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Exact solution of inventory replenishment policy for a linear trend in demand – two-equation model

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Abstract

This study presents a model for obtaining the exact optimal solution of inventory replenishment policy problem. The proposed model, called the "two-equation model", includes two governing equations and a time-frequency algorithm: the first equation determines the optimal replenishment times for a specified number of replenishments; the second equation determines the optimal number of replenishments. The time-frequency algorithm includes two main procedures to solve the first and second equations. In contrast to many approximation approaches, this model exactly solves the first equation, a simultaneous non-linear equations system using a generalized matrix-based solver. This study also examines the classical no-shortage inventory replenishment policy for linear increasing and decreasing trend demand. According to these results, the solution given by the two-equation model is an exact solution.

Keywords: Inventory; Replenishment; Linear trend in demand

1. Introduction

The conventional approach used to determine the optimal re-order point and re-order quantity is the economic order quantity (EOQ) model. However, many constraints must be considered when applying the EOQ model. One of these constraints is the assumption of a constant demand rate, which implies that the EOQ model cannot be applied to a time-dependent demand rate issue. But in an actual situation, the demand rate of a product usually is function of time. Therefore, Donaldson and others [1–17] removed the constant demand rate constraint and initiated fundamental research into time-dependent demand rate issue.

Donaldson [1] was concerned to find the lowest total cost replenishment policy for an item with a linear increasing trend demand; however, shortage was not permitted. He adopted a complicated analytical approach to first determine the optimal number of replenishments, then he determined the replenishment times accordingly. His computational analysis is not simple: it develops tabular aids, but needs interpolation. Henery [2] extended Donaldson's results by examining a general log-concave

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demand rate function. Since Donaldson's analytical approach requires complex calculations, Silver and others [3–16] proposed various simpler approaches giving approximations. Unlike the approximation methods [3–16], Hariga [17] developed an analytical procedure that iteratively determines the optimal replenishment schedule for both increasing and decreasing demand patterns. Hariga [18] provides a review of these previous studies.

Donaldson's approach [1] provided the analytical solution for the no-shortage linear trend in demand replenishment policy. To the best of the authors' knowledge, from 1977 until now, Donaldson's approach has been applied only to cases of linear increasing trend demand without shortage [1], and with shortages [19,20]. Although Silver and others [3–16] provided various simpler approaches than Donaldson's, their approaches gave approximations, rather than exact optimal solutions. Even if these approximation approaches are applied to more complicated replenishment policy problems, their solutions are still only approximate, not exact. The iterative procedures proposed by Henery [2] and Hariga [17] could provide the optimal solution, but a recursive formula for solving the replenishment times was needed. Deriving such a recursive formula, however, complicates their approaches and is subjected to the type of demand rate function $f(t)$, particularly for more complex demand rate functions such as the non-linear demand rate function. Additionally, they used the boundary condition ($t_0 = 0$ or $t_n = H$) to judge convergence and generally the convergence criterion is not equal to zero. Therefore, their approaches may lead to the result that the terminating time of the last replenishment period is not equal to the planning horizon after convergence.

Much research into replenishment policy over the past two decades has continued to focus on fundamental methodology. This implies that a powerful tool is still needed for investigating realistic replenishment and production policies. Providing a powerful methodology for replenishment policy research is obviously the most important issue: this is also the purpose of this study. This study focuses on fundamental methodology research for inventory replenishment policy to provide a simple and accurate method for obtaining an exact optimal solution. This study also examines the classical no-shortage inventory replenishment policy for linear increasing and decreasing trends in demand. The computational results demonstrate that the solution given by the two-equation model can approach the accuracy of Donaldson's analytical solution [1]. To distinguish the numerical solution given by the two-equation model from Donaldson's analytical solution and the approximation methods, the numerical solution is referred to as an exact solution that approaches the accuracy of the analytical solution. The rest of this paper will explain this approach in detail.

2. Assumptions and notations

A deterministic inventory replenishment policy problem considered in this paper is based on the work of Donaldson [1] with the following assumptions and notations:

Assumptions:
(1) A single item is considered.
(2) Rate of demand increases or decreases linearly with time.
(3) Shortages and deterioration are not allowed.
(4) Planning horizon is finite.
(5) Lead-time of inventory replenishment is zero.
(6) Minimization of total cost is the objective.

