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Nuclear Physics B 625 (2002) 327–344

NUCLEAR PHYSICS B

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$\mathbf{CP}^n$ model on fuzzy sphere

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Received 22 May 2001; accepted 14 January 2002

Abstract

We construct the $\mathbf{CP}^n$ model on fuzzy sphere. The Bogomolny bound is saturated by (anti-)self-dual solitons and the general solutions of BPS equation are constructed. The dimension of moduli space describing the BPS solution on fuzzy sphere is exactly the same as that of the commutative sphere or the (noncommutative) plane. We show that in the soliton backgrounds, the number of zero modes of Dirac operator on fuzzy sphere, Atiyah–Singer index, is exactly given by the topological charge of the background solitons.

PACS: 04.60.Kz; 12.10.-g

1. Introduction

A noncommutative space is obtained by quantizing a given space with its symplectic structure, treating it as a phase space. Also field theories can be formulated on a noncommutative space. Noncommutative field theory means that fields are defined as functions over noncommutative spaces. At the algebraic level, the fields become operators acting on a Hilbert space as a representation space of the noncommutative space. Since the noncommutative space resembles a quantized phase space, the idea of localization in ordinary field theory is lost. The notion of a point is replaced by that of a state in representation space.

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0550-3213/02/$ – see front matter
PII: S0550-3213(02)00028-7

Quantum field theory on a noncommutative space has been proved to be useful in understanding various physical phenomena, like as various limits of M(atrix) theory compactification [1,2], low energy effective field theory of D-branes with constant Neveu–Schwarz B-field background [3,4], and quantum Hall effect [5]. Although noncommutative field theories are nonlocal, they appear to be highly constrained deformation of local field theory. Thus it may help understanding nonlocality at short distances in quantum gravity.

The fuzzy sphere is constructed by introducing a cut-off parameter N for angular momentum of the spherical harmonics: $\{\widehat{Y}_{lm}; l \leqslant N\}$ [6]. Thus the number of independent functions is $\sum_{l=0}^{N}(2l+1) = (N+1)^2$. In order for this set of functions to form a closed algebra, the functions are replaced by $(N+1) \times (N+1)$ hermitian matrices and then the algebra on the fuzzy sphere is closed [7]. Consequently, the algebra on the fuzzy sphere becomes noncommutative. The commutative sphere is recovered for $N \to \infty$. One of the attractive features of the fuzzy sphere is that it is covariant with respect to $SO(3)$ like the commutative sphere.

Recently, it has been shown that the fuzzy sphere is a natural candidate for the quantum geometry due to stringy effects in the AdS/CFT duality [8] and that the field theories on fuzzy sphere appear naturally from D-brane world-volume theory [9,10] and matrix theory with some backgrounds [11]. Interestingly, it was argued based on the $SU(2)$ WZW model that the RR charges of spherical D2-branes are only defined modulo some integer [10], which are $U(1)$ charges defined on D2-brane world-volume (fuzzy sphere). This was confirmed using K-theory calculation in [12]. Many efforts to construct field theories on the fuzzy sphere were also pursued in [13–15].

In this paper we construct the $\mathbf{CP}^n$ model on fuzzy sphere. In Section 2, the fuzzy sphere is constructed by using the noncommutative version of Hopf fibration $\pi : \mathbf{S}^3 \to \mathbf{S}^2$, which is essentially based on the Holstein–Primakoff realization of $SU(2)$ algebra [16]. Based on this realization, the derivative operators on fuzzy sphere are defined. In Section 3, the $\mathbf{CP}^n$ model on fuzzy sphere is constructed. Our present construction of $\mathbf{CP}^n$ model closely follows that of Berg and Lüscher [17], in fact, the noncommutative generalization of them. It is shown in Section 4 that the Bogomolny bound is saturated by (anti-)self-dual solitons and the general solutions of BPS equation are constructed. The dimension of moduli space describing the BPS solution on fuzzy sphere is exactly the same as that of the commutative sphere [17] or the (noncommutative) plane [18]. In Section 5 we show that in the soliton backgrounds, the Atiyah–Singer index, that is the number of zero modes of Dirac operator on fuzzy sphere, is exactly given by the topological charge of the background solitons. In Section 6 we address a topological issue on the BPS solitons on fuzzy sphere and some other issues related to our work. In Appendix A, we explain the fuzzy spherical harmonics $\widehat{Y}_{lm}$, the Clebsch–Gordan decomposition of tensor products, and the Casimir operator of $SU(2)$.

2. Fuzzy sphere from Hopf fibration

The algebra of the fuzzy sphere [6] is generated by $\hat{r}_a$ satisfying the commutation relations

$$\left[\hat{r}_a, \hat{r}_b\right] = i\alpha\epsilon_{abc}\hat{r}_c \quad (a,b,c = 1,2,3) \tag{2.1}$$

as well as the following condition for $\hat{r}_a$:

$$\hat{r}_a \hat{r}_a = R^2. \tag{2.2}$$

The noncommutative coordinates of (2.1) can be represented by the generators of the $(N+1)$-dimensional irreducible representation of $SU(2)$

$$\hat{r}_a = \alpha \hat{L}_a, \tag{2.3}$$

where

$$\left[\widehat{L}_a, \widehat{L}_b\right] = i\epsilon_{abc}\widehat{L}_c. \tag{2.4}$$

