Global Optimal Fuzzy Tracker Design Based on Local Concept Approach

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***Abstract*—In this paper, we propose a global optimal fuzzy tracking controller, implemented by fuzzily blending the individual local fuzzy tracking laws, for continuous and discrete-time fuzzy systems with the aim of solving, respectively, the continuous and discrete-time quadratic tracking problems with moving or model-following targets under finite or infinite horizon (time). The differential or recursive Riccati equations, and more, the differential or difference equations in tracing the variation of the target are derived. Moreover, in the case of time-invariant fuzzy tracking systems, we show that the optimal tracking controller can be obtained by just solving algebraic Riccati equations and algebraic matrix equations. Grounding on this, several fascinating characteristics of the resultant closed-loop continuous or discrete time-invariant fuzzy tracking systems can be elicited easily. The stability of both closed-loop fuzzy tracking systems can be ensured by the designed optimal fuzzy tracking controllers. The optimal closed-loop fuzzy tracking systems cannot only be guaranteed to be exponentially stable, but also be stabilized to any desired degree. Moreover, the resulting closed-loop fuzzy tracking systems possess infinite gain margin; that is, their stability is guaranteed no matter how large the feedback gain becomes. Two examples are given to illustrate the performance of the proposed optimal fuzzy tracker design schemes and to demonstrate the proved stability properties.**

***Index Terms*—Degree of stability, exponentially stable, gain margin, global minimum, model-following, moving target, Riccati equation.**

I. INTRODUCTION

ALTHOUGH the work in fuzzy modeling and fuzzy control has been quite matured [1]–[8], the field of optimal fuzzy control is nearly open [9]. In particular, although fuzzy logic concept has been introduced into tracking control [10]–[15], the field of *theoretical* approach of *optimal fuzzy tracking control* is fully open. The goal of this work is to propose a design scheme of the global optimal fuzzy tracking controller to control and stabilize a discrete-time or continuous fuzzy system in solving, respectively, the discrete-time or continuous quadratic tracking problems with moving or model-following targets under finite or infinite horizon.

To date, the fuzzy tracking controller has been used in conceptual design only, and has always been grounded on a conventional tracker. For example, Ott, *et al.* [13] included fuzzy logic into an α–β tracker algorithm; Lea, *et al.* [11] used fuzzy concept to develop the software algorithm of a camera tracking system. No theoretical demonstration has been developed for fuzzy tracking controller design in the literature.

Stability and *optimality* are the most important requirements for any control system. Most of the existed works on the stability analysis of fuzzy control are based on Takagi–Sugeno (T–S)-type fuzzy model combined with the parallel distribution compensation (PDC) concept [1], and apply Lyapunov's method to do stability analysis. Tanaka and coworkers reduced the stability analysis and control design problems to linear matrix inequality (LMI) problems [2], [4]. They also dealt with uncertainty issue [3]. This approach had been applied to several control problems such as control of chaos [4] and of articulated vehicle [5]. A frequency shaping method for systematic design of fuzzy controllers was also done by them [16]. Sun, *et al.* developed a separation scheme to design fuzzy observer and fuzzy controller independently [6]. Methods based on grid-point approach [17] and circle criteria [18], [19] were introduced to do stability analysis of fuzzy control, as well. Wang adopted a supervisory controller and introduced stability and robustness measures [20]. Cao proposed a decomposition principle to design a discrete-time fuzzy control system and an equivalent principle to do stability analysis [8]. On the issue of optimal fuzzy control, Wang developed an *optimal* fuzzy controller to stabilize a *linear continuous* time-invariant system via the Pontryagin minimum principle [9]. Although fuzzy control of linear systems could be a good *starting point* for a better understanding of some issues in fuzzy control synthesis, it does not have many practical implications since using the fuzzy controller designed for a linear system directly as the controller may not be a good choice [9]. Moreover, the cited stability criteria may be simple, but rough to do systematic analysis and also may result in a controller with less flexibility. Tanaka and coworkers [21], [22] tried to obtain a fuzzy controller to minimize the upper bound of the quadratic performance function by LMI approach based on the *assumption of local-linear-feedback-gain control structure*. Nevertheless, no theoretical analysis on this design scheme of optimal-fuzzy-control structure was proposed.

In our previous paper [23], we proposed a global optimal and stable fuzzy controller design method for both continuous and discrete-time fuzzy systems under both finite and infinite horizons. Several fascinating characteristics, exponential stability, finite energy, any prescribed degree of stability and infinite gain margin, have been shown to exist in the closed-loop fuzzy systems for the infinite-horizon optimal control problem [23], [24]. In this paper, we shall develop the relative theories and techniques for the fuzzy tracking problems.

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Technical contributions of this paper can be described as follows. Based on the local concept approach, the global optimal fuzzy controllers with the aim of tracking moving or model-following targets under finite or infinite horizon (time) for both continuous and discrete-time fuzzy systems are theoretically derived. The optimal closed-loop time-invariant fuzzy tracking systems are guaranteed to be exponentially stable. Furthermore, we elicit that the proposed fuzzy tracking controllers can stabilize fuzzy tracking systems to any prescribed degree of stability, and the corresponding closed-loop fuzzy tracking systems possess infinite gain margin. The design methodologies are illustrated by examples.

II. System Representation and Problem Statement

We adopt the following T–S type fuzzy model as the fuzzy tracking system describing the given nonlinear plant:

$$\begin{aligned} R^i: &\text{ If } x_1 \text{ is } T_{1i},\dots,x_n \text{ is } T_{ni}, \quad \text{then} \\ &SX(t) = A_i(t)X(t) + B_i(t)u(t), \qquad i=1,\dots,r \\ &Y(t) = C(t)X(t) \end{aligned} \tag{1}$$

where R^i denotes the ith rule of the fuzzy model; $x_1,\dots,x_n$ are system states; $T_{1i},\dots,T_{ni}$ are the input fuzzy terms in the ith rule; $SX(t)$ denotes $\dot{X}(t)$ for continuous case and $X(t+1)$ for discrete case; $X(t) = [x_1,\dots,x_n]^t \in \Re^n$ is the state vector, $Y(t) = [y_1,\dots,y_{n'}]^t \in \Re^{n'}$ is the system output vector, and $u(t) \in \Re^m$ is the system input (i.e., control output); and $A_i(t), B_i(t)$ and $C(t)$ are, respectively, $n\times n, n\times m$ and $n'\times n$ matrices. The desired tracking controller is then assumed to be in a rule-based nonlinear fuzzy inference form of

$$\begin{aligned} R^i:\ &\text{If } y_1 \text{ is } S_{1i},\dots,y_{n'} \text{ is } S_{n'i}, \\ &\text{then } u(t) = r_i(t), \qquad i=1,\dots,\delta \end{aligned} \tag{2}$$

where $y_1,\dots,y_{n'}$ are the elements of output vector $Y(t), S_{1i},\dots,S_{n'i}$ are the input fuzzy terms in the ith control rule, and the plant input (i.e., control output) vector $u(t)$ or $r_i(t)$ is in $\Re^m$ space.

Then, *the optimal fuzzy tracker design scheme is to control the fuzzy tracking system in such a way to push the output $Y(t)$ close to any desired target $Y^d(t)$ without excessive control-energy consumption*. We describe the quadratic optimal fuzzy tracking control problem as follows.

Problem 1: Given the rule-based fuzzy tracking system in (1) with $X(t_0) = X_0 \in \Re^n$ and a rule-based fuzzy tracking controller in (2), find the individual optimal tracking law, $r_i^*(\cdot), i=1,\dots,\delta$, such that the composed optimal tracking controller, $u^*(\cdot)$, can minimize the quadratic cost functional [25], $J(u(\cdot))$, over all possible inputs $[u(\cdot)]$ of class piecewise-continuous (PC)

$$\begin{aligned} J(u(\cdot)) = \sum_{t=t_0}^{t_1-1} &[u^t(t)S(t)u(t) + X^t(t)L_1(t)X(t) \\ &+ (Y(t)-Y^d(t))^t L_2(t)(Y(t)-Y^d(t))] \end{aligned} \tag{3}$$

$$\begin{aligned} J(u(\cdot)) = \int_{t_0}^{t_1} &[u^t(t)S(t)u(t) + X^t(t)L_1(t)X(t) \\ &+ (Y(t)-Y^d(t))^t L_2(t)(Y(t)-Y^d(t))]\, dt \end{aligned} \tag{4}$$

for discrete-time and continuous systems, respectively, where

$$\begin{aligned} L_1(t) = [I_n - C^t(t)(C(t)C^t(t))^{-1}C(t)]^t L_3(t) \\ \times [I_n - C^t(t)(C(t)C^t(t))^{-1}C(t)] \end{aligned} \tag{5}$$

$S(t), L_2(t)$ and $L_3(t)$ are, respectively, $m\times m, n'\times n'$ and $n\times n$ nonnegative symmetric matrices; $X^t(t)L_1(t)X(t)$ is the state-trajectory penalty to produce smooth response; $u^t(t)S(t)u(t)$ is fuel consumption; and the last term in $J(u(\cdot))$ is related to error cost. Moreover, the performance index in (3) and (4) with $L_1(t)$ in (5) can be, respectively, rewritten as [25]

$$\begin{aligned} J(u(\cdot)) = \sum_{t=t_0}^{t_1-1} &[u^t(t)S(t)u(t) + (X(t) \\ &- X^d(t))^t L(t)(X(t)-X^d(t))] \end{aligned} \tag{6}$$

$$\begin{aligned} J(u(\cdot)) = \int_{t_0}^{t_1} &[u^t(t)S(t)u(t) + (X(t) \\ &- X^d(t))^t L(t)(X(t)-X^d(t))] \end{aligned} \tag{7}$$

where $L(t) = L_1(t) + C^t(t)L_2(t)C(t)$ and the desired trajectory $X^d(t) = C^t(t)[C(t)C^t(t)]^{-1}Y^d(t)$.

Adopting the same local-concept-based optimization technique in our previous papers [23], [24], we know the optimal global decisions for the quadratic fuzzy tracking problem can be regarded as a series of optimal global decisions based on the following successively on-going local quadratic optimal fuzzy tracking issue with the initial state resulting from the previous decision. The time dependence is denoted by lower index for notation simplification.

Problem 2: Given the fuzzy tracking subsystem

$$SX_l = A_{i_l}X_l + B_{i_l}r_{i_l}, \quad i=1,\dots,r \tag{8}$$

with the initial state resulting from the previous decision, i.e., $X_{0_t} = X_t^*$

1) find the optimal local decision at time $t, r_{i_t}^*$, for minimizing the cost functional

$$J_t(r_i(\cdot)) = \sum_{t}^{t_1-1} \left[\left(X_l - X_l^d\right)^t L_l \left(X_l - X_l^d\right) + r_{i_l}^t S_l r_{i_l}\right] \tag{9}$$

$$J_t(r_i(\cdot)) = \int_{t}^{t_1} \left[\left(X_l - X_l^d\right)^t L_l \left(X_l - X_l^d\right) + r_{i_l}^t S_l r_{i_l}\right] dt \tag{10}$$

2) obtain the optimal global decision at time t, u_t^*, for minimizing the cost functional $J_t(u(\cdot))$ in (9) or (10) by fuzzily blending each local decision, i.e., $u_t^* = \sum_{i=1}^r h_i(X_t^*)r_{i_t}^*$.

Notice that the next-decision initial state is $SX_t^* = \sum_{i=1}^r h_i(X_t^*)(A_{i_t}X_t^* + B_{i_t}r_{i_t}^*)$ instead of $SX_t^* = \sum_{i=1}^r h_i(X_t^*)(A_{i_t}X_t^* + B_{i_t}u_t^*)$, since there exists the one-to-one correspondence relationship between each fuzzy tracking subsystem and the corresponding fuzzy tracking law.

III. Optimal Fuzzy Tracker Design

We shall design the optimal trackers for the discrete-time systems in Section III-A and for continuous systems in Section III-B.

A. Optimal Fuzzy Tracker for Discrete-Time Fuzzy System

We are going to design the optimal fuzzy tracking controllers for the discrete-time fuzzy tracking systems with moving target in Section III-A.1 and for that with model-following target in Section III-A.2.