Notations:

W	Total cost including replenishment cost and holding cost.
$W^*(n)$	Lowest total cost for n replenishments.
c_1	Replenishment cost per order.
c_2	Inventory holding cost per unit per year.
D	Total demand during the planning horizon.
H	Planning horizon.
$f(t)$	Demand rate function $f(t) = a + bt$, $f(t) \geqslant 0$.
a	Demand rate at $t = 0$.
b	Rate of demand change per unit of time, $b > 0$ for linear increasing trend and $b < 0$ for linear decreasing trend.
α	Ratio of a to b, $\alpha = a/b$.

$Q_i(t)$	Inventory function of ith replenishment period, $Q_i(t)=\int_t^{t_i} f(u)\,\mathrm{d}u$, $t_{i-1}\leqslant t\leqslant t_i$.
t_i	Terminating time of ith replenishment period or starting time of $(i+1)$th replenishment period.
t_m	$t_i\leqslant t_m\leqslant t_{i+1}, f(t_m)=\dfrac{\int_{t_i}^{t_{i+1}} f(u)\,\mathrm{d}u}{t_{i+1}-t_i}$
A_i, D_i, B_i	Coefficients of simultaneous equations system of variables t_{i-1}, t_i, t_{i+1}, respectively.
n	Number of replenishments.
n^*	Number of replenishments obtained by EOQ model.
n_0	Optimal number of replenishments.
$[x]$	Gauss function of x.
s	Decreasing direction index for total cost.
EPS	Convergence criterion of t_i's iteration.
k	Number index of t_i's iteration.

3. Two-equation model

In this study, the total cost includes the replenishment cost and holding cost, and can be expressed as follows:

$$W=nc_1+c_2\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} Q_i(u)\,\mathrm{d}u. \tag{1}$$

In Eq. (1),

$$Q_i(t)=\int_t^{t_i} f(u)\,\mathrm{d}u. \tag{2}$$

In Eq. (2), $Q_i(t)$ represents the inventory function after the ith replenishment and prior to the $(i+1)$th replenishment. In the classical problem of no-shortage inventory replenishment for a linear trend, $Q_i(t)$ is determined using the following differential equation and condition:

$$\frac{\mathrm{d}Q_i(t)}{\mathrm{d}t}=-f(t),\quad t_{i-1}\leqslant t\leqslant t_i, \tag{3}$$

$$Q_i(t_i)=0. \tag{4}$$

From (1), the total cost W is a function of replenishment times t_i $(i=1,2,\ldots,n-1)$ and number of replenishments n for a given c_1, c_2 and $f(t)$. Since there are two kinds of variable in (1), i.e. t_i and n, two equations are needed to obtain the lowest total cost for the replenishment policy problem. One of these two equations determines the optimal replenishment times for a specified number of replenishments; the other determines the optimal number of replenishments.

The two-equation model proposed herein includes two main procedures to solve the inventory replenishment policy problem. The first procedure determines the optimal replenishment times for a specified number of replenishments; the second procedure determines the optimal number of replenishments. Each procedure has one equation. The two procedures and relative equations for solving the problem of no-shortage inventory replenishment with a linear trend demand are presented in the next subsections.

3.1. *First procedure*: *Determine the optimal replenishment times under a specified number of replenishments n*

To obtain the optimal replenishment times under a specified number of replenishments n, the first and second derivative of W against t_i are

$$\frac{\partial W}{\partial t_i}=c_2\left[(t_i-t_{i-1})f(t_i)-\int_{t_i}^{t_{i+1}} f(u)\,\mathrm{d}u\right],\quad i=1,2,\ldots,n-1, \tag{5}$$

$$\frac{\partial^2 W}{\partial t_i^2}=c_2\left[2f(t_i)+(t_i-t_{i-1})\frac{\mathrm{d}f(t_i)}{\mathrm{d}t_i}\right]. \tag{6}$$

Let $\partial W/\partial t_i=0$; then

$$\int_{t_i}^{t_{i+1}} f(u)\mathrm{d}u=(t_i-t_{i-1})f(t_i),\quad i=1,2,\ldots,n-1. \tag{7}$$