Since the second Casimir of $SU(2)$ in the $(N+1)$-dimensional irreducible representation is given by $N(N+2)/4$, thus α and R are related by the following relation

$$R^2 = \alpha^2 \frac{N(N+2)}{4}. \tag{2.5}$$

In the $\alpha \to 0$ limit, $\hat{r}_a$ describe commutative sphere:

$$r_1 = R\sin\theta\cos\phi, \qquad r_2 = R\sin\theta\sin\phi, \qquad r_3 = R\cos\theta. \tag{2.6}$$

Since $\mathbf{S}^2$ is not parallelizable unlike $\mathbf{S}^3 \simeq SU(2)$, the module of derivations on $\mathbf{S}^2$ is not free [6]. If we enlarge the coordinate space from $\mathbf{S}^2$ to $\mathbf{S}^3$ by the addition of a $U(1)$ gauge degree of freedom, we can have a free module of the derivations (acting on $\mathbf{S}^3$). This is a well-known construction, called the Hopf fibration of $\mathbf{S}^2$. Indeed $\mathbf{S}^3$ can be regarded as a principal fiber bundle with base space $\mathbf{S}^2$ and a $U(1)$ structure group. Equivalently,

$$\mathbf{S}^2 \simeq SU(2)/U(1), \tag{2.7}$$

where $U(1)$ is the subgroup of $SU(2)$. A complex scalar field on $\mathbf{S}^2$ can then be identified with a smooth section of this bundle.

The Hopf fibration $\pi : \mathbf{S}^3 \to \mathbf{S}^2$ can be generalized to the noncommutative space $\mathbf{C}^2$ satisfying the relations

$$[a_\alpha, a_\beta] = \left[a_\alpha^\dagger, a_\beta^\dagger\right] = 0, \qquad \left[a_\alpha, a_\beta^\dagger\right] = \delta_{\alpha\beta} \quad (\alpha, \beta, = 1, 2) \tag{2.8}$$

as follows

$$\widehat{L}_a = \frac{1}{2}\xi^\dagger \sigma_a \xi, \qquad \xi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{2.9}$$

where σ_a are the Pauli matrices and ξ is an $SU(2)$ spinor with the normalization $\xi^\dagger\xi = N$. (Based on this Hopf fibration, topologically nontrivial field configurations on fuzzy sphere were discussed in [14,15].) It is straightforward to check for $\widehat{L}_a$'s of (2.9) to satisfy the $SU(2)$ algebra (2.4). Now the $SU(2)$ generators are given by $\mathbf{C}^2$ coordinates as

$$\widehat{L}_1 = \frac{1}{2}\left(a_1 a_2^\dagger + a_1^\dagger a_2\right), \qquad \widehat{L}_2 = \frac{i}{2}\left(a_1 a_2^\dagger - a_1^\dagger a_2\right), \qquad \widehat{L}_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \tag{2.10}$$

which is, in fact, the Schwinger realization of $SU(2)$ algebra. The associated ladder operators are defined as

$$\widehat{L}_+ = \widehat{L}_1 + i\widehat{L}_2 = a_1^\dagger a_2, \qquad \widehat{L}_- = \widehat{L}_1 - i\widehat{L}_2 = a_1 a_2^\dagger \tag{2.11}$$

and their communication relations are

$$\big[\widehat{L}_+, \widehat{L}_-\big] = 2\widehat{L}_3, \qquad \big[\widehat{L}_3, \widehat{L}_\pm\big] = \pm\widehat{L}_\pm. \tag{2.12}$$

Note that the $SU(2)$ generators in (2.9) are invariant under the transformation

$$\xi \to e^{i\psi}\xi, \qquad \xi^\dagger \to \xi^\dagger e^{-i\psi}, \tag{2.13}$$

showing that the fiber is $U(1)$.

The $(N+1)$-dimensional irreducible representation of $SU(2)$, denoted as $\mathcal{H}_N$, can be given by the following orthonormal basis

$$|n\rangle = \left|\frac{N}{2}, n - \frac{N}{2}\right\rangle = \frac{(a_1^\dagger)^n (a_2^\dagger)^{N-n}}{\sqrt{n!(N-n)!}}\,|0\rangle_{12} \quad (n = 0, 1, \ldots, N), \tag{2.14}$$

where $|j, m\rangle$ $(j \in \mathbf{Z}/2)$ is a spherical harmonics and $|0\rangle_{12}$ is the vacuum defined by $a_1|0\rangle_{12} = a_2|0\rangle_{12} = 0$. Let $\mathcal{A}_N$ be operator algebra acting on the $(N+1)$-dimensional Hilbert space $\mathcal{H}_N$, which can be identified with the algebra $\mathrm{Mat}(N+1)$ of the complex $(N+1)\times(N+1)$ matrices. Then the integration over the fuzzy sphere is given by the trace over $\mathcal{H}_N$ and defined by[1]

$$\mathrm{Tr}\,\mathcal{O} = \frac{1}{N+1}\sum_{n=0}^{N}\langle n|\mathcal{O}|n\rangle, \tag{2.15}$$

where $\mathcal{O} \in \mathcal{A}_N$.