1) Moving-Target Tracking Problem: In this subsection, we shall discuss the finite-horizon tracking problem first, and then generalize the results into infinite-horizon tracking solutions. The local quadratic optimization problem is obviously the same as the general linear quadratic optimal tracking control problem. Therefore, it is reasonable that solving the optimal tracking problem for each fuzzy subsystem can be achieved in the way of the conventional approach. We can use calculus-of-variations method combined with Lagrange-multiplier method to derive the local optimal fuzzy tracking law, and then, *fuzzily blend* these tracking laws to achieve the *global optimal* fuzzy tracking controller.

Theorem 1 (Time-Varying Finite-Horizon Case): For the discrete-time fuzzy tracking system and discrete-time fuzzy tracking controller represented, respectively, by (1) and (2), let $A_i(t), B_i(t), C(t), S(t), L_2(t), L_3(t)$ be given matrices, then the follwoing hold.

1) The local optimal tracking law is

$$r_i^*(t) = G_i^1(t)X^*(t) + r_i^{\text{ext}}(t), \quad i = 1, \ldots, r \tag{11}$$

and their "blending" global optimal tracking controller

$$u^*(t) = \sum_{i=1}^r h_i(X^*(t))\left[G_i^1(t)X^*(t) + r_i^{\text{ext}}(t)\right] \tag{12}$$

minimizes $J(u(\cdot))$ in (3), where $G_i^1(t)$, the local feedback gain, and $r_i^{\text{ext}}(t)$, the introduced external local input, are given by

$$\begin{aligned} G_i^1(t) = &-S^{-1}(t)B_i^t(t)\pi_i(t+1)\left[I_n + B_i(t)S^{-1}(t)\right. \\ &\left.\times B_i^t(t)\pi_i(t+1)\right]^{-1} A_i(t) \end{aligned} \tag{13}$$

$$\begin{aligned} r_i^{\text{ext}}(t) = &-S^{-1}(t)B_i^t(t)\left[I_n + \pi_i(t+1)\right. \\ &\left.\times B_i(t)S^{-1}(t)B_i^t(t)\right]^{-1} b(t+1) \end{aligned} \tag{14}$$

where $\pi_i(t+1)$ is the symmetric positive–semidefinite solution of the following recursive Riccati equation:

$$\begin{aligned} K(t) = &L(t) + A_i^t(t)K(t+1) \\ &\times\left[I_n + B_i(t)S^{-1}(t)B_i^t(t)K(t+1)\right]^{-1} A_i(t) \end{aligned} \tag{15}$$

with $K(t_1) = \mathbf{0}_{n\times n}$, zero matrix of dimension $n\times n$, and $b(t+1)$ being the introduced target-dependent variable satisfying

$$\begin{aligned} b(t) = &A_i^t(t)\left[I_n + \pi_i(t+1)B_i(t)S^{-1}(t)B_i^t(t)\right]^{-1} b(t+1) \\ &- L(t)X^d(t), b(t_1) = \mathbf{0}_{n\times 1}. \end{aligned} \tag{16}$$

2) The optimal feedback subsystem is

$$\begin{aligned} X^*(t+1) = &\left[I_n + B_i(t)S^{-1}(t)B_i^t(t)\pi_i(t+1)\right]^{-1} \\ &\times A_i(t)X^*(t) + B_i(t)r_i^{\text{ext}}(t) \end{aligned} \tag{17}$$

and then the global optimal closed-loop tracking system is

$$\begin{aligned} X^*(t+1) = &\sum_{i=1}^r h_i(X^*(t))\left[I_n + B_i(t)S^{-1}(t)\right. \\ &\left.\times B_i^t(t)\pi_i(t+1)\right]^{-1} A_i(t)X^*(t) \\ &+ \sum_{i=1}^r h_i(X^*(t))B_i(t)r_i^{\text{ext}}(t). \end{aligned} \tag{18}$$

Proof: The proof is similar to the derivation in our previous papers [23], [24].

So far, we have solved the optimal finite-horizon fuzzy tracking problem by finding the optimal solution to the general time-varying case. We are now concerned with the infinite-horizon tracking problem, which is the case that the operating time goes to infinity or is much larger than the time-constant of the dynamic system. In other words, the performance index is

$$\begin{aligned} J(u(\cdot)) = &\sum_{t=t_0}^{\infty}[u^t(t)S(t)u(t) + (X(t) \\ &- X^d(t))^t L(t)(X(t) - X^d(t))]. \end{aligned} \tag{19}$$

We are eager to know if a time-invariant fuzzy tracking system can give rise to a time-invariant *linear* optimal tracking law with regard to each subsystem, and then generate a more implementable and important design scheme. The following theorem demonstrates that a time-invariant fuzzy tracking system cannot give rise to the time-invariant linear optimal fuzzy tracking law except in the case of constant target. □

Theorem 2 (Time-Invariant Infinite-Horizon Case): Consider the discrete time-invariant fuzzy tracking system and discrete-time fuzzy tracking controller described, respectively, by (1) and (2). If (A_i, B_i) is completely controllable (c.c.) and (A_i, C) is completely observable (c.o.) for all $i = 1, \ldots, r$, then the following hold.

1) The local optimal tracking law is

$$r_i^*(t) = \bar{G}_i^1 X^*(t) + \bar{r}_i^{\text{ext}}(t), \quad i = 1, \ldots, r \tag{20}$$

and their "blending" global optimal tracking controller

$$u^*(t) = \sum_{i=1}^r h_i(X^*(t))\left[\bar{G}_i^1 X^*(t) + \bar{r}_i^{\text{ext}}(t)\right] \tag{21}$$

minimizes $J(u(\cdot))$ in (19), where the local constant feedback gain, $\bar{G}_i^1$, and the external local input, $\bar{r}_i^{\text{ext}}(t)$, are calculated by

$$\bar{G}_i^1 = -S^{-1}B_i^t\bar{\pi}_i\left[I_n + B_iS^{-1}B_i^t\bar{\pi}_i\right]^{-1} A_i \tag{22}$$

$$\bar{r}_i^{\text{ext}}(t) = -S^{-1}B_i^t\left[I_n + \bar{\pi}_iB_iS^{-1}B_i^t\right]^{-1}\bar{b}(t+1) \tag{23}$$

$$\bar{b}(t) = -\sum_{j=0}^{\infty} A_i^{j^t}\left[I_n + \bar{\pi}_iB_iS^{-1}B_i^t\right]^{-j} LX^d(t+j) \tag{24}$$

where $\bar{\pi}_i$ is the unique symmetric positive semidefinite solution of the following discrete-time algebraic Riccati equation:

$$K = L + A_i^tK\left[I_n + B_iS^{-1}B_i^tK\right]^{-1} A_i. \tag{25}$$

2) The optimal feedback subsystem is

$$X^*(t+1) = \left[I_n + B_i S^{-1} B_i^t \bar{\pi}_i\right]^{-1} A_i X^*(t) + B_i \bar{r}_i^{\text{ext}}(t) \tag{26}$$

and then the global optimal closed-loop tracking system is

$$X^*(t+1) = \sum_{i=1}^{r} h_i(X^*(t)) \left[I_n + B_i S^{-1} B_i^t \bar{\pi}_i\right]^{-1} A_i X^*(t) + \sum_{i=1}^{r} h_i(X^*(t)) B_i \bar{r}_i^{\text{ext}}(t). \tag{27}$$

Proof:

1) The optimal solution indeed follows the optimal solution in Theorem 1 except that all the parameters in (1), (2), and (19) are now constant. It is easy to show that the asymptotic solution of the recursive Riccati equation in (15) is also the steady-state solution, i.e., $\lim_{t_1\to\infty} \pi_i(k,t_1) = \bar{\pi}_i$, and this solution results in the asymptotic solution of the recursive equation in (16) which is equivalent to $\bar{b}(t)$ in (24), i.e., $\lim_{t_1\to\infty} b(t) = \bar{b}(t)$.
2) Moreover, according to Lemma 2 in the Appendix, we know that (A_i, B_i) being c.c. and (A_i, C) being c.o., $\forall i = 1,\ldots,r$, guarantee the existence of an unique symmetric positive semidefinite solution of the algebraic Riccati equation in (25). Hence, the proof is completed. □

2) Model-Following Tracking Problem: In this subsection, we are devoted to the model-following tracking problem, where the tracked target is the response of some reference model. Similar to the previous subsection, the finite-horizon tracking problem is discussed first. The derived optimal solutions can then be generalized into those for the infinite-horizon problem as we did in the last subsection. We adopt the same T-S type fuzzy tracking system as in Section II, and thereupon, the standard model-following tracking problem can be described as the following issue.

Problem 3: Given a discrete-time fuzzy tracking system and a fuzzy tracking controller, respectively, in (1) and (2) with $X(t_0) = X_0 \in \Re^n$ and $t \in [t_0, t_1 - 1]$, find $u^*(\cdot)$ to minimize $J(u(\cdot))$ in (3), where the desired output $Y^d(t)$ is the response of a linear system or model

$$\begin{aligned} z_1(t+1) &= F_1(t) z_1(t) + J_1(t)\nu(t) \\ Y^d(t) &= E_1(t) z_1(t) \end{aligned} \tag{28}$$

with $z_1(t_0) = z_{10}$, to the command input $\nu(t) \in \Re^{m'}$, which is the zero-input response of the system: $z_2(t+1) = F_2(t) z_2(t)$ and $\nu(t) = E_2(t) z_2(t)$ with $z_2(t_0) = z_{20}$ [25], where $z_1(t) \in \Re^h$ and $z_2(t) \in \Re^{h'}$ are system states; $F_1(t), J_1(t), E_1(t), F_2(t)$ and $E_2(t)$ are matrices of $h\times h, h\times m', n'\times h, h'\times h'$ and $m'\times h'$, respectively.

Accordingly, the desired tracked system, via letting $Z(t) = [z_1^t(t) \;\; z_2^t(t)]^t$, can be rewritten as the following augmented system [25]:

$$\begin{aligned} Z(t+1) &= \begin{bmatrix} F_1(t) & J_1(t)E_2(t) \\ \mathbf{0}_{h'\times h} & F_2(t) \end{bmatrix} Z(t) = F(t)Z(t) \\ Y^d(t) &= [E_1(t) \quad \mathbf{0}_{n'\times h'}] Z(t) = E(t)Z(t). \end{aligned} \tag{29}$$

We further define a new variable $\tilde{X}(t) = [X^t(t) \;\; Z^t(t)]^t$ [25], and then Problem 3 can be simplified as the following issue.

Problem 4: Given a fuzzy tracking system

$$\tilde{X}(t+1) = \sum_{i=1}^{r} h_i(\tilde{X}(t)) \tilde{A}_i(t) \tilde{X}(t) + \sum_{i=1}^{r} h_i(\tilde{X}(t)) \tilde{B}_i(t) u(t) \tag{30}$$

with $\tilde{X}(t_0) = \tilde{X}_0 \in \Re^{n+h+h'}$ and $h_i(\tilde{X}(t)) = h_i(X(t)), t \in [t_0, t_1 - 1]$, find $u^*(\cdot)$ to minimize

$$J(u(\cdot)) = \sum_{t=t_0}^{t_1-1} [\tilde{X}^t(t)\tilde{L}(t)\tilde{X}(t) + u^t(t)S(t)u(t)] \tag{31}$$

where the parameters are as shown in the equation at the bottom of the page. Notice that the fuzzy tracking controller is $u(t) = \sum_{i=1}^{r} h_i(\tilde{X}(t)) r_i(t)$.

Obviously, the optimal solutions for the augmented optimal quadratic tracking problem in Problem 4 follow from Theorem 1 except that $X^d(\cdot)$ in Section III-A.1 are zero vectors now. Then, via complicated matrix manipulations, we can obtain the optimal solutions for the original optimal quadratic tracking problem in Problem 3 as follows. The identity input weighting factor is set to get more concise formula in the remainder of this section, i.e., $S(\cdot) = I_m$ for all time steps.