The linear increasing and decreasing trend demand functions $f(t)=a+bt$ are both log-concave, i.e. the derivative of $\log(f(t))$ is a decreasing function of time. If inequality (4) in Henery [2] appears as (8), then Eq. (6) becomes Eq. (9).

$$f(t_i)+(t_i-t_{i-1})\frac{\mathrm{d}f(t_i)}{\mathrm{d}t_i}>f(t_{i+1}), \tag{8}$$

$$\frac{\partial^2 W}{\partial t_i^2}=c_2\left[f(t_i)+f(t_i)+(t_i-t_{i-1})\frac{\mathrm{d}f(t_i)}{\mathrm{d}t_i}\right]>c_2[f(t_i)+f(t_{i+1})]>0. \tag{9}$$

Because $\partial^2 W/\partial t_i^2 > c_2[f(t_i) + f(t_{i+1})] > 0$,

$\partial^2 W/\partial t_{i-1}\partial t_i = -c_2 f(t_i)$,

$\partial^2 W/\partial t_i \partial t_{i+1} = -c_2 f(t_{i+1})$,

and for $j \neq i-1, i, i+1$,

$\partial^2 W/\partial t_i \partial t_j = 0$,

then we can prove that the Hessian matrix H_M associated with W is positive definite by taking any non-zero vector Z, and proving that $Z^T H_M Z > 0$ is always true. Therefore, under a specified number of replenishments n, the replenishment times t_i ($i = 1, 2, \ldots, n-1$), which are satisfied with Eq. (7) and $t_0 = 0$, $t_n = H$ will be unique and can then produce the lowest total cost $W^*(n)$, as obtained from Eq. (1). Eq. (7) is the governing equation for determining replenishment times, and is the, so-called, first equation in the two-equation model.

Eq. (7) is a simultaneous non-linear equations system with t_i as variables. This feature has not been pointed out by past research. Therefore, until now, there has been no research that solved (7) using the general numerical techniques normally used for solving a simultaneous non-linear equations system. Even in the approaches presented by Henery [2] and Hariga [17], they used a recursive formula to solve (7).

3.2. *Second procedure*: *Determine the optimal number of replenishments n_0*

Optimal replenishment times t_i and lowest total cost $W^*(n)$ under any number of replenishments n, can be obtained using the first procedure. If we can obtain the optimal number of replenishments n_0, then we can obtain the optimal replenishment times and the lowest total cost for the replenishment policy problem using the first procedure. To obtain the optimal number of replenishments, the second procedure is carried out from an appropriate n until the lowest total cost $W^*(n)$ stops decreasing and starts to increase, i.e. if Eq. (10) is satisfied, the optimal number of replenishments $n_0 = n - 1$:

$$\mathrm{Min}\{W^*(n-2), W^*(n-1), W^*(n)\} = W^*(n-1). \tag{10}$$

Eq. (10) is the governing equation to determine the optimal number of replenishments and is the so-called second equation in the two-equation model. Clearly, if we can solve Eq. (7) exactly, and calculate the lowest total cost $W^*(n)$ under the number of replenishments n from an appropriate n until Eq. (10) is satisfied, we can obtain the exact optimal solution of the replenishment policy problem.

The classical no-shortage inventory replenishment policy with a linear trend in demand is governed by the first and second equations stated above. If we can solve the first equation exactly, under any number of replenishments, we can easily obtain the exact optimal solution for the replenishment policy problem. This is the basic concept of the two-equation model. For more realistic situations such as the replenishment policy that considers inventory with deterioration, allowable shortages, etc., the first equation will be more complex than Eq. (7). The concept of the two-equation model and the proposed time-frequency algorithm that is listed in the next section, are still adequate for obtaining the optimal solution in such situations.