Let us consider a scalar field Φ on the noncommutative $\mathbf{C}^2$ defined by (2.8) (or on $\mathbf{S}^3$ after the restriction $\xi^\dagger\xi = N$) of the form [14]

$$\Phi = \sum \Phi_{m_1 m_2 n_1 n_2} a_1^{\dagger m_1} a_2^{\dagger m_2} a_1^{n_1} a_2^{n_2}. \tag{2.16}$$

The above scalar field Φ can be classified according to the $U(1)$ gauge transformation (2.13) and the set of fields Φ with definite $U(1)$ charge k will be denoted as Φ_k:

$$\Phi_k \to \Phi_k e^{-ik\psi}, \tag{2.17}$$

where $k = m_1 + m_2 - n_1 - n_2 \in \mathbf{Z}$. Indeed the field Φ is a section of the $U(1)$ bundle over $\mathbf{S}^2$ with a definite value k along the fiber and the number k labels the equivalence classes (homotopy classes) of Φ in (2.16) according to the Hopf fibration (2.7). In Section 4, we will show that this number is related to the topological charge of $\mathbf{CP}^n$ solitons, $|Q| = k$.

If we define an operator denoted as $\widehat{K}_3$ for later use

$$\widehat{K}_3 = -N + \big(a_1^\dagger a_1 + a_2^\dagger a_2\big), \tag{2.18}$$

[1] In the commutative limit, Tr over matrices is mapped to the integration over functions as $\mathrm{Tr} \to \int \frac{d\Omega}{4\pi}$.

we see that Φ_k in (2.17) is an eigenfunction of this operator, namely

$$\big[\widehat{K}_3, \Phi_k\big] = k\Phi_k. \tag{2.19}$$

By the analogy from [17], $\widehat{K}_3$ may be identified with a derivation along the fiber, i.e., a Killing vector along $U(1) \subset SU(2)$. For this reason we need another two derivations (tangent to $\mathbf{S}^2 \subset SU(2)$) to form a closed $SU(2)$ algebra. There is a unique (up to sign) choice on the generators, which is essentially based on the Holstein–Primakoff realization of $SU(2)$ algebra [16] for two atomic spins, and they are given by

$$\widehat{K}_+ = a_1^\dagger \sqrt{N - a_1^\dagger a_1} + a_2^\dagger \sqrt{N - a_2^\dagger a_2}, \qquad \widehat{K}_- = K_+^\dagger, \tag{2.20}$$

where

$$\widehat{K}_\pm = \widehat{K}_1 \pm i\widehat{K}_2. \tag{2.21}$$

It is straightforward to check the $SU(2)$ algebra

$$\big[\widehat{K}_a, \widehat{K}_b\big] = i\epsilon_{abc}\widehat{K}_c, \tag{2.22}$$

or

$$\big[\widehat{K}_+, \widehat{K}_-\big] = 2\widehat{K}_3, \qquad \big[\widehat{K}_3, \widehat{K}_\pm\big] = \pm\widehat{K}_\pm. \tag{2.23}$$

Then the derivatives of an operator $\mathcal{O}$ are defined by the adjoint action of $\widehat{K}_a$:[2]

$$\widehat{\nabla}_a \mathcal{O} = i\big[\widehat{K}_a, \mathcal{O}\big]. \tag{2.24}$$

Thus the generators $\widehat{K}_a$ are the derivations acting on the Hopf bundle (2.7).[3]

As pointed out in [14] (also in [17]), for the description of topologically nontrivial field configurations (with $k \neq 0$), the field Φ in (2.16) needs to be a mapping from $\mathcal{H}_N$ to $\mathcal{H}_M$ where $M - N \neq 0$. For Φ_k in (2.17), $M - N = k$. However, note that, although $\widehat{K}_\pm : \Phi_k \to \Phi_{k\pm1}$ and $\Phi_k : \mathcal{H}_N \to \mathcal{H}_{N+k}$, the relevant quantities such as the action and topological charge densities are the operations keeping the representation space: $\mathcal{H}_N \to \mathcal{H}_N$, so the trace (2.15) of these densities is well-defined.

3. $\mathbf{CP}^n$ model on fuzzy sphere

The $\mathbf{CP}^n$ model manifold is defined by an $(n+1)$-dimensional complex vector $\Phi = (\phi_1, \phi_2, \ldots, \phi_{n+1})$ of unit length with the equivalence relation under the overall phase rotation $\Phi \sim e^{i\theta}\Phi$ [20–22]. This complex projective space of real dimensions $2n$ is equivalent of the coset space $U(n+1)/U(1) \times U(n)$.

[2] Since the coordinates of fuzzy sphere (or $\mathbf{S}^3$) are Lie algebra elements, the derivatives on the fuzzy sphere (or $SU(2)$ manifold) can be defined as usual as an endomorphism or an adjoint operation of Lie algebra like as (2.24) [19]. Then the Leibnitz rule should be obvious in this definition: $\widehat{\nabla}_a(\mathcal{O}_1\mathcal{O}_2) = (\widehat{\nabla}_a\mathcal{O}_1)\mathcal{O}_2 + \mathcal{O}_1(\widehat{\nabla}_a\mathcal{O}_2)$.

[3] The reason $\widehat{L}_a$'s are not chosen as the derivations is that, for the fibration (2.7), $\widehat{L}_a$'s are not proper derivative to expose topologically nontrivial field configurations since $\widehat{L}_a : \Phi_k \to \Phi_k$. In this sense, $\widehat{K}_a$'s are more appropriate for our problem since they allow us to directly separate the $U(1)$ symmetry (2.13) from $SU(2)$.

For the purpose of manifestly $SU(2)$ invariant action, we will not impose the condition (2.19) for a moment. Later we will describe how to naturally project the theory onto the fuzzy sphere by the restriction (2.19) which specifies the homotopy class of Φ in terms of $U(1)$ fiber.