Theorem 3 (Time-Varying Finite-Horizon Case): For the fuzzy tracking system and fuzzy tracking controller represented, respectively, by (1) and (2), let the desired trajectory, $X^d(t)$, come from $Y^d(t) = CX^d(t)$, where $Y^d(t)$ is the output of the tracked model in (28). Then, the following hold.

1) The local optimal tracking law is

$$r_i^*(t) = G_i^1(t) X^*(t) + G_i^2(t) Z(t), \quad i = 1,\ldots,r \tag{32}$$

and their "blending" global optimal tracking controller

$$u^*(t) = \sum_{i=1}^{r} h_i(X^*(t)) \left[G_i^1(t) X^*(t) + G_i^2(t) Z(t)\right] \tag{33}$$

$$\tilde{B}_i(t) = \begin{bmatrix} B_i(t) \\ \mathbf{0}_{(h+h')\times m} \end{bmatrix}, \quad \tilde{A}_i(t) = \begin{bmatrix} A_i(t) & \mathbf{0}_{n\times(h+h')} \\ \mathbf{0}_{(h+h')\times n} & F(t) \end{bmatrix} \quad \text{and}$$

$$\tilde{L}(t) = \begin{bmatrix} L(t) & -L(t)C^t(t)[C(t)C^t(t)]^{-1}E(t) \\ -E^t(t)[C(t)C^t(t)]^{-1}C(t)L(t) & E^t(t)[C(t)C^t(t)]^{-1}C(t)L(t)C^t(t)[C(t)C^t(t)]^{-1}E(t) \end{bmatrix}.$$

minimizes $J(u(\cdot))$ in (3), where the feedback gain, $G_i^1(t)$, and the introduced matrix, $G_i^2(t)$, are calculated by

$$G_i^1(t) = -B_i^t(t)\pi_i(t+1)\left[I_n + B_i(t)B_i^t(t)\pi_i(t+1)\right]^{-1} A_i(t) \quad (34)$$

$$G_i^2(t) = -B_i^t(t)\left[I_n + \pi_i(t+1)B_i(t)B_i^t(t)\right]^{-1} \pi_i^{21^t}(t+1)F(t) \quad (35)$$

where $\pi_i(t+1)$ is the symmetric positive–semidefinite solution of the following recursive Riccati equation:

$$K(t) = L(t) + A_i^t(t)K(t+1) \times \left[I_n + B_i(t)B_i^t(t)K(t+1)\right]^{-1} A_i(t) \quad (36)$$

with $K(t_1) = \mathbf{0}_{n\times n}$, and $\pi_i^{21}(t+1)$ satisfies

$$K_{21}(t) = F^t(t)K_{21}(t+1)\left[I_n + B_i(t)B_i^t(t)\pi_i(t+1)\right]^{-1} \times A_i(t) - E^t(t)[C(t)C^t(t)]^{-1}C(t)L(t) \quad (37)$$

with $K_{21}(t_1) = \mathbf{0}_{(h+h')\times n}$.

2) The optimal feedback subsystem is

$$X^*(t+1) = \left[I_n + B_i(t)B_i^t(t)\pi^i(t+1)\right]^{-1} A_i(t)X^*(t) + B_i(t)G_i^2(t)Z(t) \quad (38)$$

and then the global optimal closed-loop tracking system is

$$X^*(t+1) = \sum_{i=1}^{r} h_i(X^*(t))\left[I_n + B_i(t)B_i^t(t)\pi_i(t+1)\right]^{-1} \times A_i(t)X^*(t) + \sum_{i=1}^{r} h_i(X^*(t))B_i(t)G_i^2(t)Z(t). \quad (39)$$

Proof:

1) For notation simplification, the identity and zero matrices of any dimension will be denoted by I and $\mathbf{0}$, respectively. We can still, based on the inference in Section II, decompose the quadratic tracking problem in Problem 4 into r linear quadratic tracking problems as in Problem 2 except that $X^d(\cdot)$ are zero vectors now. Then, grounding on Theorem 1, we have the following local optimal solution:

$$\tilde{X}^*(t+1) = \left[I + \tilde{B}_i(t)\tilde{B}_i^t(t)\tilde{\pi}_i(t+1)\right]^{-1} \tilde{A}_i(t)\tilde{X}^*(t) \quad (40)$$

$$r_i^*(t) = -\tilde{B}_i^t(t)\tilde{\pi}_i(t+1)\left[I + \tilde{B}_i(t) \times \tilde{B}_i^t(t)\tilde{\pi}_i(t+1)\right]^{-1} \tilde{A}_i(t)\tilde{X}^*(t) \quad (41)$$

where $\tilde{\pi}_i(t)$ is the symmetric positive–semidefinite solution of the the following recursive Riccati equation:

$$\tilde{K}(t) = \tilde{A}_i^t(t)\tilde{K}(t+1)\left[I + \tilde{B}_i(t)\tilde{B}_i^t(t) \times \tilde{K}(t+1)\right]^{-1} \tilde{A}_i(t) + \tilde{L}(t), \quad \tilde{K}(t_1) = \mathbf{0}. \quad (42)$$

2) Now, let

$$\tilde{K}(t) = \begin{bmatrix} K(t) & K_{21}^t(t) \\ K_{21}(t) & K_{22}(t) \end{bmatrix}.$$

We obtain $r_i^*(t)$ in (32) from (41) via

$$\begin{bmatrix} I + B_iB_i^tK & B_iB_i^tK_{21}^t \\ \mathbf{0} & I \end{bmatrix}^{-1} = \begin{bmatrix} [I + B_iB_i^tK]^{-1} & -[I + B_iB_i^tK]^{-1}B_iB_i^tK_{21}^t \\ \mathbf{0} & I \end{bmatrix} \quad (43)$$

and $B_i^tK_{21}^t - B_i^tK[I + B_iB_i^tK]^{-1}B_iB_i^tK_{21}^t = B_i^t[I + KB_iB_i^t]^{-1}K_{21}^t$, where the time-dependence is omitted for notation simplification; $K(t)$ in (36) and K_{21} in (37) are derived from (42) with the aid of $I - K[I + B_iB_i^tK]^{-1}B_iB_i^t = [I + KB_iB_i^t]^{-1}$; and then we have $X^*(t)$ in (38) from (40).

3) We then fuzzily blend the r local optimal tracking laws and optimal tracking subsystems to obtain the corresponding optimal tracking controller $u^*(t)$ in (33) and the optimal trajectory $X^*(t)$ in (39), respectively. □

The scheme of generalizing the optimal tracking solution from finite-horizon problem to infinite-horizon problem for model-following target is just the same as that for moving target in Section III-A1. Therefore, we only summarizes the solutions of the infinite-horizon problem with respect to the model-following tracking issue as follows.

Theorem 4 (Time-Invariant Infinite-Horizon Case): For the time-invariant fuzzy tracking system and fuzzy tracking controller represented, respectively, by (1) and (2), let the desired trajectory, $X^d(t)$, come from $Y^d(t) = CX^d(t)$, where $Y^d(t)$ is the output of the tracked model in (28). If (A_i, B_i) is c.c. and (A_i, C) is c.o., for all $i = 1, \ldots, r$, then, the "blending" global optimal tracking controller is

$$u^*(t) = \sum_{i=1}^{r} h_i(X^*(t))\left[\bar{G}_i^1X^*(t) + \bar{G}_i^2Z(t)\right] \quad (44)$$

which minimizes $J(u(\cdot))$ in (19); and the corresponding global optimal closed-loop tracking system is

$$X^*(t+1) = \sum_{i=1}^{r} h_i(X^*(t))[I_n + B_iB_i^t\bar{\pi}_i]^{-1}A_iX^*(t) + \sum_{i=1}^{r} h_i(X^*(t))B_i\bar{G}_i^2Z(t) \quad (45)$$

where $\bar{G}_i^1 = -B_i^t\bar{\pi}_i[I_n + B_iB_i^t\bar{\pi}_i]^{-1}A_i$, $\bar{G}_i^2 = -B_i^t[I_n + \bar{\pi}_iB_iB_i^t]^{-1}\bar{\pi}_i^{21^t}(t+1)F, \bar{\pi}_i$ is the symmetric positive–semidefinite solution of the recursive Riccati equation

$$K = L + A_i^tK\left[I_n + B_iB_i^tK\right]^{-1} A_i \quad (46)$$

and $\bar{\pi}_i^{21}$ satisfies

$$K_{21} = F^tK_{21}\left[I_n + B_iB_i^t\bar{\pi}_i\right]^{-1} A_i - E^t[CC^t]^{-1}CL. \quad (47)$$

Proof: The proof, grounded on Theorem 3, follows the same generalization of the finite-horizon case to the infinite-horizon case as that in Theorem 2. □

B. Optimal Fuzzy Tracker for Continuous Fuzzy System

We are now going to design the optimal fuzzy tracking controllers for the continuous fuzzy tracking systems with moving target in Section III-B1 and for that with model-following target in Section III-B2.

1) Moving-Target Tracking Problem: As remarked in Section II, we know that, via describing the fuzzy system from the local inspection, the nonlinear quadratic optimal fuzzy tracking problem in Problem 1 can be described by the linear quadratic problem in Problem 2. Hence, the continuous fuzzy tracking problem can be solved by obtaining the local optimal decision or the tracking solutions for each fuzzy *subsystem* by the conventional approach first, and then fuzzily blending the local solutions to obtain the global optimal decision or the optimal solutions for the entire continuous fuzzy system as follows.

Theorem 5 (Time-Varying Finite-Horizon Case): For the continuous fuzzy tracking system and fuzzy tracking controller represented, respectively, by (1) and (2), let $A_i(t), B_i(t), C(t), S(t), L_2(t), L_3(t)$ be given matrices, then the following hold.

1) The local optimal tracking law is

$$r_i^*(t) = -S^{-1}(t)B_i^t(t)[\pi_i(t,t_1)X^*(t) + b(t)], \quad i = 1,\ldots,r \tag{48}$$

and their "blending" global optimal tracking controller

$$u^*(t) = -\sum_{i=1}^{r} h_i(X^*(t))S^{-1}(t)B_i^t(t)[\pi_i(t,t_1)X^*(t) + b(t)] \tag{49}$$

minimizes $J(u(\cdot))$ in (4), where $\pi_i(t,t_1)$ is the symmetric positive–semidefinite solution of the following differential Riccati equation:

$$-\dot{K}(t) = L(t) + A_i^t(t)K(t) + K(t)A_i^t(t) - K(t)B_i(t)S^{-1}(t)B_i^t(t)K(t) \tag{50}$$

with $K(t_1) = \mathbf{0}_{n\times n}$, zero matrix of dimension $n\times n$, and $b(t)$ is the introduced target-dependent variable satisfying

$$\dot{b}(t) = -\left[A_i(t) - B_i(t)S^{-1}(t)B_i^t(t)\pi_i(t,t_1)\right]^t b(t) + L(t)X^d(t), b(t_1) = \mathbf{0}_{n\times 1}. \tag{51}$$

2) The optimal feedback subsystem is

$$\dot{X}^*(t) = \left[A_i(t) - B_i(t)S^{-1}(t)B_i^t(t)\pi_i(t,t_1)\right]X^*(t) - B_i(t)S^{-1}(t)B_i^t(t)b(t) \tag{52}$$

and then the global optimal closed-loop tracking system is

$$\dot{X}^*(t) = \sum_{i=1}^{r} h_i(X^*(t))[A_i(t) - B_i(t)S^{-1}(t)B_i^t(t)\pi_i(t,t_1)] \times X^*(t) - \sum_{i=1}^{r} h_i(X^*(t))B_i(t)S^{-1}(t)B_i^t(t)b(t). \tag{53}$$

Proof: Grounding on the calculus-of-variations method, we introduce the Lagrange multiplier $p(t)$ and let $p(t) = K(t)X^*(t) + b(t)$, whence we obtain the local optimal tracking law $r_i^*(t)$ in (48), and the optimal feedback subsystem $X^*(t)$ in (52), where $K(t)$ and $b(t)$ satisfy, respectively, (50) and (51) with $K(t_1) = \mathbf{0}_{n\times n}$ and $b(t_1) = \mathbf{0}_{n\times 1}$. Accordingly, we fuzzily blend the local optimal results to get the corresponding optimal tracking controller $u^*(t)$ in (49) and the optimal tracking trajectory $X^*(t)$ in (53). □

The aforementioned theorem covers the general continuous time-varying quadratic optimal problems, where the horizon t_1 is fixed and $t_0 \in [0, t_1)$ is arbitrary. We are now concerned with the time-invariant infinite-horizon tracking problem. In other words, the performance index is

$$J(u(\cdot)) = \int_{t_0}^{\infty} [u^t(t)Su(t) + (X(t) - X^d(t))^t L(X(t) - X^d(t))]\, dt. \tag{54}$$

We shall demonstrate that, even in the infinite-horizon situation, a time-invariant fuzzy tracking system cannot give rise to the time-invariant linear optimal fuzzy tracking law except in the case of constant target. □

Theorem 6 (Time-Invariant Infinite-Horizon Case): Consider the continuous time-invariant fuzzy tracking system and fuzzy tracking controller described, respectively, by (1) and (2). If (A_i, B_i) is c.c. and (A_i, C) is c.o. for all $i = 1,\ldots,r$, then the following hold.