4. Numerical technique and algorithm

In a general case, Eq. (7) is a simultaneous non-linear equations system; it is also difficult to solve. Therefore, the related investigations [3–16] did not solve (7) directly, but used various approximation methods. In fact, we can solve Eq. (7) by using a simple numerical method. For example, substituting $f(t) = a + bt$ into Eq. (7), we can obtain Eq. (11). Eq. (11) is a simultaneous non-linear equations system of variables t_i ($i = 1, 2, \ldots, n-1$), and can be solved directly using the general numerical techniques used for solving a simultaneous non-linear equations system. In an iterative process for solving t_i, the value of t_{i-1} and t_{i+1} in Eq. (11) can be substituted with the last respective iterative value. Treating Eq. (11) thus, it can be transformed into a quadratic equation of t_i under each iteration; we can then solve t_i by taking the positive root of the quadratic equation directly, as in Eq. (12):

$$3t_i^2 + (4\alpha - 2t_{i-1})t_i - (2\alpha t_{i-1} + 2\alpha t_{i+1} + t_{i+1}^2) = 0, \quad i = 1, 2, \ldots, n-1, \tag{11}$$

$$t_i = -\alpha + \frac{(t_{i-1}+\alpha) + \sqrt{(t_{i-1}+\alpha)^2 + 3(t_{i+1}+\alpha)^2}}{3},$$
$$i = 1, 2, \ldots, n-1. \quad (12)$$

The numerical method stated as above is very simple but subjected to the type of demand rate function $f(t)$. Therefore, this study first proposes a matrix-based solver to solve (7) directly. The first step is to linearize Eq. (7), i.e. according to the mean value theorem of integration, the left-hand side of Eq. (7) can be written as (13):

$$\int_{t_i}^{t_{i+1}} f(u)\,\mathrm{d}u = (t_{i+1} - t_i) f(t_m), \quad t_i \leqslant t_m \leqslant t_{i+1}. \quad (13)$$

By substituting Eq. (13) into Eq. (7), Eq. (7) can be simplified as Eq. (14), that is a simultaneous linear equations system:

$$A_i t_{i-1} + D_i t_i + B_i t_{i+1} = 0, \quad i = 1, 2, \ldots, n-1. \quad (14)$$

In Eq. (14), the coefficients of variable t_{i-1}, t_i, t_{i+1} are

$$A_i = -f(t_i), \quad B_i = -f(t_m) = \frac{\int_{t_i}^{t_{i+1}} f(u)\,\mathrm{d}u}{t_{i+1} - t_i},$$
$$D_i = -A_i - B_i. \quad (15)$$

The combination of Eq. (14) with boundary conditions $t_0 = 0$, $t_n = H$ can be formulated as matrix form Eq. (16):

$$\begin{bmatrix} 1 & & & & & & \\ A_1 & D_1 & B_1 & & & & \\ & & \cdots & & & & \\ & & A_i & D_i & B_i & & \\ & & & & \cdots & & \\ & & & & A_{n-1} & D_{n-1} & B_{n-1} \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ t_1 \\ .. \\ t_i \\ .. \\ t_{n-1} \\ t_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ .. \\ 0 \\ .. \\ 0 \\ H \end{bmatrix}. \quad (16)$$

Because the coefficients in Eq. (16) are relative to replenishment times t_i, Eq. (16) must be solved by the iterative method. In the iteration process, the value of t_i and t_{i+1} in Eq. (15) can be substituted with the last iterative value, respectively. In addition, the integration of coefficient B_i can be calculated directly or by using a numerical integration method if necessary.

Eq. (16) is a simultaneous equations system with tri-diagonal band matrix, t_i, which can be easily solved during each iteration. The coefficients A_i, B_i, D_i in Eq. (15) can be calculated for any type of demand rate function $f(t)$. Even for the more complex first equation, a new expression of coefficients A_i, B_i, D_i like Eq. (15) can be found, and Eq. (16) still can be used to solve t_i.

The optimal replenishment times t_i and lowest total cost $W^*(n)$ under any number of replenishments n can be calculated by using the first procedure. The second procedure can be carried out from $n = 1$ until the lowest total cost $W^*(n)$ does not further decrease and starts to increase. Obviously, precisely estimating the optimal number of replenishments as the starting number of replenishments for the second procedure would allow us to markedly reduce the total computation time. Herein, we use an EOQ-based estimator to estimate the optimal number of replenishments, i.e. using $[n^*]$ as the starting number of replenishments for the second procedure. The notation $[x]$ represents the Gauss function of x and n^* is given by

$$n^* = \sqrt{\frac{c_2 DH}{2c_1}}. \quad (17)$$

Solving the first and second equations produces the optimal replenishment policy. This study proposes a time-frequency algorithm that includes two main procedures to solve the first and second equations and thus obtain the optimal replenishment policy. The time-frequency algorithm is as follows:

Step 1. Set $n = [n^*]$, $s = 1$.