Since the derivation of the field variable $\Phi(\hat{r}_a)$ is given by the adjoint action of $\widehat{K}_a$ as in (2.24), the natural action for the $\mathbf{CP}^n$ model turns out to be

$$S = \mathrm{Tr}\big[\big(\widehat{\nabla}_a\Phi\big)^\dagger\big(\widehat{\nabla}_a\Phi\big) + \big(\Phi^\dagger\widehat{\nabla}_a\Phi\big)\big(\Phi^\dagger\widehat{\nabla}_a\Phi\big)\big], \tag{3.1}$$

with the constraint

$$\Phi^\dagger\Phi = 1. \tag{3.2}$$

Precisely speaking, the normalization in (3.2) means that $\Phi^\dagger\Phi$ is an identity operator acting on the representation space $\mathcal{H}_N$. This theory has a global $U(n+1)$ symmetry and a local $U(1)$ symmetry

$$\Phi(\hat{r}) \to \Phi(\hat{r})g(\hat{r}), \quad g(\hat{r}) \in U(1), \tag{3.3}$$

which removes the degrees of freedom for an overall $U(1)$ phase of Φ. The $U(1)$ gauge transformation acts on the right-hand side, which leaves the constraint (3.2) invariant. This ordering of the gauge transformation is the key point which makes the whole theory work [18].

The above action with the constraint (3.2) can be rewritten as

$$S = \mathrm{Tr}\big[\big(\widehat{D}_a\Phi\big)^\dagger\big(\widehat{D}_a\Phi\big) + \lambda\big(\Phi^\dagger\Phi - 1\big)\big], \tag{3.4}$$

with

$$\widehat{D}_a\Phi = \widehat{\nabla}_a\Phi - i\Phi\widehat{A}_a, \tag{3.5}$$

where $\widehat{A}_a(\hat{r})$ is the $U(1)$ gauge field without kinetic term and $\lambda(\hat{r})$ is a Lagrange multiplier to incorporate the constraint (3.2). As there are no derivatives of $\widehat{A}_a$, one can solve the $\widehat{A}_a$ equation to get

$$\widehat{A}_a = -i\Phi^\dagger\widehat{\nabla}_a\Phi. \tag{3.6}$$

Thus $\Phi^\dagger\widehat{D}_a\Phi = 0$. Note that $\widehat{D}_3\Phi = 0$ if we require the condition (2.19).

This action is invariant under the local gauge transformation defined by (3.3) and

$$\widehat{A}_a \to g^\dagger\widehat{A}_a g - ig^\dagger\widehat{\nabla}_a g. \tag{3.7}$$

Since the field strength of the gauge field $\widehat{A}_a$ is defined as the curvature tensor of the covariant derivative $\widehat{D}$ by $[\widehat{D}_a, \widehat{D}_b]\Phi = -i\Phi\widehat{F}_{ab} - \epsilon_{abc}\widehat{D}_c\Phi$, then one can find

$$\begin{aligned}\widehat{F}_{ab} &= \widehat{\nabla}_a\widehat{A}_b - \widehat{\nabla}_b\widehat{A}_a + i\big[\widehat{A}_a, \widehat{A}_b\big] + \epsilon_{abc}\widehat{A}_c \\ &= -i\big[\big(\widehat{D}_a\Phi\big)^\dagger\widehat{D}_b\Phi - \big(\widehat{D}_b\Phi\big)^\dagger\widehat{D}_a\Phi\big].\end{aligned} \tag{3.8}$$

In order to solve the constraint (3.2), it is convenient to parameterize the field as follows

$$\Phi = W\frac{1}{\sqrt{W^\dagger W}}, \tag{3.9}$$

where W is an $(n+1)$-dimensional vector. We also introduce an $(n+1)$-dimensional projection operator

$$P \equiv 1 - \Phi\Phi^\dagger = 1 - W\frac{1}{W^\dagger W}W^\dagger, \tag{3.10}$$

whose kernel is a one-dimensional space generated by W vector. In terms of these field variables, the action (3.4) becomes

$$S = \mathrm{Tr}\left(\frac{1}{W^\dagger W}\widehat{\nabla}_a W^\dagger P \widehat{\nabla}_a W\right). \tag{3.11}$$

Then, one can check that the above action has a local scaling symmetry, $W \to W\Delta(\hat{r})$, as on the (non)commutative plane [18]. In addition, there is a still local $U(1)$ gauge symmetry $W \to We^{i\Lambda(\hat{r})}$.

From the field equation for Φ

$$\widehat{D}_a\widehat{D}_a\Phi - \Phi\lambda = 0, \tag{3.12}$$

we can deduce

$$\lambda = \Phi^\dagger\widehat{D}_a\widehat{D}_a\Phi = -\big(\widehat{D}_a\Phi\big)^\dagger\widehat{D}_a\Phi, \tag{3.13}$$

and the field equation becomes

$$\widehat{D}_a\widehat{D}_a\Phi + \Phi\big(\widehat{D}_a\Phi\big)^\dagger\widehat{D}_a\Phi = 0. \tag{3.14}$$