1) The local optimal tracking law is

$$r_i^*(t) = -S^{-1}B_i^t[\bar{\pi}_i X^*(t) + \bar{b}(t)], \qquad i = 1,\ldots,r \tag{55}$$

and their "blending" global optimal tracking controller

$$u^*(t) = -\sum_{i=1}^{r} h_i(X^*(t))S^{-1}B_i^t[\bar{\pi}_i X^*(t) + \bar{b}(t)] \tag{56}$$

minimizes $J(u(\cdot))$ in (54), where $\bar{\pi}_i$ is the unique symmetric positive–semidefinite solution of the following steady state Riccati equation (SSRE):

$$A_i^t K + KA_i - KB_iS^{-1}B_i^tK + L = \mathbf{0}_{n\times n} \tag{57}$$

and the target-dependent variable is

$$\bar{b}(t) = -\int_t^{\infty} e^{[A_i - B_iS^{-1}B_i^t\bar{\pi}_i]^t(\tau - t)} \cdot LX^d(\tau)\, d\tau. \tag{58}$$

2) The optimal feedback subsystem is

$$\dot{X}^*(t) = \left[A_i - B_iS^{-1}B_i^t\bar{\pi}_i\right]X^*(t) - B_iS^{-1}B_i^t\bar{b}(t) \tag{59}$$

and then the global optimal closed-loop tracking system is

$$\dot{X}^*(t) = \sum_{i=1}^{r} h_i(X^*(t))\left[A_i - B_iS^{-1}B_i^t\bar{\pi}_i\right]X^*(t) - \sum_{i=1}^{r} h_i(X^*(t))B_iS^{-1}B_i^t\bar{b}(t). \tag{60}$$

Proof: Based on the optimal solutions in Theorem 5, we can obtain the infinite-horizon time-invariant optimal solution by letting $t_1 \to \infty$. Moreover, we know that the asymptotic solution of the differential Riccati equation in (50) is also the solution of the algebraic Riccatic equation in (57), i.e., $\lim_{t_1\to\infty} \pi_i(k,t_1) = \bar{\pi}_i$, and this solution results in the asymptotic solution of the differential equation in (51) which is equivalent to $\bar{b}(t)$ in (58), i.e., $\lim_{t_1\to\infty} b(t) = \bar{b}(t)$. Furthermore, since (A_i, B_i) is c.c. and (A_i, C) is c.o., for all $i = 1,\ldots,r$, we know, via the linear quadratic theory [25], that the symmetric positive–semidefinite solution of the algebraic Riccati equation in (57) uniquely exists. □

2) Model-Following Tracking Problem: We are now concerned with the fuzzy tracking problem with the target from the response of some reference model, i.e., the model-following tracking problem. The optimal solutions for the finite-horizon tracking problem are derived first, and then they are generalized into the infinite-horizon situation as we did in the last subsection. The same T–S type fuzzy tracking system as in Section II is adopted, and a continuous model-following tracking problem is formulated as follows.

Problem 5: Given a continuous fuzzy tracking system and a fuzzy tracking controller, respectively, in (1) and (2) with $X(t_0) = X_0 \in \Re^n$ and $t \in [t_0, t_1]$, find the individual optimal tracking law, $r_i^*(\cdot), i = 1, \ldots, \delta$, such that the composed optimal tracking controller, $u^*(\cdot)$, can minimize $J(u(\cdot))$ in (4), where the desired output $Y^d(t)$ is the response of a linear system or model,

$$\begin{aligned} \dot{z_1}(t) &= F_1(t)z_1(t) + J_1(t)\nu(t), \\ Y^d(t) &= E_1(t)z_1(t) \end{aligned} \tag{61}$$

with $z_1(t_0) = z_{10}$, to the command input $\nu(t) \in \Re^{m'}$, which is the zero-input response of the system: $\dot{z_2}(t) = F_2(t)z_2(t)$ and $\nu(t) = E_2(t)z_2(t)$ with $z_2(t_0) = z_{20}$ [25], where $z_1(t) \in \Re^h$ and $z_2(t) \in \Re^{h'}$ are system states; $F_1(t), J_1(t), E_1(t), F_2(t)$ and $E_2(t)$ are matrices of $h \times h, h \times m', n' \times h, h' \times h'$ and $m' \times h'$, respectively.

Similar to Section III-A2, we can get a more concise problem formulation.

Problem 6: Given a fuzzy tracking system

$$\dot{\tilde{X}}(t) = \sum_{i=1}^{r} h_i(\tilde{X}(t)) \left[\tilde{A}_i(t)\tilde{X}(t) + \tilde{B}_i(t) \sum_{j=1}^{\delta} w_j(Y(t)) r_j(t) \right] \tag{62}$$

with $\tilde{X}(t_0) = \tilde{X}_0 \in \Re^{n+h+h'}$ and $h_i(\tilde{X}(t)) = h_i(X(t)), t \in [t_0, t_1]$, find the individual optimal tracking law, $r_i^*(\cdot), i = 1, \ldots, \delta$, to minimize

$$J(r_i(\cdot)) = \int_{t_0}^{t_1} \left[\tilde{X}^t(t)\tilde{L}(t)\tilde{X}(t) + \sum_{i=1}^{\delta}\sum_{j=1}^{\delta} w_i(Y(t))w_j(Y(t))r_i^t(t)S(t)r_j(t) \right] dt \tag{63}$$

where the parameters are shown in the equation at the bottom of the page.

Obviously, the optimal solutions for the augmented optimal quadratic tracking problem in Problem 6 are the same as those in Theorem 5 by setting $X^d(\cdot)$ as zero vectors now. Then via further matrix manipulations, we can obtain the optimal solutions for the original optimal quadratic tracking problem in Problem 5 as follows.

Theorem 7 (Time-Varying Finite-Horizon Case): For the continuous fuzzy tracking system and the fuzzy tracking controller represented, respectively, by (1) and (2), let the desired trajectory, $X^d(t)$, come from $Y^d(t) = CX^d(t)$, where $Y^d(t)$ is the output of the tracked model in (61). Then, the following hold.

1) The local optimal tracking law is

$$\begin{aligned} r_i^*(t) = &-S^{-1}(t)B_i^t(t)\pi_i(t,t_1)X^*(t) \\ &- S^{-1}(t)B_i^t(t)\pi_i^{21^t}(t,t_1)Z(t), \qquad i = 1, \ldots, r \end{aligned} \tag{64}$$

and their "blending" global optimal tracking controller

$$\begin{aligned} u^*(t) = \sum_{i=1}^{r} h_i(X^*(t)) \big[&-S^{-1}(t)B_i^t(t)\pi_i(t,t_1)X^*(t) \\ &- S^{-1}(t)B_i^t(t)\pi_i^{21^t}(t,t_1)Z(t) \big] \end{aligned} \tag{65}$$

minimizes $J(u(\cdot))$ in (4), where $\pi_i^{21}(t,t_1)$ satisfies

$$\begin{aligned} -\dot{K}_{21}(t) &= F^t(t)K_{21}(t) + K_{21}(t)A_i(t) \\ &\quad - K_{21}(t)B_i(t)S^{-1}(t)B_i^t(t)\pi_i(t,t_1) \\ &\quad - E^t(t)[C(t)C^t(t)]^{-1}C(t)L(t) \\ K_{21}(t_1) &= \mathbf{0}_{(h+h')\times n} \end{aligned} \tag{66}$$

and $\pi_i(t,t_1)$ is the symmetric positive–semidefinite solution of the differential Riccati equation in (50) with $K(t_1) = \mathbf{0}_{n\times n}$.

2) The optimal feedback subsystem is

$$\begin{aligned} \dot{X}^*(t) = &(A_i(t) - B_i(t)S^{-1}(t)B_i^t(t)\pi_i(t,t_1))X^*(t) \\ &- B_i(t)S^{-1}(t)B_i^t(t)\pi_i^{21^t}(t,t_1)Z(t) \end{aligned} \tag{67}$$

and then the global optimal closed-loop tracking system is

$$\begin{aligned} \dot{X}^*(t) = &\sum_{i=1}^{r} h_i(X^*(t))(A_i(t) \\ &- B_i(t)S^{-1}(t)B_i^t(t)\pi_i(t,t_1))X^*(t) \\ &- \sum_{i=1}^{r} h_i(X^*(t))B_i(t)S^{-1}(t)B_i^t(t)\pi_i^{21^t}(t,t_1)Z(t). \end{aligned} \tag{68}$$

Proof: The proof is similar to that of Theorem 3. □

We now let the finite horizon t_1 approach infinity in order to get the infinite-horizon optimal solutions as follows.

Theorem 8 (Time-Invariant Infinite-Horizon Case): For the continuous time-invariant fuzzy tracking system and the fuzzy tracking controller represented, respectively, by (1) and (2), let the desired trajectory, $X^d(t)$, come from $Y^d(t) = CX^d(t)$,

$$\tilde{B}_i(t) = \begin{bmatrix} B_i(t) \\ \mathbf{0}_{(h+h')\times m} \end{bmatrix}, \quad \tilde{A}_i(t) = \begin{bmatrix} A_i(t) & \mathbf{0}_{n\times(h+h')} \\ \mathbf{0}_{(h+h')\times n} & F(t) \end{bmatrix}, \quad \text{and}$$

$$\tilde{L}(t) = \begin{bmatrix} L(t) & -L(t)C^t(t)[C(t)C^t(t)]^{-1}E(t) \\ -E^t(t)[C(t)C^t(t)]^{-1}C(t)L(t) & E^t(t)[C(t)C^t(t)]^{-1}C(t)L(t)C^t(t)[C(t)C^t(t)]^{-1}E(t) \end{bmatrix}.$$

where $Y^d(t)$ is the output of the tracked model in (61). If (A_i, B_i) is c.c. and (A_i, C) is c.o., for all $i = 1, \ldots, r$, then the following hold.

