relation (A17) becomes

$$v_1^{-1} = \text{Max}\{\|a\mathbf{P}_1\mathbf{Y}_1^{-1/2}x + b\mathbf{P}_2^{-1}\mathbf{Y}_1^{-1/2}x\|\ \|\mathbf{Y}_1^{-1/2}x\|\}. \tag{A31}$$

Defining the vector y as

$$y = a^{1/2}\mathbf{Q}_1^{1/2}\mathbf{Y}_1^{-1/2}x, \tag{A32}$$

relation (A31) becomes

$$\nu_1^{-1} = \text{Max}\,\{\|\mathbf{P}_1\mathbf{Q}_1^{-1/2}y+(b/a)\mathbf{P}_2^{-1}\mathbf{Q}_1^{-1/2}y\|\;\|\mathbf{Q}_1^{-1/2}y\|\}, \tag{A33}$$

or

$$\nu_1^{-2} = \text{Max}\,\{y'\mathbf{Q}_1^{-1/2}\{\mathbf{P}_1+(b/a)\mathbf{P}_2^{-1}\}^2\mathbf{Q}_1^{-1/2}yy'\mathbf{Q}_1^{-1}y\}. \tag{A34}$$

Thus, the sufficient condition for justifying $\nu_1 \geqslant \mu_1$ is the existence of parameters $a, b > 0$ such that

$$\mu_1^2 y'\mathbf{Q}_1^{-1/2}\{\mathbf{P}_1+(b/a)\mathbf{P}_2^{-1}\}^2\mathbf{Q}_1^{-1/2}yy'\mathbf{Q}_1^{-1}y \leqslant 1, \tag{A35}$$

for all vectors $y \in \mathbb{R}^n$ defined by (A32).

Since it is given in (A12) that

$$\mathbf{Y}_1 = a\mathbf{Q}_1 + b\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1},$$

then vector y defined in (A32) is restricted by the following relation:

$$y'y+(b/a)y'\mathbf{Q}_1^{-1/2}\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}\mathbf{Q}_1^{-1/2}y = 1. \tag{A36}$$

Thus, the sufficient condition (A35) for justifying $\nu_1 \geqslant \mu_1$ is equivalently expressed by

$$\mu_1^2 z'\mathbf{Q}_1^{-1/2}\{\mathbf{P}_1+(b/a)\mathbf{P}_2^{-1}\}^2\mathbf{Q}_1^{-1/2}zz'\mathbf{Q}_1^{-1}z \leqslant \{z'z+(b/a)z'\mathbf{Q}_1^{-1/2}\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}\mathbf{Q}_1^{-1/2}z\}^2, \tag{A37}$$

for all $z \in \mathbb{R}^n$ with $\|z\| = 1$.

For each vector $z \in \mathbb{R}^n$ with $\|z\| = 1$, relation (A37) is sufficiently justified by the existence of a scaling factor s such that

$$sz'\mathbf{Q}_1^{-1/2}\{\mathbf{P}_1+(b/a)\mathbf{P}_2^{-1}\}^2\mathbf{Q}_1^{-1/2}z+(1/s)\mu_1^2 z'\mathbf{Q}_1^{-1}z \leqslant 2z'z+2(b/a)z'\mathbf{Q}_1^{-1/2}\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}\mathbf{Q}_1^{-1/2}z. \tag{A38}$$

Collecting terms in (A38) that are multiplied by the powers of (b/a), we have

$$(b/a)^2 sz'\mathbf{Q}_1^{-1/2}\mathbf{P}_2^{-2}\mathbf{Q}_1^{-1/2}z + (b/a)z'\mathbf{Q}_1^{-1/2}\{s\mathbf{P}_1\mathbf{P}_2^{-1}+s\mathbf{P}_2^{-1}\mathbf{P}_1-2\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}\}\mathbf{Q}_1^{-1/2}z \leqslant 2z'z-(1/s)\mu_1^2 z'\mathbf{Q}_1^{-1}z-sz'\mathbf{Q}_1^{-1/2}\mathbf{P}_1^2\mathbf{Q}_1^{-1/2}z. \tag{A39}$$