4. BPS solitons

As in the (non)commutative case, the $\mathbf{CP}^n$ model on the fuzzy sphere has the Bogomolny bound. Let's consider the inequality

$$\frac{1}{2}\mathrm{Tr}\big\{\big(\widehat{D}_i\Phi \pm i\epsilon_{ij}\widehat{D}_j\Phi\big)^\dagger\big(\widehat{D}_i\Phi \pm i\epsilon_{ik}\widehat{D}_k\Phi\big)\big\} + \mathrm{Tr}\big\{\big(\widehat{D}_3\Phi\big)^\dagger\big(\widehat{D}_3\Phi\big)\big\} \geqslant 0, \tag{4.1}$$

where $i, j = 1, 2$. Expanding this inequality we obtain

$$S = \mathrm{Tr}\big\{\big(\widehat{D}_i\Phi\big)^\dagger\big(\widehat{D}_i\Phi\big) + \big(\widehat{D}_3\Phi\big)^\dagger\big(\widehat{D}_3\Phi\big)\big\} \geqslant \mp i\epsilon_{ij}\,\mathrm{Tr}\big\{\big(\widehat{D}_i\Phi\big)^\dagger\big(\widehat{D}_j\Phi\big)\big\} \equiv \pm Q, \tag{4.2}$$

where the $U(1)$ gauge invariant "topological charge" is

$$Q = -i\epsilon_{ij}\,\mathrm{Tr}\big\{\big(\widehat{D}_i\Phi\big)^\dagger\big(\widehat{D}_j\Phi\big)\big\} = \mathrm{Tr}\,\widehat{F}_{12}. \tag{4.3}$$

The Bogomolny bound of the (Euclidean) action is saturated by the configuration which satisfies the (anti-)self-dual equations

$$\widehat{D}_i\Phi \pm i\epsilon_{ij}\widehat{D}_j\Phi = 0 \quad \text{or} \quad \widehat{D}_\pm\Phi = 0, \tag{4.4}$$

and

$$\widehat{D}_3\Phi = 0. \tag{4.5}$$

The gauge covariant condition (4.5) is equivalent to (2.19). Since the field Φ corresponds to a section of the bundle (2.7), (4.5) is a natural requirement. In this way we can naturally project the theory onto the fuzzy sphere by specifying the topological sector of Φ although the field Φ is defined on $\mathbf{S}^3$ rather than $\mathbf{S}^2$.

In terms of W variables in (3.9), the (anti-)self-dual equation reduces to

$$\widehat{D}_{\pm}\Phi = P\big(\widehat{\nabla}_{\pm}W\big)\big(W^{\dagger}W\big)^{-1/2} = 0, \tag{4.6}$$

and the topological charge can be rewritten as

$$Q = \frac{1}{2}\,\mathrm{Tr}\left\{\frac{1}{W^{\dagger}W}\left(\widehat{\nabla}_{+}W^{\dagger}P\widehat{\nabla}_{-}W - \widehat{\nabla}_{-}W^{\dagger}P\widehat{\nabla}_{+}W\right)\right\}. \tag{4.7}$$

Eq. (4.6) is equivalent to $\widehat{\nabla}_{\pm}W = Wc(\hat{r})$ for an arbitrary complex function $c(\hat{r})$. To find the (anti-)self-dual configurations, the scale and the gauge symmetries can be used to put $c(\hat{r}) = 0$, ending with a pure "(anti-)holomorphic equation"

$$\widehat{\nabla}_{\pm}W = i\big[\widehat{K}_{\pm}, W\big] = 0. \tag{4.8}$$

Using the definitions of $\widehat{K}_{\pm}$ operators in (2.20), one can easily find the general BPS solutions satisfying (4.5) and (4.8) as[4]

$$W_k^{+} = \sum_{l=0}^{k} c_{kl}^{+}\big(a_1^{\dagger}\big)^{l}\big(a_2^{\dagger}\big)^{k-l}\sqrt{\prod_{n=0}^{l-1}\big(N-n-a_1^{\dagger}a_1\big)\big(N-k+n-a_2^{\dagger}a_2\big)}, \tag{4.9}$$

$$W_k^{-} = \sum_{l=0}^{k} c_{kl}^{-}\sqrt{\prod_{n=0}^{l-1}\big(N-n-a_1^{\dagger}a_1\big)\big(N-k+n-a_2^{\dagger}a_2\big)}\,(a_1)^{l}(a_2)^{k-l}, \tag{4.10}$$

where $k \leqslant N$. These solutions are eigenstates of $\widehat{K}_3$ with the eigenvalue $\pm k$ and possess $2k(n+1)+2n$ real parameters $c_{kl}^{\pm}$ to specify the BPS solutions.

Let us take the commutative limit, $\alpha \to 0$ (or $N \to \infty$), where a_1 and a_2 become usual complex variables in $\mathbf{C}^2$. And introduce the stereographic projection of the (commutative) sphere to the complex plane given by

$$z = R\,\frac{r_1 + ir_2}{R - r_3} = R\,\frac{a_1^{\dagger}}{a_2^{\dagger}}, \qquad \bar{z} = R\,\frac{r_1 - ir_2}{R - r_3} = R\,\frac{a_1}{a_2}, \tag{4.11}$$

where the relations (2.3) and (2.10) are used. Then the solutions (4.9) and (4.10) can be rearranged into the standard form [21] in the commutative limit up to a scale factor which can be scaled away using the scale symmetry ($W \to W\Delta$):

$$W_k^{+} = a_{+k}^{0}\prod_{l=1}^{k}\big(z - a_{+k}^{l}\big), \qquad W_k^{-} = a_{-k}^{0}\prod_{l=1}^{k}\big(\bar{z} - a_{-k}^{l}\big). \tag{4.12}$$

[4] The following formula may be useful to find explicit solutions: $af(a^{\dagger}a) = f(a^{\dagger}a+1)a$, $a^{\dagger}f(a^{\dagger}a) = f(a^{\dagger}a-1)a^{\dagger}$ for operators a and $a^{\dagger}$ satisfying (2.8) and a nonsingular function $f(a^{\dagger}a)$.