1) The local optimal tracking law is

$$r_i^*(t) = -S^{-1}B_i^t\bar{\pi}_i X^*(t) - S^{-1}B_i^t\bar{\pi}_i^{21^t}(t)Z(t), \ \ i = 1, \ldots, r \tag{69}$$

and their "blending" global optimal tracking controller

$$u^*(t) = \sum_{i=1}^{r} h_i(X^*(t)) \times \left[-S^{-1}B_i^t\bar{\pi}_i X^*(t) - S^{-1}B_i^t\bar{\pi}_i^{21^t}(t)Z(t)\right] \tag{70}$$

minimizes $J(u(\cdot))$ in (54), where $\bar{\pi}_i$ is the unique symmetric positive–semidefinite solution of the SSRE in (57), and

$$\bar{\pi}_i^{21}(t) = \lim_{t_1\to\infty} \pi_i^{21}(t, t_1) = -\int_t^{\infty} e^{F^t(\tau-t)} \cdot E^t(CC^t)^{-1}CL \cdot e^{(A_i - B_iS^{-1}B_i^t\bar{\pi}_i)(\tau-t)} \cdot d\tau. \tag{71}$$

2) The optimal feedback subsystem is

$$\dot{X}^*(t) = (A_i - B_iS^{-1}B_i^t\bar{\pi}_i)X^*(t) - B_iS^{-1}B_i^t\bar{\pi}_i^{21^t}(t)Z(t) \tag{72}$$

and then the global optimal closed-loop tracking system is

$$\dot{X}^*(t) = \sum_{i=1}^{r} h_i(X^*(t))\left(A_i - B_iS^{-1}B_i^t\bar{\pi}_i\right)X^*(t) - \sum_{i=1}^{r} h_i(X^*(t))B_iS^{-1}B_i^t\bar{\pi}_i^{21^t}(t)Z(t). \tag{73}$$

Proof: The proof, grounded on Theorem 7, follows the same generalization of the finite-horizon case to the infinite-horizon case as that in Theorem 6. □

IV. Stability and Gain Margin

So far, the design scheme of the fuzzy trackers for both continuous and discrete-time fuzzy systems have been developed. We are now devoted to the stability analysis of both kinds of resultant closed-loop fuzzy tracking systems. In this section, we shall show that the designed fuzzy tracking controllers can not only exponentially stabilize the fuzzy tracking system, but also form a closed-loop time-invariant fuzzy tracking system with any desired degree of stability. We are also concerned with the range of the feedback gain, *gain margin*, to which we can increase under the stability consideration.

In other words, for the continuous case, we discuss the stability of the following two systems:

$$\dot{X}^*(t) = \sum_{i=1}^{r} h_i(X^*(t))\left(A_i - B_iS^{-1}B_i^t\bar{\pi}_i\right)X^*(t) - \sum_{i=1}^{r} h_i(X^*(t))B_iS^{-1}B_i^t\overline{b(t)} \tag{74}$$

for the moving-target tracking problem, and

$$\dot{X}^*(t) = \sum_{i=1}^{r} h_i(X^*(t))\left(A_i - B_iS^{-1}B_i^t\bar{\pi}_i\right)X^*(t) - \sum_{i=1}^{r} h_i(X^*(t))B_iS^{-1}B_i^t\bar{\pi}_i^{21^t}(t)Z(t) \tag{75}$$

for the model-following-target tracking issue. Since $S^{-1}B_i^t\bar{b}(t)$ in (74) and $S^{-1}B_i^t\bar{\pi}_i^{21^t}(t)Z(t)$ in (75) are both associated with the target only, they can be regarded as external local inputs, $\bar{r}_i^{\text{ext}}(t)$. Therefore, we can unify these two equations into

$$\dot{X}^*(t) = \sum_{i=1}^{r} h_i(X^*(t)) \times \left[(A_i - B_iS^{-1}B_i^t\bar{\pi}_i)X^*(t) - B_i\bar{r}_i^{\text{ext}}(t)\right] \tag{76}$$

which is a nonlinear system constituted by a set of linear fuzzy subsystems. Then, based on the *converse theorem* of Lyapunov stability theory [28], we know the stability of the nonlinear tracking system in (76) is coincident with that of the corresponding linearized system (with regard to X_o)

$$\dot{X}^*(t) = \sum_{i=1}^{r} h_i(X_o)\left(A_i - B_iS^{-1}B_i^t\bar{\pi}_i\right)X^*(t) - \sum_{i=1}^{r} h_i(X_o)B_i\bar{r}_i^{\text{ext}}(t). \tag{77}$$

Therefore, the stability of the fuzzy system in (76) is governed by the term $\sum_{i=1}^{r} h_i(X(t))(A_i - B_iS^{-1}B_i^t\bar{\pi}_i)$, which also handles the stability of the following zero-input fuzzy system:

$$\dot{X}^*(t) = \sum_{i=1}^{r} h_i(X^*(t))\left(A_i - B_iS^{-1}B_i^t\bar{\pi}_i\right)X^*(t). \tag{78}$$

Hence, we shall focus only on discussing the stability of the zero-input fuzzy system in the above to demonstrate the stability of the resultant closed-loop fuzzy tracking system in (76) or (74) and (75).

Furthermore, we have demonstrated in our previous paper [23] that the aforementioned zero-input fuzzy system in (78) is exponentially stable, and possesses any degree of stability and infinite gain margin if each subsystem is c.c. and c.o. (well-behaved). Therefore, we conclude that the resultant continuous closed-loop fuzzy tracking system in (76) or (74) and (75) also possess such fantastic characteristics.

We now step for analyzing the stability property of the resultant discrete-time closed-loop fuzzy tracking system. In other words, the stability of the following two systems are discussed first:

$$X^*(t+1) = \sum_{i=1}^{r} h_i(X^*(t))\left(\left[I_n + B_iS^{-1}B_i^t\bar{\pi}_i\right]^{-1} \times A_iX^*(t) + B_i\bar{r}_i^{\text{ext}}(t)\right) \tag{79}$$

for the moving-target tracking problem, and

$$X^*(t+1) = \sum_{i=1}^{r} h_i(X^*(t))\left(\left[I_n + B_iB_i^t\bar{\pi}_i\right]^{-1} \times A_iX^*(t) + B_i\bar{G}_i^2Z(t)\right) \tag{80}$$

for the model-following-target tracking issue. Since $\bar{G}_i^2 Z(t)$ in (80) is associated with the target only and can be regarded as an external local input, $\bar{r}_i^{\rm ext}(t)$, (79) and (80) can be unified into one equation [(79)] by setting $S = I_m$.

Grounding on the converse theorem, we know the stability characteristics for both discrete-time issues of moving-target tracking and model-following-target tracking can be guaranteed by the stability of the following zero-input fuzzy system:

$$X^*(t+1) = \sum_{i=1}^{r} h_i(X^*(t)) \left[I_n + B_i S^{-1} B_i^t \bar{\pi}_i\right]^{-1} A_i X^*(t) \tag{81}$$

which has been demonstrated to be exponentially stable and to possess any degree of stability for each well-behaved subsystem in [24].

In the remainder of this section, we shall examine another characteristic, *gain margin*, of the resultant *discrete-time* closed-loop fuzzy tracking system. Recall that the gain margin of a closed-loop system is the amount by which the loop gain can be changed until the system becomes unstable [25]. As we remarked earlier, for a time-invariant well-behaved fuzzy tracking subsystem, the designed global optimal tracking controller in (21) and (44) can be unified into $u^*(t) = \sum_{i=1}^{r} h_i(X^*(t))[\bar{G}_i^1 X^*(t) + \bar{r}_i^{\rm ext}(t)]$. In order to measure the gain margin, we consider a corresponding tracking controller $u(t) = \sum_{i=1}^{r} h_i(X(t))[\beta \bar{G}_i^1 X(t) + \bar{r}_i^{\rm ext}(t)]$. *The gain margin of the closed-loop fuzzy tracking system is defined as the amount by which* β *can be increased until the system becomes unstable*. Now, let $v(t) \triangleq u(t)/\beta = \sum_{i=1}^{r} h_i(X(t))[\bar{G}_i^1 X(t) + \bar{r}_i^{\prime\rm ext}(t)]$, where $\bar{r}_i^{\prime\rm ext}(t) = \bar{r}_i^{\rm ext}(t)/\beta$, and then we have, by setting the input weighting factor to be one for convenience

$$\begin{aligned} J(u(\cdot)) &= \sum_{t=t_0}^{\infty} (e^t(t) L e(t) + u^t(t) S u(t)) \\ &= \sum_{t=t_0}^{\infty} (e^t(t) L e(t) + \beta^2 v^t(t) S v(t)) \end{aligned} \tag{82}$$

where $e(t) = X(t) - X^d(t)$ and $\beta \geq 1$. We further consider

$$J(v(\cdot)) = \sum_{t=t_0}^{\infty} (q e^t(t) L e(t) + v^t(t) S v(t)), \quad q > 0. \tag{83}$$

Notice that $J(u(\cdot)) = \beta^2 J(v(\cdot))$ and $q = 1/\beta^2$. Comparing (83) to (82), we find that the larger the β is, the smaller the q is, which means that when q goes to zero, the gain margin of the closed-loop fuzzy tracking system becomes infinite. Now, we shall show that the resulting closed-loop fuzzy tracking system possesses an infinite gain margin.

Lemma 1: Consider a linear time-invariant dynamical fuzzy subsystem

$$\begin{aligned} X(t+1) &= A_i X(t) + B_i r_i(t) \\ Y(t) &= C X(t) \end{aligned} \tag{84}$$

with $X(t_0)$ known. If (A_i, B_i) is c.c., (A_i, C) is c.o., and $\hat{\bar{\pi}}_i(q)$ is the positive–semidefinite solution of the modified discrete-time algebraic Riccati equation

$$K(q) = qL + A_i^t K(q) \left[I_n + B_i S^{-1} B_i^t K(q)\right]^{-1} A_i \tag{85}$$

where $K(q)$ is the dependent variable of the algebraic equation, then $\lim_{q\to 0} \hat{\bar{\pi}}_i(q)$ exists and is equal to $\hat{\bar{\pi}}_i(0)$, which is the symmetric positive–semidefinite solution of the modified discrete-time Riccati equation

$$K(0) = A_i^t K(0) \left[I_n + B_i S^{-1} B_i^t K(0)\right]^{-1} A_i. \tag{86}$$

Proof: See the Appendix. □

Theorem 9 (Gain Margin for Discre-Time Case): Consider the discrete time-invariant fuzzy tracking system and fuzzy tracking controller described, respectively, by (1) and (2) with model-following target or moving target. If (A_i, B_i) is c.c., (A_i, C) is c.o. and $\lim_{\beta\to\infty}(\beta - 1)\rho(B_i S^{-1} B_i^t \hat{\bar{\pi}}_i(1/\beta^2)) < 2$, for all $i = 1, \ldots, r$, where $\rho(B_i S^{-1} B_i^t \hat{\bar{\pi}}_i(1/\beta^2))$ denotes the spectral radius of $B_i S^{-1} B_i^t \hat{\bar{\pi}}_i(1/\beta^2)$, then the optimal fuzzy tracking controller

$$\begin{aligned} u^*(t) &= \sum_{i=1}^{r} h_i(X^*(t)) \\ &\times \left(-S^{-1} B_i^t \bar{\pi}_i \left[I_n + B_i S^{-1} B_i^t \bar{\pi}_i\right]^{-1} A_i X^* + \bar{r}_i^{\rm ext}(t)\right) \end{aligned} \tag{87}$$

generates a closed-loop fuzzy tracking system in (79) with an infinite-gain margin, where $\bar{r}_i^{\rm ext}(t)$ is equal to (23) for the moving-target problem, or equal to $\bar{G}_i^2 Z(t)$ with $\bar{G}_i^2$ in Theorem 4 for the model-following-target issue. That is, the modified closed-loop fuzzy tracking system

$$\begin{aligned} X^*(t+1) = &\sum_{i=1}^{r} h_i(X^*(t)) \Big\{ \Big(I_n - \beta B_i S^{-1} B_i^t \hat{\bar{\pi}}_i(q) \\ &\times \left[I_n + B_i S^{-1} B_i^t \hat{\bar{\pi}}_i(q)\right]^{-1}\Big) A_i X^*(t) \Big\} \\ &+ \sum_{i=1}^{r} h_i(X^*(t)) B_i \bar{r}_i^{\rm ext}(t) \end{aligned} \tag{88}$$

is always stable for any $\beta \geq 1$, where $q = 1/\beta^2$ and $\hat{\bar{\pi}}_i(q)$ is the positive–semidefinite solution of the modified discrete-time Riccati equation in (85).