Denote relation (A39) as

$$\mathbf{C}_1(z)(b/a)^2+\mathbf{C}_2(z)(b/a) \leqslant \mathbf{C}_3(z). \tag{A40}$$

While $\mathbf{C}_1(z) > 0$ is recognized, the existence of parameters $a, b > 0$ in (A40) is sufficiently justified by either (1) $\mathbf{C}_3(z) > 0$, or (2) $\mathbf{C}_2(z) < 0$ with $\mathbf{C}_3(z) = 0$. In the next stage, we shall prove that coefficients in the polynomial (A40) do fulfill the sufficient condition.

Stage 2. For each vector $z \in \mathbb{R}^n$ with $\|z\| = 1$, let the scaling factor s be given such that

$$sz'\mathbf{Q}_1^{-1/2}\mathbf{P}_1^2\mathbf{Q}_1^{-1/2}z = 1. \tag{A41}$$

Knowing from (A17) that

$$\mu_1^{-1} = \|\mathbf{P}_1\mathbf{Q}_1^{-1/2}x_1\|\;\|\mathbf{Q}_1^{-1/2}x_1\| \geqslant \|\mathbf{P}_1\mathbf{Q}_1^{-1/2}z\|\;\|\mathbf{Q}_1^{-1/2}z\|,$$

we have

$$1 \geqslant \mu_1^2 z'\mathbf{Q}_1^{-1}zz'\mathbf{Q}_1^{-1/2}\mathbf{P}_1^2\mathbf{Q}_1^{-1/2}z = (1/s)\mu_1^2 z'\mathbf{Q}_1^{-1}z. \tag{A42}$$

Combining (A41) and (A42), we have

$$2z'z-(1/s)\mu_1^2 z'\mathbf{Q}_1^{-1}z - sz'\mathbf{Q}_1^{-1/2}\mathbf{P}_1^2\mathbf{Q}_1^{-1/2}z \geqslant 0, \tag{A43}$$

where the equality holds for the case of $z = x_1$.

The condition given in (A28) is

$$\mu_1 v_1'\mathbf{P}_2^{-1}w_1 < w_1'\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}w_1,$$

where, by definition given in (A5),

$$\mu_1^{-1} = \|\mathbf{P}_1\mathbf{Q}_1^{-1/2}x_1\|\,\|\mathbf{Q}_1^{-1/2}x_1\|,$$
$$v_1 = \mathbf{P}_1\mathbf{Q}_1^{-1/2}x_1/\|\mathbf{P}_1\mathbf{Q}_1^{-1/2}x_1\|,$$

and

$$w_1 = \mathbf{Q}_1^{-1/2}x_1/\|\mathbf{Q}_1^{-1/2}x_1\|.$$

Thus, relation (A28) becomes

$$x_1'\mathbf{Q}_1^{-1/2}\mathbf{P}_1\mathbf{P}_2^{-1}\mathbf{Q}_1^{-1/2}x_1/\|\mathbf{P}_1\mathbf{Q}_1^{-1/2}x_1\|^2 < x_1'\mathbf{Q}_1^{-1/2}\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}\mathbf{Q}_1^{-1/2}x_1. \tag{A44}$$

While $z = x_1$, the scaling factor s in (A41) becomes

$$sx_1'\mathbf{Q}_1^{-1/2}\mathbf{P}_1^2\mathbf{Q}_1^{-1/2}x_1 = 1, \tag{A45}$$

and relation (A44) leads to

$$x_1'\mathbf{Q}_1^{-1/2}\{s\mathbf{P}_1\mathbf{P}_2^{-1}+s\mathbf{P}_2^{-1}\mathbf{P}_1-2\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}\}\mathbf{Q}_1^{-1/2}x_1 < 0. \tag{A46}$$

Therefore, the coefficients of the polynomial (A40) fulfill the sufficient condition for the existence of parameters $a, b > 0$ that justify the validity of relation (A39). ■

Remark (A2)

Lemma A3 provides the fact that the unstructured robustness bound can be improved by interpolating Lyapunov equations, i.e. Max $\{v_1, v_2\} \geqslant$ Max $\{\mu_1, \mu_2\}$ if a certain numerical property of the Lyapunov matrices is fulfilled. However, the exact values of applicable interpolating parameters remain unknown, and the degree of improvement in the unstructured robustness bound is not predictable. This is a common difficulty of analyses by use of the Lyapunov-based techniques, since the exact numerical property of a Lyapunov equation is not analytically resolvable.