Thus it is clear that the parameters $c_{kl}^{\pm}$ in (4.9) can be interpreted as the moduli of k BPS solitons on the fuzzy sphere and the dimension of moduli space of k solitons is exactly the same as that of the commutative sphere or the (noncommutative) plane.

The vacuum moduli space has $2n$ parameters for $\mathbf{CP}^n$ space but, in commutative and large radius limit, they have infinite inertia due to the volume factor. For fuzzy sphere, since the volume is finite and there are only finite number of states, it is interesting to study the moduli space dynamics of solitons, including these vacuum moduli.

Now let us calculate the topological charge Q defined by (4.7) for the BPS solutions (4.9). For the configurations satisfying (4.8), we first note that Eq. (4.7) can be rewritten as

$$Q_{\mathrm{s}} = \frac{1}{2}\,\mathrm{Tr}\left\{\widehat{\nabla}_+\left(\frac{1}{W^\dagger W}\,W^\dagger\widehat{\nabla}_- W\right) - \frac{1}{W^\dagger W}\,W^\dagger\big(\widehat{\nabla}_+\widehat{\nabla}_- - \widehat{\nabla}_-\widehat{\nabla}_+\big)W\right\} \tag{4.13}$$

for the soliton (4.9) satisfying $\widehat{\nabla}_+ W = 0$, and

$$Q_{\mathrm{as}} = \frac{1}{2}\,\mathrm{Tr}\left\{-\widehat{\nabla}_-\left(\frac{1}{W^\dagger W}\,W^\dagger\widehat{\nabla}_+ W\right) + \frac{1}{W^\dagger W}\,W^\dagger\big(\widehat{\nabla}_+\widehat{\nabla}_- - \widehat{\nabla}_-\widehat{\nabla}_+\big)W\right\} \tag{4.14}$$

for the antisoliton (4.10) satisfying $\widehat{\nabla}_- W = 0$. On the fuzzy sphere, the traces of total derivative in (4.13) and (4.14) are always zero because $\mathcal{H}_N$ is of finite dimension, while, on the (noncommutative) plane, the topological charge is coming from them [18]. On the other hand, the second terms in (4.13) and (4.14) on the fuzzy sphere become $\mp 2i\widehat{\nabla}_3$, which gives the topological charge, while, on the (noncommutative) plane, they vanish. Thus we have shown that the topological charge (4.13) or (4.14) is an integer number given by an eigenvalue of $\widehat{K}_3$;

$$Q_{\mathrm{s}} = k, \qquad Q_{\mathrm{as}} = -k. \tag{4.15}$$

In Section 6 we will speculate on a topological issue related to the charge Q of BPS solitons on fuzzy sphere [23].

5. Atiyah–Singer index on fuzzy sphere

In this section we will calculate the zero modes of Dirac operator on fuzzy sphere under the BPS background defined by (4.4) and (4.5). We will show that the Atiyah–Singer index, that is the number of the zero modes of Dirac operator, is exactly given by the topological charge $|Q|$. Thus this result presumably implies that the Atiyah–Singer index theorem is still valid on fuzzy sphere. (For discussions about the Atiyah–Singer index on commutative sphere in the case of gauge theory, see [24].)

We will take a spinor field $\Psi(\hat{r})$ as follows

$$\Psi(\hat{r}) = \Psi^+(\hat{r})b + \Psi^-(\hat{r})b^\dagger, \tag{5.1}$$

where $\Psi^\pm(\hat{r})$ are bosonic fields defined on the fuzzy sphere and b, $b^\dagger$ are Grassmannian operators satisfying the following anticommutation relations

$$\{b, b\} = \{b^\dagger, b^\dagger\} = 0, \qquad \{b, b^\dagger\} = 1. \tag{5.2}$$

As shown in [14] and [15], using these anticommuting operators b and $b^\dagger$, the Clifford algebra appropriate for the fuzzy sphere can be constructed:

$$\Gamma_+ = b\{b, \bullet\}, \qquad \Gamma_- = b^\dagger\{b^\dagger, \bullet\}, \qquad \Gamma_3 = b\{b^\dagger, \bullet\} - b^\dagger\{b, \bullet\}, \tag{5.3}$$

where the Gamma matrices $\Gamma_\pm$, Γ_3 act on a Grassmannian operator $\zeta = b$ or $b^\dagger$, e.g., $\Gamma_+\zeta = b\{b, \zeta\}$, etc. Then it is easy to check the Clifford algebra

$$\Gamma_\pm^2 = 0, \qquad \Gamma_3^2 = 1, \tag{5.4}$$

and

$$\Gamma_+\Gamma_- + \Gamma_-\Gamma_+ = 1, \qquad \Gamma_\pm\Gamma_3 + \Gamma_3\Gamma_\pm = 0, \qquad \Gamma_+\Gamma_- - \Gamma_-\Gamma_+ = \Gamma_3. \tag{5.5}$$

Thanks to the last relation in (5.5), we can regard Γ_3 as the chirality operator on the fuzzy sphere. So Ψ^+ and Ψ^- in (5.1) are the components with the chirality $+1$ and -1, respectively.