Proof: As we know, the stability of the modified nonlinear fuzzy tracking system in (88) is coincident with that of the following zero-input fuzzy system:

$$\begin{aligned} X^*(t+1) = &\sum_{i=1}^{r} h_i(X^*(t)) \Big\{ \Big(I_n - \beta B_i S^{-1} B_i^t \hat{\bar{\pi}}_i(q) \\ &\times \left[I_n + B_i S^{-1} B_i^t \hat{\bar{\pi}}_i(q)\right]^{-1}\Big) A_i X^*(t) \Big\}. \end{aligned} \tag{89}$$

Let $\hat{A}c_i, i = 1, \ldots, r$, denote the subsystem matrix of the fuzzy system in (89), i.e., $\hat{A}c_i \triangleq (I_n - \beta B_i S^{-1} B_i^t \hat{\bar{\pi}}_i(q)[I_n + B_i S^{-1} B_i^t \hat{\bar{\pi}}_i(q)]^{-1}) A_i$, and then, (89) can be rewritten as

$$X^*(t+1) = \sum_{i=1}^{r} h_i(X^*(t)) \hat{A}c_i X^*(t). \tag{90}$$

Notice that $\sum_{i=1}^{r} h_i(X^*(t)) = 1$. We shall show that each fuzzy subsystem in (89) or (90) is exponentially stable for any $\beta \geq 1$. Furthermore, demonstrate that the entire zero-input fuzzy systems in (89) or (90) are also exponentially stable for any $\beta \geq 1$; and then, prove that the modified closed-loop fuzzy tracking system in (88) is always stable for any $\beta \geq 1$.

1) Via Lemma 2 in the Appendix and Lemma 1, we know $\rho([I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)]^{-1} A_i) < 1$ and $\hat{\hat{\pi}}_i(q)$ is always available even in the case of infinite gain margin. We shall show $\rho\{(I_n - \beta B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(1/\beta^2)[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(1/\beta^2)]^{-1}) A_i\} < 1$ for all $\beta \geq 1$. Let (λ_1, v_1) denote the eigenpair of $[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)]^{-1} A_i$, i.e., $[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)]^{-1} A_i \cdot v_1 = \lambda_1 v_1$. By (85), we have $(\hat{\hat{\pi}}_i(q) - qL) v_1 = \lambda_1 A_i^t \hat{\hat{\pi}}_i(q) v_1$. Hence, we have $|\hat{\hat{\pi}}_i(q) - qL - \lambda_1 A_i^t \hat{\hat{\pi}}_i(q)| = 0$. Therefore, for all $\lambda_1 \in \sigma([I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)]^{-1} A_i), \lambda_1$ is also an eigenvalue of $(\hat{\hat{\pi}}_i(q) - qL)(A_i^t \hat{\hat{\pi}}_i(q))^{-1}$, which is equivalent to $A_i^t \hat{\hat{\pi}}_i(q)[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)]^{-1} A_i (A_i^t \hat{\hat{\pi}}_i(q))^{-1}$. To ensure this, $A_i^t \hat{\hat{\pi}}_i(q)$ commutes with $[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)]^{-1} A_i$ obviously, i.e.,

$$
\begin{aligned}
&A_i^t \hat{\hat{\pi}}_i(q) \cdot \left[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)\right]^{-1} A_i \\
&\quad = \left[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)\right]^{-1} A_i \cdot A_i^t \hat{\hat{\pi}}_i(q). \\
&A_i A_i^t \hat{\hat{\pi}}_i(q) \cdot \left[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)\right]^{-1} \cdot A_i \\
&\quad = A_i \cdot \left[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)\right]^{-1} \cdot A_i A_i^t \hat{\hat{\pi}}_i(q)
\end{aligned}
$$

and then, A_i commutes with $[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)]^{-1}$ or, more precisely, with $B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)$.

2) Accordingly, $[I_n + (1-\beta) B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)]$ commutes with $[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)]^{-1} A_i$. Recall that if A and B are commutative operators, then $\rho(AB) \leq \rho(A)\rho(B)$. Hence, we have

$$
\begin{aligned}
\rho&\left\{ \left(I_n - \beta B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q) \left[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)\right]^{-1}\right) A_i \right\} \\
&= \rho\left\{ \left[I_n + (1-\beta) B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)\right] \right. \\
&\quad \left. \times \left[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)\right]^{-1} A_i \right\} \\
&\leq \rho\left\{I_n + (1-\beta) B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)\right\} \\
&\quad \cdot \rho\left\{ \left[I_n + B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)\right]^{-1} A_i \right\} \\
&< \rho\left\{I_n - (\beta - 1) B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(q)\right\} < 1, \quad \forall q = 1/\beta^2 \geq 0
\end{aligned}
\tag{91}
$$

since $1 > 1 - (\beta - 1)\lambda(B_i S^{-1} B_i^t \hat{\hat{\pi}}_i(1/\beta^2)) > 1 - \lim_{\beta \to \infty} (\beta - 1)\lambda(B_i S^{-1} B_i^t \hat{\hat{\pi}}_\infty^i(1/\beta^2)) > -1$. So, the spectrum of the subsystem matrix, characterizing the dynamical behavior of each subsystem in (89) or (90), is always located in the unit disc of the complex space; in other words, each fuzzy subsystem in (89) or (90) is exponentially stable for any $\beta \geq 1$.

3) Then, we can use the mathematical induction method to demonstrate that there exist constant $m > 0$ and $\eta < 1$ such that the states of the entire fuzzy systems in (89) or (90) satisfy

$$
\|X^*(t)\| \leq m \|X_0\| \eta^{k-k_0}, \quad \forall k, k_0 \geq 0, \quad \forall X_0 \in \Re^n \tag{92}
$$

in other words, the zero-input fuzzy system in (89) is exponentially stable for any $\beta \geq 1$. Hence, the stability of the modified nonlinear fuzzy tracking system in (88) is ensured positively for all $\beta \geq 1$, and accordingly, our resultant closed-loop fuzzy tracking systems in (79) or (80) possess infinite-gain margin.

□

V. Numerical Simulations

In this section, a simple nonlinear mass-spring-damper mechanical system for continuous case, and an optimal backing up control of a computer simulated trunk-trailer for discrete-time case is adopted as the tracking system to illustrate the proposed optimal fuzzy tracking control scheme and its theoretic aspect.

A. Discrete-Time Tracking System

A computer simulated trunk-trailer system is used as a tracking system to track a moving target or a model-following target. The computer simulated truck-trailer physical system was described by Tanaka and Sano [29] as

$$
\begin{aligned}
x_1(t+1) &= (1 - v \cdot t'/L') x_1(t) + v \cdot t'/l \cdot u(t) \\
x_2(t+1) &= x_2(t) + v \cdot t'/L' \cdot x_1(t) \\
x_3(t+1) &= x_3(t) + v \cdot t' \cdot \sin(x_2(t) + v \cdot t'/2L' \cdot x_1(t))
\end{aligned}
$$

where l is the length of truck, L' is the length of trailer, t' is the sampling time, and v is the constant speed of the backward movement. Then, they used the following fuzzy model to represent the aforementioned mathematical model:

R^1 : If $z(t) \equiv x_2(t) + v \cdot t'/\{2L'\} \cdot x_1(t)$ is about 0, then $X(t+1) = A_1 X(t) + B_1 u(t)$

R^2 : If $z(t) \equiv x_2(t) + v \cdot t'/\{2L'\} \cdot x_1(t)$ is about π or $-\pi$, then $X(t+1) = A_2 X(t) + B_2 u(t)$,

and the system output is $Y(t) = CX(t)$ with $C = [0\ \ 0\ \ 1], l = 2.8, L' = 5.5, v = -1.0, t' = 2.0$ and $X(t) = [x_1(t)\ \ x_2(t)\ \ x_3(t)]^t$, where

$$
A_1 = \begin{bmatrix} 1.3636 & 0 & 0 \\ -0.3636 & 1.0 & 0 \\ 0.0120 & -2.0 & 1.0 \end{bmatrix}
$$

$$
A_2 = \begin{bmatrix} 1.3636 & 0 & 0 \\ -0.3636 & 1.0 & 0 \\ 0 & -0.0064 & 1.0 \end{bmatrix}
$$

$$
B_1 = B_2 = \begin{bmatrix} -0.7143 \\ 0 \\ 0 \end{bmatrix}.
$$

Grounding on this fuzzy system, we assume our fuzzy tracking controller as

R^1 : If $z(t) \equiv x_2(t) + v \cdot t'/\{2L'\} \cdot x_1(t)$ is about 0, then $u(t) = r_1(t)$

R^2 : If $z(t) \equiv x_2(t) + v \cdot t'/\{2L'\} \cdot x_1(t)$ is about π or $-\pi$, then $u(t) = r_2(t)$.

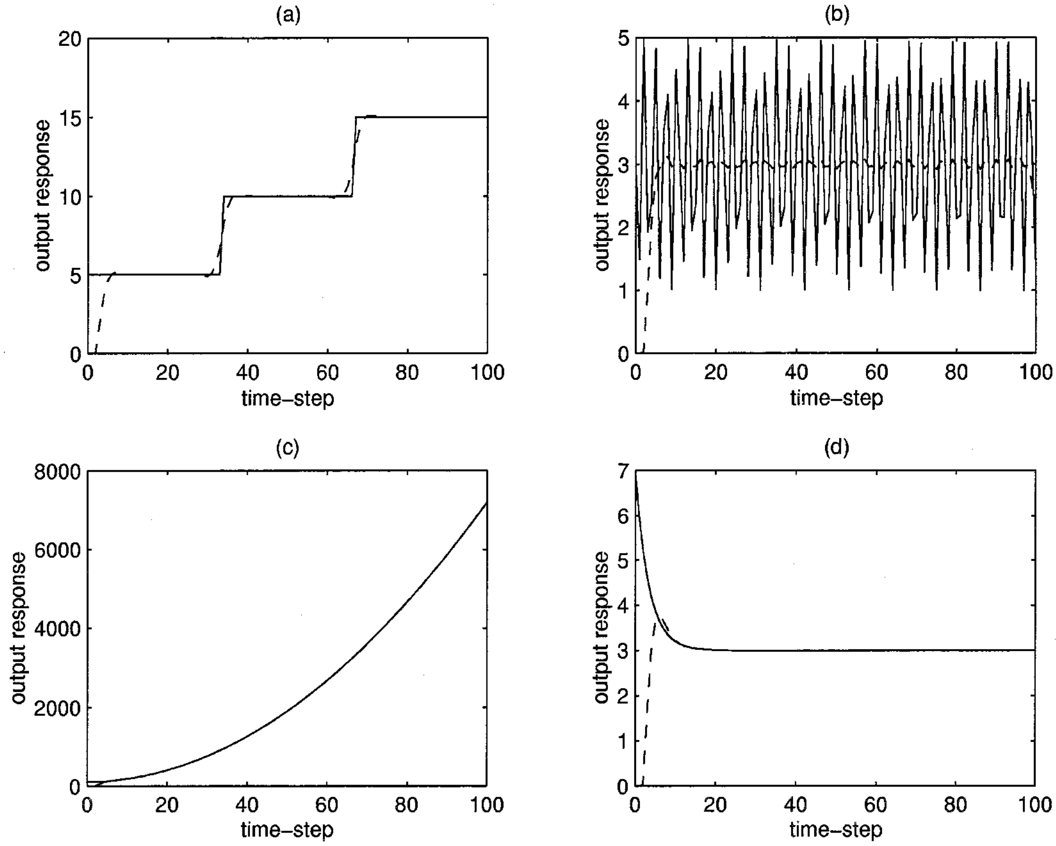

Fig. 1. Output responses (denoted by dashed line) of the discrete-time fuzzy tracking system with the designed *finite-horizon* optimal fuzzy tracking controllers in Section III-A for various *moving targets* (denoted by solid line), where (a) $Y^d(t)$ being a stepwise target. (b) $Y^d(t) = 3 + 2\sin 4k$. (c) $Y^d(t) = 100 + k + 0.7k^2$. (d) $Y^d(t) = 3 + 4e^{-0.3k}$.