Step 2. Calculate $t_i = iH/n, i = 0, 1, 2, \ldots, n$.

Step 3. Solve $t_i, i = 1, 2, \ldots, n-1$, from (16) by iteration, and calculate $W^*(n)$ from (1). If $n = [n^*]$ then set $s = s$, $n = n + s$ and go to Step 2 else go to step 4.

Step 4. If $\text{Min}\{W^*(n-2s), W^*(n-s), W^*(n)\} = W^*(n-s)$ then go to Step 6 else go to Step 5.

Step 5. If $\text{Min}\{W^*(n-s), W^*(n)\} = W^*(n)$ then set $s = s$, $n = n + s$ and go to Step 2. If $\text{Min}\{W^*(n-s), W^*(n)\} = W^*(n-s)$ then set $s = -s$, $n = n + 2s$ and go to Step 2.

Step 6. Optimal number of replenishments $n_0 = n - s$, optimal replenishment times t_i, $i = 0, 1, \ldots, n_0-1$.

In the above algorithm, the s value is the decreasing direction index of $W^*(n)$; $s = +1$ indicates that $W^*(n)$ decreases as n increases. Moreover, $s = -1$ indicates that $W^*(n)$ decreases as n decreases.

In Step 3 of the algorithm, the criterion for t_i's convergence EPS must be specified according to the requirement for accuracy. Accordingly, the judgement of convergence is

$$\text{Max}\left|\frac{t_i^{k+1} - t_i^k}{H}\right| \leqslant \text{EPS}, \quad i = 1, 2, \ldots, n-1. \tag{18}$$

In inequality (18), subscripts k and $k+1$ represent the number index of iteration. If inequality (18) is satisfied after iterating $k+1$ times, the iteration has converged.

It is necessary to explain that the value of EPS will affect the number of effective decimal places of computed optimal replenishment times. Theoretically, when the value of EPS approaches 0, the solution given by the proposed approach will be identical to the analytical solution. However, letting EPS approach zero is unnecessary; that would only increase computation time and, besides, there are only a finite number of decimal places in the optimal solution needed in practical applications, even when using the analytical solution.

5. Model comparison

This study examines the classical no-shortage inventory replenishment policy using the 12 linear increasing demand numerical examples used by Amrani and Rand and others [12,13], and with the linear decreasing demand numerical example used by Phelps and Hariga [7,17]. Table 1 presents a set of 12 sample problems, which is the same as in [12,13]. Moreover, Table 2 summarizes the calculated results.

Table 1
The parameters of the 12 linear increasing trend sample problems

No.	a	b	H	c_1	c_2
1	0	900	1	9	2
2	0	900	2	9	2
3	0	100	4	100	2
4	0	1600	3	42	0.56
5	6	1	11	30	1
6	6	1	11	50	1
7	6	1	11	60	1
8	6	1	11	70	1
9	6	1	11	90	1
10	100	150	1	30	2
11	100	150	1.5	30	2
12	100	150	2	30	2

Table 2 clearly demonstrates that the optimal total cost of replenishment policy, calculated using the approach proposed herein, are identical to the solutions using Donaldson's analytical approach [1]. Obviously, the solution given by the proposed method is superior to the approximation methods [3–16].

For a more detailed examination, Tables 3 and 4 compare the optimal replenishment times using Donaldson's method, and the first sample problem in Table 1 using the approach proposed herein. In this comparison, EPS of the two-equation model are taken as 5E-6 and 5E-14 for Tables 3 and 4, respectively.