Now we will consider massless fermions interacting with the $\mathbf{CP}^n$ fields satisfying (4.4) and (4.5). The fermions Ψ are vectors in $\mathbf{C}^{n+1}$ like Φ constrained by [21]

$$\Phi^\dagger\Psi = \overline{\Psi}\Phi = 0, \tag{5.6}$$

where $\overline{\Psi} = \Gamma_3\Psi^\dagger = (\Psi^-)^\dagger b - (\Psi^+)^\dagger b^\dagger$. For the purpose of manifestly $SU(2)$ invariant action as in Section 3, we will first consider the Dirac action on $\mathbf{S}^3$ (constructed by the noncommutative $\mathbf{C}^2$ generated by (2.8) with the constraint $\xi^\dagger\xi = N$) and later project the theory onto fuzzy sphere. The covariant derivative about a spinor Ψ coupled to the gauge field $\widehat{A}_a$ given by (3.6) is defined as follows

$$\mathcal{D}\Psi = \Gamma^a\widehat{D}_a\Psi = \Gamma^a\big(\widehat{\nabla}_a\Psi - i\Psi\widehat{A}_a\big). \tag{5.7}$$

The relevant Dirac action for our problem turns out to be[5]

$$S_D = \mathrm{Tr}\big[\overline{\Psi}i\mathcal{D}\Psi + \bar{\lambda}\Phi^\dagger\Psi + \overline{\Psi}\Phi\lambda\big], \tag{5.8}$$

where $\bar{\lambda}$ and λ are Lagrange multipliers to incorporate the constraints (5.6). The above Dirac action has a global $U(n+1)$ symmetry and a local $U(1)$ symmetry

$$\Psi(\hat{r}) \to \Psi(\hat{r})g(\hat{r}), \tag{5.9}$$

together with the transformation (3.3).

From the equation of motion for Ψ,

$$i\mathcal{D}\Psi + \Phi\lambda = 0, \tag{5.10}$$

we can deduce

$$\lambda = -i\Phi^\dagger\mathcal{D}\Psi, \tag{5.11}$$

[5] Here we are implicitly assuming the trace over the fermionic Fock space generated by b and $b^\dagger$, $\{|\nu\rangle, \nu = 0, 1\}$.

and the resulting Dirac equation becomes

$$\left(1-\Phi\Phi^{\dagger}\right)\mathcal{D}\Psi=0 \quad \text{or} \quad P\mathcal{D}\Psi=0, \tag{5.12}$$

where P is the projection operator defined by (3.10).

The constraints in (5.6) can be solved by introducing a spinor, $\eta(\hat{r})=\eta^{+}(\hat{r})b+\eta^{-}(\hat{r})b^{\dagger}$, of the form

$$\Psi=P\eta, \tag{5.13}$$

where the spinor η is an unconstrained vector in $\mathbf{C}^{n+1}$. In terms of η variables, the action (5.8) can be rewritten as

$$S_D=\mathrm{Tr}\left\{\bar{\eta}Pi\mathcal{D}(P\eta)\right\}. \tag{5.14}$$

The equation of motion in terms of η becomes

$$P\mathcal{D}(P\eta)=0. \tag{5.15}$$

It is obvious that this action has the $U(1)$ gauge symmetry as well as an additional symmetry given by the shift of fermions,

$$\eta\to\eta+W\chi, \tag{5.16}$$

where $\chi=\chi^{+}b+\chi^{-}b^{\dagger}$ is an arbitrary spinor. This shift symmetry is a fermionic partner of the scale symmetry in the bosonic action (3.11).

The fields $\Psi^{\pm}$ can be expanded on the noncommutative $\mathbf{C}^2$ similarly to (2.16) [14]

$$\Psi^{\pm}=\sum\Psi^{\pm}_{m_1m_2n_1n_2}a_1^{\dagger m_1}a_2^{\dagger m_2}a_1^{n_1}a_2^{n_2}. \tag{5.17}$$

The chiral fields $\Psi^{\pm}$ can also be classified according to the $U(1)$ gauge transformation (2.13) and the set of the fields with definite $U(1)$ charge k will be denoted as $\Psi^{\pm}_k$. Then the chiral fields $\Psi^{\pm}_k$ transform under the $U(1)$ gauge transformation (2.13) as

$$\Psi^{\pm}_k\to\Psi^{\pm}_k e^{-ik\psi}. \tag{5.18}$$

Now we will take gauge covariant projection onto the fuzzy sphere:

$$\widehat{D}_3\Psi_k=0, \tag{5.19}$$

which is equivalent to

$$\left[\widehat{K}_3,\Psi_k\right]=k\Psi_k. \tag{5.20}$$

In the projected subspace (5.19), the action is invariant with respect to the chiral transformation [15]

$$\Psi_k(\hat{r})\to e^{i\alpha\Gamma_3}\Psi_k(\hat{r}), \qquad \overline{\Psi}_k(\hat{r})\to\overline{\Psi}_k(\hat{r})e^{i\alpha\Gamma_3} \tag{5.21}$$

which is the result of the second Clifford algebra in (5.5).