With the chosen membership functions [29], the firing strengths are

$$\begin{aligned} h_1(X(t)) &= \alpha_1(t) = (1 - 1/(1 + \exp(-3(z(t) - \pi/2)))) \\ &\quad \cdot (1/(1 + \exp(-3(z(t) + \pi/2)))) \\ h_2(X(t)) &= \alpha_2(t) = 1 - \alpha_1(t) \end{aligned}$$

which, in this case, are also the normalized firing-strengths of the rules for the fuzzy system and controller. The performance index for the finite-horizon tracking problem is set as

$$J(u(\cdot)) = \sum_{t=0}^{100} [e^t(t)Le(t) + u^t(t)Su(t)] \tag{93}$$

and that for the infinite-horizon tracking problem is

$$J(u(\cdot)) = \sum_{t=0}^{\infty} [e^t(t)Le(t) + u^t(t)Su(t)]. \tag{94}$$

Now, we can design the optimal fuzzy tracking controllers for the trunk-trailer tracking system in both cases of moving target and modeling-following target by the proposed design scheme in Section III-A.

Though the fuzzy subsystem is unstable (the spectrum of system matrix $\sigma(A_i) = \{1, 1, 1.36\}, i = 1, 2$), it is time-invariant and well-behaved; i.e., the fuzzy subsystem is c.c. and c.o. ($\mathrm{rank}[\lambda I_3 - A_i B_i] = \mathrm{rank}[\lambda I_3 - A_i C] = 3$, for all $\lambda \in \sigma(A_i)$). Then, given $L_3 = I_3$ and $L_2 = 1$ in (3), the unique symmetric positive–semidefinite solution of the discrete-time algebraic Riccati equation in (25) or (46) is

$$\bar{\pi}_1 = \begin{bmatrix} 3.7628 & -11.2435 & 1.9448 \\ -11.2435 & 53.2545 & -10.9962 \\ 1.9448 & -10.9962 & 3.8906 \end{bmatrix}$$
$$\bar{\pi}_2 = \begin{bmatrix} 1.5003 & -1.2627 & 1.2329 \\ -1.2627 & 4.4386 & -4.3549 \\ 1.2329 & -4.3549 & 161.8765 \end{bmatrix}.$$

For the moving-target tracking problem, we can obtain, based on Theorems 1 and 2, the optimal trajectory of the closed-loop fuzzy tracking system with the designed optimal fuzzy tracking controller. The output responses of the resultant closed-loop fuzzy tracking system for various targets are shown in Fig. 1 for the finite-horizon problem. The output responses for the infinite-horizon problem are quite similar to those shown in Fig. 1. As for the model-following-target case, since each fuzzy subsystem is well-behaved as mentioned above, the optimal fuzzy tracking controller and the corresponding tracking trajectory can be obtained according to Theorems 3 and 4. Fig. 2 shows the finite-horizon optimal output responses of the resultant closed-loop fuzzy tracking system for the targets from the tracked model in (29) with various parameters $((F_1, F_2) = (1, 1), (1, 0.9), (0.85, 0.85)$ and $(0.45, 0.45))$. The output responses for the infinite-horizon problem under the same simulation situations are very close to those shown in Fig. 2. Our simulation results also show that the designed

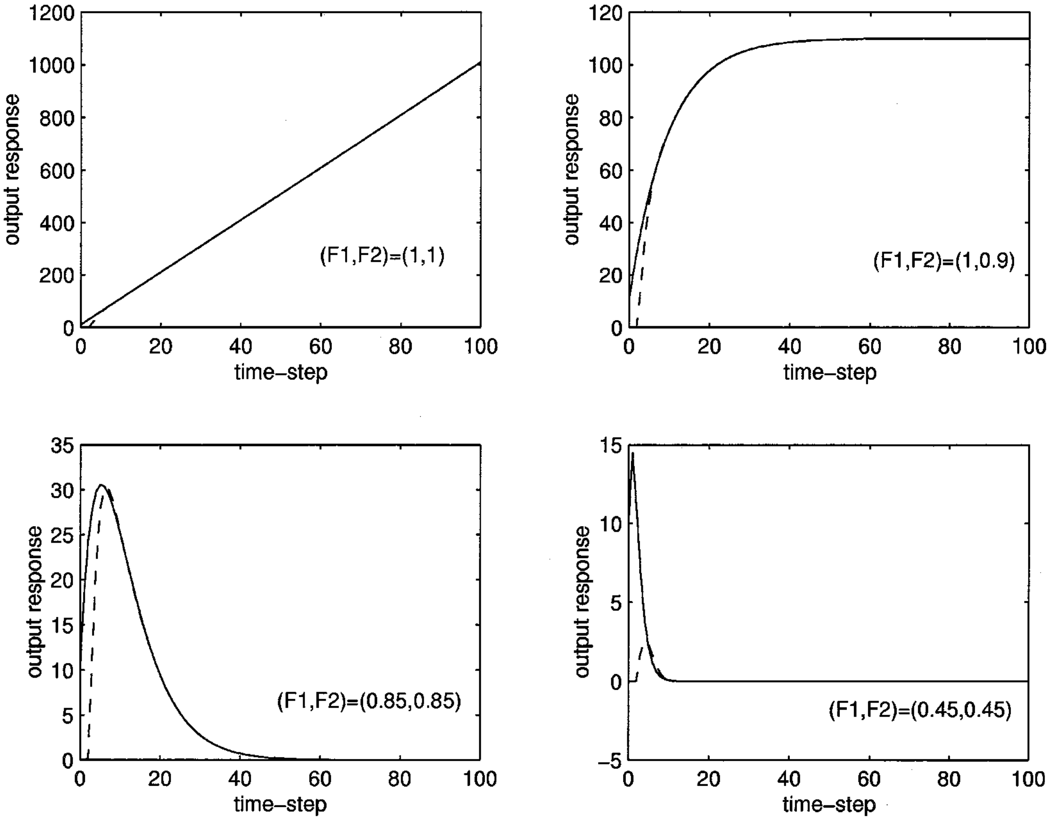

Fig. 2. Output responses (denoted by dashed line) of the discrete-time fuzzy tracking system with the designed *finite-horizon* optimal fuzzy tracking controllers in Section III-B for the targets (denoted by solid line) from the *tracked model* with four different sets of parameters: $(F_1, F_2) = (1, 1), (1, 0.9), (0.85, 0.85)$ and $(0.45, 0.45)$.

optimal fuzzy tracking controllers can efficiently push the simulated trunk-trailer system to trace the targets as soon as possible.

B. Continuous Tracking System

In this section, we adopt the following simple nonlinear mass-spring-damper mechanical system to track a moving or a model-following target:

$$M\ddot{x} + g(x, \dot{x}) + f(x) = \phi(\dot{x})u$$

where M is the mass and u is the force; $f(x)$ and $g(x, \dot{x})$ are the nonlinear or uncertain terms with respect to the spring and the damper, respectively; and $\phi(\dot{x})$ is the nonlinear term with respect to the input term. The tracking system can be rewritten as [3]

$$\ddot{x} = -0.1\dot{x}^3 - 0.02x - 0.67x^3 + u$$

where $x \in [-1.5\ \ 1.5]$ and $\dot{x} \in [-1.5\ \ 1.5]$. Accordingly, we model this nonlinear system as [3]

$$\begin{aligned} R^i:\ & \text{If } x(t) \text{ is } F_1^i \quad \text{and} \quad \dot{x}(t) \text{ is } F_2^i \\ & \text{then } \dot{X}(t) = A_i X(t) + B_i u(t), \quad i = 1, \ldots, 4 \end{aligned}$$

where $F_1^2 = F_1^1, F_1^4 = F_1^3, F_2^3 = F_2^1$ and $F_2^4 = F_2^2$, and the system output is $Y(t) = CX(t)$ with $C = [0\ \ 1]$ for every rule, where $X(t) = [\dot{x}(t)\ \ x(t)]^t, B_i = [1\ \ 0]^t, i = 1, \ldots, 4$

$$A_1 = \begin{bmatrix} 0 & -0.02 \\ 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} -0.225 & -0.02 \\ 1 & 0 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0 & -1.5275 \\ 1 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} -0.225 & -1.5275 \\ 1 & 0 \end{bmatrix}$$

and the membership functions of the fuzzy terms are $\mu_{F_1^1}(X(t)) = 1 - (x^2(t)/2.25), \mu_{F_1^3}(X(t)) = (x^2(t)/2.25)$, $\mu_{F_2^1}(X(t)) = 1 - (\dot{x}^2(t)/2.25)$, and $\mu_{F_2^2}(X(t)) = (\dot{x}^2(t)/2.25)$. The firing-strengths of the rules are $\alpha_i(X(t)) = \mu_{F_1^i}(X(t)) \cdot \mu_{F_2^i}(X(t)), i = 1, \ldots, 4$.

We then assume the desired fuzzy tracking controller is

$$\begin{aligned} R^i:\ & \text{If } x(t) \text{ is } F_1^i \quad \text{and} \quad \dot{x}(t) \text{ is } F_2^i, \\ & \qquad \text{then } u(t) = r_i(t), \qquad i = 1, \ldots, 4. \end{aligned}$$

Also, we set the performance index for the finite-horizon tracking problem as

$$J(u(\cdot)) = \int_0^{40} [e^t(t)Le(t) + u^t(t)Su(t)]\, dt \tag{95}$$

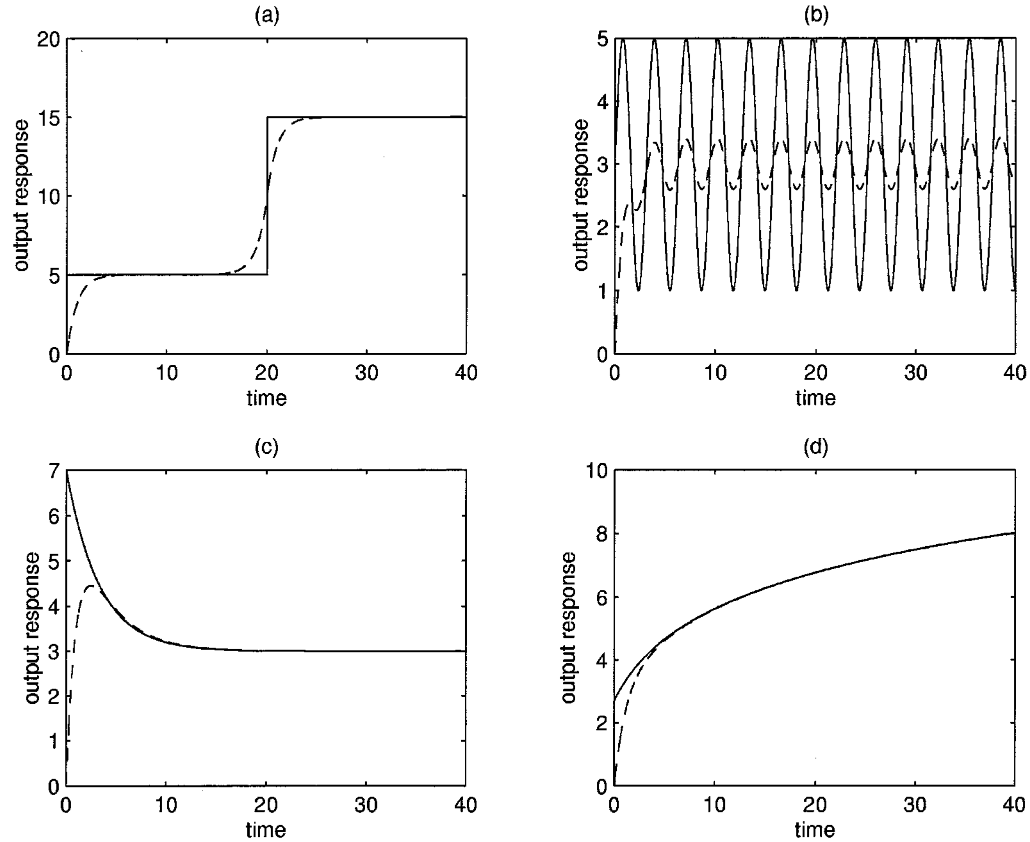

Fig. 3. Output responses (denoted by dashed line) of the continuous fuzzy tracking system with the designed *finite-horizon* optimal fuzzy tracking controllers in Section III.1 for various *moving targets* (denoted by solid line), where (a) $Y^d(t)$ being a stepwise target. (b) $Y^d(t) = 3 + 2\sin 2t$. (c) $Y^d(t) = 3 + 4e^{-0.3t}$. (d) $Y^d(t) = 0.5 + 2\log(3+t)$.

and that for the infinite-horizon tracking problem as

$$J(u(\cdot)) = \int_0^\infty [e^t(t)Le(t) + u^t(t)Su(t)]\,dt \tag{96}$$

where $e(t) = X(t) - X^d(t)$.