Tables 3 and 4 reveal that the smaller the EPS, the closer are the optimal replenishment times obtained using the approach proposed herein and Donaldson's analytical solution. For example, for EPS = 5E-6, the optimal replenishment times obtained by this study are correct to six decimal places; for EPS = 5E-14, the optimal replenishment times obtained by this study are correct to 14 decimal places. The deviations between Donaldson's analytical solution and the solution using the two-

Table 2
The results for the sample problems given in Table 1

No.	Donaldson [1][a]	Percentage increase above the optimal						
		Silver [3]	Goyal and Gommes [4]	Ritchie [5]	Tsado [6]	Amrani and Rand [12]	Yang and Rand [13]	Proposed
1	62.63	1.2555	0.1209	4.6229	4.6673	0.1029	0.0608	0.0000
2	172.89	2.1104	1.6275	0.5959	1.3381	0.5959	0.0463	0.0000
3	561.30	5.3479	1.3820	0.2016	10.0957	0.2016	0.0155	0.0000
4	1744.94	1.5876	0.6230	0.0357	5.4091	0.0356	0.0029	0.0000
5	291.21	7.6566	0.2085	0.0258	1.0968	0.0258	0.0019	0.0000
6	378.05	11.7907	0.5622	0.1748	2.0514	0.1748	0.0043	0.0000
7	418.05	0.5930	3.4220	2.2302	6.6906	0.5930	0.0307	0.0000
8	450.84	4.2506	9.3362	7.3633	14.3612	4.2506	0.0907	0.0000
9	510.84	14.4429	0.3953	0.0690	2.0304	0.0690	0.0010	0.0000
10	75.21	0.8682	4.0666	2.7269	7.8399	0.8682	0.0176	0.0000
11	121.23	5.4776	11.3779	9.1443	0.4576	0.4576	0.0100	0.0000
12	173.82	6.0923	0.3529	0.0529	1.7656	0.0529	0.0005	0.0000
Overall	4961.01	4.8847	2.0181	1.2601	5.6409	0.5418	0.0171	0.0000

[a]Normalized total cost (= total cost/c_2) has been round off to two decimal places for comparison reason.

Table 3
Optimal replenishment times comparison for problem 1 (EPS = 5E-6)

Order no.	Donaldson [1] (analytical solution)	Proposed	
		Replenishment times	Error of convergence
1	0.0000000	0.0000000	0.0000000
2	0.2300529	0.2300526	0.0000003
3	0.3984633	0.3984629	0.0000004
4	0.5412797	0.5412793	0.0000004
5	0.6690224	0.6690220	0.0000003
6	0.7864581	0.7864579	0.0000003
7	0.8962326	0.8962325	0.0000001
Cost[a]	62.6302052	62.6302052	0.0000000

[a]Normalized total cost (= total cost/c_2).

equation model are listed in the error of convergence column. The error comes from the criterion that t_i's convergence EPS does not equal 0, not from an approximation assumption. Clearly, the solution accuracy of the two-equation model approaches the accuracy of the analytical solution, unlike the solutions provided using the approximation approaches [3–16]. Obviously, when it is difficult or impossible to obtain the analytical solution, the solution obtained may be accurate enough to stand for the analytical solution under an appropriate requirement for accuracy. This accounts for why the two-equation model solution can be called an exact solution.

Table 5 lists the computation sequence, total number of computations for the first procedure and optimal number of replenishments for the set of sample problems given in Table 1.

According to this table, the total number of computations for the first procedure needed by the approach proposed herein is markedly reduced when applying the EOQ-based estimator in the second procedure. The second procedure has a good starting number of replenishments, thereby allowing us to avoid much unnecessary computation. For example, as found in the second sample problem in Table 1, the EOQ-based estimator provides a starting number of replenishments $[n^*] = 20$; the sequence of computation is 20-21-19, and the optimal number of replenishments $n_0 = 20$ is obtained after three computations for the first procedure. In fact, the 12 sample problems all require only three computations for

Table 4
Optimal replenishment times comparison for problem 1 (EPS = 5E-14)

Order no.	Donaldson [1] (analytical solution)	Proposed	
		Replenishment times	Error of convergence
1	0.000000000000000	0.000000000000000	0.000000000000000
2	0.230052877859349	0.230052877859346	0.000000000000003
3	0.398463272879829	0.398463272879826	0.000000000000003
4	0.541279682057064	0.541279682057061	0.000000000000003
5	0.669022372803606	0.669022372803602	0.000000000000004
6	0.786458118055185	0.786458118055183	0.000000000000002
7	0.896232649405742	0.896232649405740	0.000000000000002
Cost[a]	62.630205178277500	62.630205178277500	0.000000000000000

[a]Normalized total cost (= total cost/c_2).