Nevertheless, the qualitative properties of interpolating Lyapunov equations, demonstrated in Lemma A2 and Lemma A3, make it possible to create the following two iterative procedures that produce less conservative unstructured robustness bounds. These algorithms are devised such that, given Lyapunov equations (A1) and (A2) in each iteration, (i) the intermediate values of unstructured robustness bounds μ_1 and μ_2 are nearly equal to each other, and (ii) Lyapunov equations (A1) and (A2) are interpolated in a balanced manner.

Algorithm A1

Step 1. Assign $\mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{I}$.

Step 2. Equate Lyapunov equations (A1) and (A2) to acquire the matrices $\mathbf{P}_1$ and $\mathbf{P}_2$.

Step 3. For $\mathbf{Y}_1$ in (A12), interpolating parameters a and b are chosen such that

$$\sigma_{\min}(a\mathbf{Q}_1) = 1, \tag{A47}$$

and

$$\sigma_{\min}(b\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}) = 1. \tag{A48}$$

Similarly, for $\mathbf{Y}_2$ in (A14), interpolating parameters c and d are chosen such that

$$\sigma_{\min}(c\mathbf{Q}_2) = 1, \tag{A49}$$

and

$$\sigma_{\min}(d\mathbf{P}_1^{-1}\mathbf{Q}_1\mathbf{P}_1^{-1}) = 1. \tag{A50}$$

Step 4. Make the replacement of

$$\mathbf{Q}_1 = \mathbf{Y}_1/\sigma_{\min}(\mathbf{Y}_1), \tag{A51}$$

and

$$\mathbf{Q}_2 = \mathbf{Y}_2/\sigma_{\min}(\mathbf{Y}_2), \tag{A52}$$

and repeat from Step 2 until a convergent condition is detected.

Algorithm A2

Step 1. Assign $\mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{I}$.

Step 2. Equate Lyapunov equations (A1) and (A2) to acquire the matrices $\mathbf{P}_1$ and $\mathbf{P}_2$.

Step 3. For $\mathbf{Y}_1$ in (A12), interpolating parameters a and b are chosen such that

$$\sigma_{\max}(a\mathbf{Q}_1)\sigma_{\min}(a\mathbf{Q}_1) = 1, \tag{A53}$$

and

$$\sigma_{\max}(b\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1})\sigma_{\min}(b\mathbf{P}_2^{-1}\mathbf{Q}_2\mathbf{P}_2^{-1}) = 1. \tag{A54}$$

Similarly, for $\mathbf{Y}_2$ in (A14), interpolating parameters c and d are chosen such that

$$\sigma_{\max}(c\mathbf{Q}_2)\sigma_{\min}(C\mathbf{Q}_2) = 1, \tag{A55}$$

and

$$\sigma_{\max}(d\mathbf{P}_1^{-1}\mathbf{Q}_1\mathbf{P}_1^{-1})\sigma_{\min}(d\mathbf{P}_1^{-1}\mathbf{Q}_1\mathbf{P}_1^{-1}) = 1. \tag{A56}$$

Step 4. Make the replacement of

$$\mathbf{Q}_1 = \mathbf{Y}_1/\sigma_{\min}(\mathbf{Y}_1), \tag{A57}$$

and

$$\mathbf{Q}_2 = \mathbf{Y}_2/\sigma_{\min}(\mathbf{Y}_2), \tag{A58}$$

and repeat from Step 2 until a convergent condition is detected.

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