Now, we can design the optimal fuzzy tracking controllers for the mass-spring-damper tracking system to follow the desired moving target or model-following target by the proposed design scheme in Section III-B. Let $L_3 = I_2$ (identity matrix of dimension 2) and $L_2 = 1$ in (4). Since each fuzzy subsystem is well-behaved ($\text{rank}[A_i \;\; A_iB_i] = 2$ and $\text{rank}[C^t \;\; A_i^tC^t]^t = 2$ for $i = 1, \ldots, 4$), we have the unique symmetric positive–semidefinite solution of the algebraic Riccati equation in (57)

$$\bar{\pi}_1 = \begin{bmatrix} 0.0326 & 0.0316 \\ 0.0316 & 1.0311 \end{bmatrix} \quad \bar{\pi}_2 = \begin{bmatrix} 0.0324 & 0.0316 \\ 0.0316 & 1.0311 \end{bmatrix}$$
$$\bar{\pi}_3 = \begin{bmatrix} 0.0326 & 0.0301 \\ 0.0301 & 1.0309 \end{bmatrix} \quad \text{and} \quad \bar{\pi}_4 = \begin{bmatrix} 0.0323 & 0.0301 \\ 0.0301 & 1.0306 \end{bmatrix}.$$

For the moving-target tracking problem, we can obtain, based on Theorems 6 and 7, the optimal trajectory of the closed-loop fuzzy tracking system with the designed optimal fuzzy tracking controller. The output responses of the resultant closed-loop fuzzy tracking system for various moving targets (Y^d being a stepwise function, $Y^d(t) = 3 + 2\sin 2t, Y^d(t) = 3 + 4e^{-0.3t}$ or $Y^d(t) = 0.5 + 2\log(3+t)$) are shown in Fig. 3 for the finite-horizon problem. The corresponding output responses for the infinite-horizon problem are very close to those shown in Fig. 3.

As for the case of model-following target, since each fuzzy subsystem is well-behaved as mentioned above, the optimal fuzzy tracking controller and the corresponding tracking trajectory can be obtained according to Theorems 7 and 8. Fig. 4 shows the finite-horizon optimal output responses of the resultant closed-loop fuzzy tracking system for the targets from the tracked model in Problem 5 with various parameters $((F_1, F_2) = (-1, -0.2), (-0.2, -1), (-5, -1/30)$ and $(-1/30, -5))$. The corresponding output responses for the infinite-horizon problem are quite similar to those shown in Fig. 4. Our simulation results also show that the designed optimal fuzzy tracking controller can efficiently push the simulated trunk-trailer system to trace the targets in a short time.

VI. Conclusion

A sufficient condition for global optimization of fuzzy control was adopted in this paper. Grounded on this condition, a *nonlinear* global optimal quadratic tracking problem can be decomposed into a set of *linear* local optimal quadratic tracking problems, and then, the local-concept-approach design scheme of global optimal fuzzy tracking controllers for both continuous and discrete-time fuzzy systems was derived theoretically. Grounding on this efficient design scheme, several fascinating characteristics have been shown to exist in both kinds of resultant closed-loop time-invariant fuzzy tracking systems. Simu-

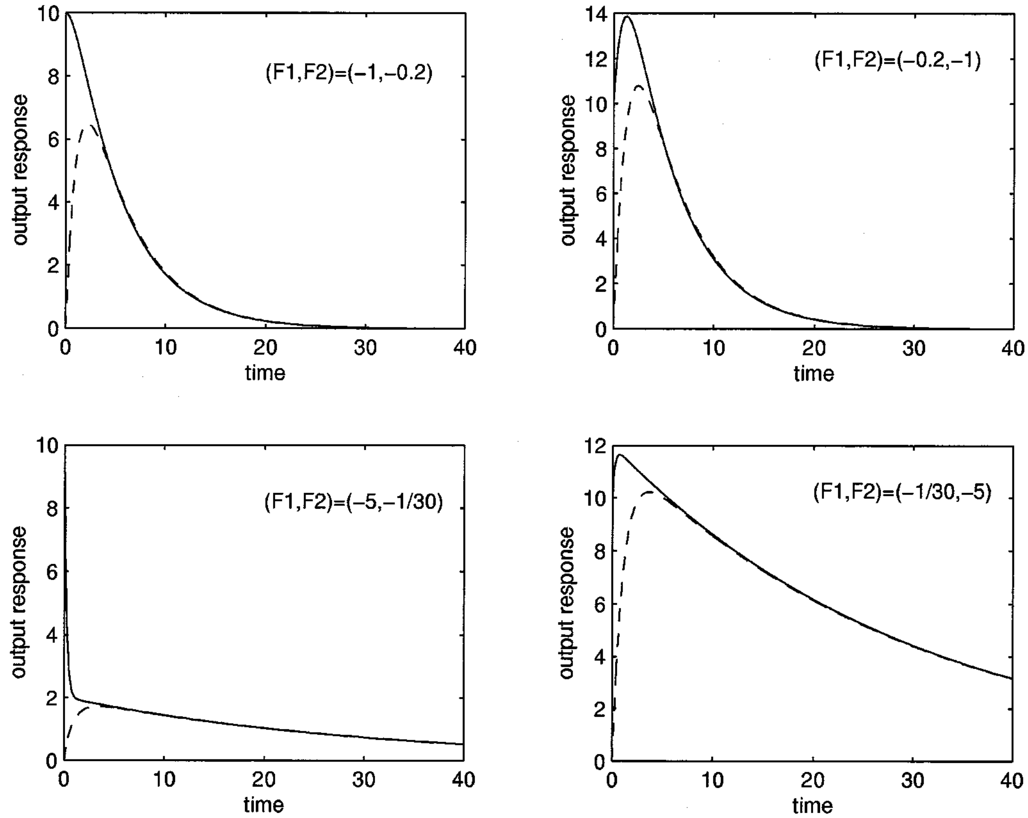

Fig. 4. Output responses (denoted by dashed line) of the continuous fuzzy tracking system with the designed *finite-horizon* optimal fuzzy tracking controllers in Section III-B for the targets (denoted by solid line) from the *tracked model* with four different sets of parameters: $(F_1, F_2) = (-1, -0.2), (-0.2, -1)$, $(-5, -1/30)$ and $(-1/30, -5)$.

lation results have manifested that the designed optimal fuzzy tracking controllers can effectively drive a fuzzy system to trace the target profile in a short time. In this paper, we consider only the noise-free tracking systems. In the future work, we shall develop theoretically sound stochastic fuzzy estimation or fuzzy filtering techniques based on the theorems developed in this paper to deal with the practical noise-contaminated systems.

APPENDIX A

Proof of Lemma 1:

1) We consider the optimal solution for minimizing

$$J_i(r_i(\cdot)) = \sum_{t=t_0}^{\infty} (qX^t(t)LX(t) + r_i^t(t)Sr_i(t)), \qquad \forall q > 0.$$

From Lemma 3, for any $q > 0$, the global minimizer is

$$\hat{r}_i^*(t) = -S^{-1}B_i^t\hat{\hat{\pi}}_i(q)\left[I_n + B_iS^{-1}B_i^t\hat{\hat{\pi}}_i(q)\right]^{-1} A_iX^*(t), \qquad \beta \ge 1, \qquad t \in [t_0, \infty)$$

where $\hat{\hat{\pi}}^i(q)$ is the symmetric positive–semidefinite solution of the modified discrete-time algebraic Riccati equation in (85), and the corresponding closed-loop system

$$\hat{X}^*(t+1) = \left[I_n + B_iS^{-1}B_i^t\hat{\hat{\pi}}_i(q)\right]^{-1} A_i\hat{X}^*(t), \qquad t \in [t_0, \infty) \tag{97}$$

is exponentially stable, i.e., $\rho\{[I_n + B_iS^{-1}B_i^t\hat{\hat{\pi}}_i(q)]^{-1} A_i\} < 1$.

2) Now, we shall check if the limit value of $\hat{\hat{\pi}}_i(q)$ exists and is equal to $\hat{\hat{\pi}}_i(0)$. For notation simplification, we use K_q and K_q^+ to denote $\hat{\hat{\pi}}_i(q)$ and $\hat{\hat{\pi}}_i(q+\epsilon)$, where $\hat{\hat{\pi}}_i(q+\epsilon)$ is the symmetric positive–semidefinite solution of the following equation:

$$K_q^+ = (q+\epsilon)L + A_i^t\left[I_n + K_q^+B_iS^{-1}B_i^t\right]^{-1} K_q^+A_i. \tag{98}$$

Define $\delta K_q \triangleq K_q^+ - K_q$, then

$$\delta K_q = \epsilon L + A_i^t\left\{K_q^+\left[I_n + B_iS^{-1}B_i^tK_q^+\right]^{-1} - \left[I_n + K_qB_iS^{-1}B_i^t\right]^{-1} K_q\right\} A_i. \tag{99}$$

Let $\bar{A}$ and $\bar{A}_+$ denote, respectively, $[I_n + B_iS^{-1}B_i^tK_q]^{-1}A_i$ and $[I_n + B_iS^{-1}B_i^tK_q^+]^{-1}A_i$, then

$$\delta K_q = \epsilon L + \bar{A}^t\delta K_q\bar{A}_+. \tag{100}$$

Let $Z_q = (\partial K_q/\partial q) = \lim_{\epsilon\to 0}(K_q^+ - K_q)/\epsilon$, then we obtain a discrete-time Lyapunov-like equation

$$Z_q = L + \bar{A}^tZ_q\bar{A}_+. \tag{101}$$

¿From (1), we know $\rho(\bar{A}), \rho(\bar{A}_+) < 1$, and accordingly, the unique solution is

$$Z_q = \sum_{t=t_0^i}^{\infty} (\bar{A}^k)^tL(\bar{A}_+^k) > 0. \tag{102}$$

In other words, $x^t Z_q x = (\partial/\partial q) x^t K_q x > 0$ for all $x \in \Re^n$. Hence, the function $x^t K_q x$ is monotonic decreasing as $q \to 0$, and bounded below by 0; i.e., $\lim_{q\to 0} x^t K_q x$ constantly exists for all $x \in \Re^n$. We can pick special x values to let $\lim_{q\to 0} K_q = K_0$, i.e., $\lim_{q\to 0} \hat{\hat{\pi}}_i(q) = \hat{\hat{\pi}}_i(0)$.

□

Lemma 2: For the discrete time-invariant fuzzy subsystem in (84), if (A_i, B_i) is stabilizable and (A_i, C) is detectable, then the following hold.

1) There exists an unique $n \times n$ symmetric positive–semidefinite solution, $\bar{\pi}_i$, of the discrete-time algebraic Riccati equation

$$K = L + A_i^t K \left[I_n + B_i S^{-1} B_i^t K\right]^{-1} A_i. \tag{103}$$

2) The asymptotically local optimal control law is

$$r_i^*(t) = -S^{-1} B_i^t \bar{\pi}_i \left[I_n + B_i S^{-1} B_i^t \bar{\pi}_i\right]^{-1} \times A_i X^*(t), \qquad t \in [t_0, \infty) \tag{104}$$

which minimizes the local quadratic functional $J_i(r_i(\cdot)) = \sum_{t=t_0}^{\infty} [X^t(t) L X(t) + r_i^t(t) S r_i(t)]$.

3) The optimal feedback fuzzy subsystem

$$X^*(t+1) = \left[I_n + B_i S^{-1} B_i^t \bar{\pi}_i\right]^{-1} A_i X^*(t), \quad t \in [t_0, \infty) \tag{105}$$

is asymptotically and exponentially stable.

Proof: This lemma is a counterpart of the classical discrete-time linear quadratic optimal control theorem. □

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