Table 5
Computation sequence and optimal number of replenishments for the problems given in Table 1

No.	Computation sequence	Total number of computations	n_0
1	7-8-6	3	7
2	20-21-19	3	20
3	5-6-7	3	6
4	12-13-11	3	12
5	4-5-6	3	5
6	3-4-5	3	4
7	3-4-5	3	4
8	3-4-2	3	3
9	2-3-4	3	3
10	2-3-4	3	3
11	3-4-5	3	4
12	5-6-7	3	6

Table 6
The results for the 13th sample problem

Order no.	Model		
	Phelps [7]	Hariga [17]	Proposed
	Replenishment times		
1	0.000	0.000	0.000
2	2.390	2.174	2.173
Normalized total cost[a]	711.937610	708.811915	708.811835

[a]Normalized total cost are calculated by using the replenishment times listed in the table, the replenishment times and total cost have been round off to three and six decimal places respectively for comparison reason (the convergent criterion of Hariga is 1E-6).

the first procedure, which is the ideal situation. This observation implies that the EOQ-estimator is fairly accurate for the optimal number of replenishments.

The next sample problem is a linear decreasing demand case used by Phelps [7] and Hariga [17]. Until now, Donaldson's analytical approach [1] and Henery's iterative approach [2] had not been applied to such a demand pattern. Table 6 summarizes the results using the approaches presented by Phelps [7], Hariga [17], and the proposed model.

Thirteenth sample problem. $a = 100$, $b = -10$, $c_1 = 30$, $c_2 = 0.2$, $H = 4.78$.

The replenishment times and total cost of the Hariga [17] and the proposed model are almost identical, but the total cost obtained by the two-equation model is the lowest in Table 6. It is obvious that the two-equation model can be applied to both increasing and decreasing demand patterns using the same Eq. (16), but Hariga's approach [17] needs two different recursive formulas to treat increasing and decreasing demand patterns.

6. Conclusions

This study proposes a "two-equation model" to solve the classical no-shortage inventory replenishment policy for linear increasing and decreasing demand. The problem is governed by the first and second equations of the two-equation model. This study demonstrates that the first equation of the two-equation model is a simultaneous non-linear equations system of replenishment times, and can be solved by a general numerical method for solving simultaneous non-linear equations system. This study also provides a generalized matrix-based solver for the first equation. Additionally, this study demonstrates that an EOQ-based estimator is fairly accurate for the optimal number of replenishments. The correctness and accuracy of the two-equation model are verified using 13 comparative examples.

Comparing the two-equation model with other optimal solution approaches, we summarize the advantages of two-equation model as follows:

(1) Deriving a recursive formula for solving t_i is not necessary: this feature reveals that the two-equation model is easy to apply.
(2) The type of demand rate function $f(t)$ is not important for the two-equation model. For a more complex demand rate function $f(t)$, the significant work for the two-equation model is to compute $B_i = -\int_{t_i}^{t_{i+1}} f(u)\,\mathrm{d}u/(t_{i+1} - t_i)$ in Eq. (15).
(3) We can use the general numerical method used to solve non linear equations system to solve Eq. (11), or use the general numerical method used to solve the linear equations system to solve the linearized Eq. (14). Therefore, a special numerical method is not necessary for the two-equation model.
(4) Since the boundary conditions $t_0 = 0$ and $t_n = H$ are imposed in the solving procedure, therefore the solution obtained by the two-equation model is the exact optimal.

Obviously, the two-equation model provides a novel means of investigating the lowest total cost replenishment policy and also production policy. The two-equation model focuses on fundamental methodology research into inventory replenishment policy. Its concept is intuitive and its algorithm is very simple. The two-equation model could potentially apply to more complex replenishment policy problems such as problems that take deterioration, allowable shortage, etc., into consideration. Our laboratory is currently studying how to apply the approach to such complicated replenishment and production policies.

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