We will solve the equation of motion (5.15) for η_k obeying $\widehat{D}_3\eta_k=0$ under the soliton background (4.9) satisfying $\widehat{\nabla}_{+}W=0$. (The analysis with the antisoliton background (4.10) ($\widehat{\nabla}_{-}W=0$) will be similar to the soliton case.) Eq. (5.15) under this

background has the following component form

$$P\left(\left[\widehat{K}_+,\eta_k^+\right]-\eta_k^+\sqrt{W^\dagger W}\left[\widehat{K}_+,\frac{1}{\sqrt{W^\dagger W}}\right]\right)=0, \tag{5.22}$$

$$P\left(\left[\widehat{K}_-,\eta_k^-\right]-\left[\widehat{K}_-,W\right]\frac{1}{W^\dagger W}W^\dagger\eta_k^-+\eta_k^-\left[\widehat{K}_-,\frac{1}{\sqrt{W^\dagger W}}\right]\sqrt{W^\dagger W}\right)=0, \tag{5.23}$$

where $PW=0$ and $[\widehat{K}_-,W^\dagger]=0$ are used.

It is easy to see that the solution of the positive chirality η_k^+ up to the shift symmetry (5.16) is given by

$$\eta_k^+=\zeta_k^+\left(a_1^\dagger,a_2^\dagger\right)\frac{1}{\sqrt{W^\dagger W}}, \tag{5.24}$$

where ζ_k^+ is a positive chirality spinor satisfying $[\widehat{K}_+,\zeta_k^+]=0$ and $[\widehat{K}_3,\zeta_k^+]=k\zeta_k^+$:

$$\zeta_k^+=\zeta_{kl}^+\left(a_1^\dagger\right)^l\left(a_2^\dagger\right)^{k-l}\sqrt{\prod_{n=0}^{l-1}\left(N-n-a_1^\dagger a_1\right)\left(N-k+n-a_2^\dagger a_2\right)},\quad l=0,\dots,k. \tag{5.25}$$

And it is also easy to check that the solution for η_k^- with the negative chirality is given by

$$\eta_k^-=W\zeta_k^-, \tag{5.26}$$

where ζ_k^- is an arbitrary negative chirality spinor satisfying $[\widehat{K}_3,\zeta_k^-]=k\zeta_k^-$. However, the solution (5.26) can be gauged away using the shift symmetry (5.16) or $\Psi_k^-=P\eta_k^-=0$, so there are no zero modes (up to the gauge and the shift symmetry) for negative chirality spinors under the soliton background.

Thanks to the finite volume of fuzzy sphere, the normalizability of zero modes (5.25) is automatically guaranteed. Note that $k+1$ zero modes in (5.25) are not independent each other. The number of independent zero modes is so k, not $k+1$. The reason is following. If we take a linear combination of these zero modes for η_k^+ to be proportional to W (there is only one such combination since we should take $\zeta_{kl}^+=c_{kl}^+\chi_k^+$ for all l), it is a trivial solution since $\Psi_k^+=P\eta_k^+=PW\chi_k^+/\sqrt{W^\dagger W}=0$. We have shown in Section 4 that this integer number is exactly the topological charge of background solitons. Thus we arrived at the Atiyah–Singer index theorem on fuzzy sphere.

For a spin complex, the Atiyah–Singer index theorem [25] states that

$$\operatorname{Index}\widehat{D}=\dim\ker\widehat{D}-\dim\ker\widehat{D}^\dagger=n_+-n_-, \tag{5.27}$$

where $\widehat{D}$ is the Dirac operator for the spin complex and $n_\pm$ is the number of normalizable zero modes of the Dirac operator of chirality ±1. For a monopole bundle $P(\mathbf{S}^2,U(1))$ which corresponds to our case (2.7), the expression (5.27) reads

$$n_+-n_-=\frac{1}{2\pi}\int_{\mathbf{S}^2}F\in\mathbf{Z}, \tag{5.28}$$

where $F=dA$ is the field strength of a (monopole) gauge field and the integer quantization is coming from the homotopy $\pi_1(U(1))=\mathbf{Z}$ [25]. In this paper, we proved

the noncommutative version of (5.28):

$$n_+ = \operatorname{Tr} \widehat{F}_{12} = Q, \tag{5.29}$$

where $n_- = 0$ for the soliton background and the $U(1)$ field strength $\widehat{F}_{12}$ is given by (3.8).

6. Discussion

In this paper we showed that the $\mathbf{CP}^n$ model on fuzzy sphere enjoys all attractive properties in commutative space. The BPS equations support (anti-)self-dual soliton solutions and the dimension of moduli space of BPS solitons is exactly the same as the commutative case. Moreover, the number of normalizable zero modes in the presence of the soliton backgrounds is exactly given by the topological charge of the solitons, thus the Atiyah–Singer index theorem remains valid even for fuzzy sphere. This seems to be in contrast to the recent results in [26] claiming that the $U(1)$ monopole charge is not integer for the fuzzy sphere at finite cut-off N. A further investigation on the difference between their case and ours should be interesting.

On fuzzy sphere, only finite number of states can be defined [6]. For example, for the fuzzy sphere with a cut off spin N, only $N+1$ states are distinguishable. So it seems to be reasonable that one cannot put too many solitons on fuzzy sphere. Moreover the generators $\widehat{K}_\pm$ in the Holstein–Primakoff realization contain the square-root factors such as $\sqrt{N - a_1^\dagger a_1}$ and $\sqrt{N - a_2^\dagger a_2}$. Thus in order to preserve the theory to be unitary, we should have an upper bound on the occupation numbers, i.e., $a_1^\dagger a_1, a_2^\dagger a_2 \leqslant N$. Then this also put an upper bound on the topological charge $|Q| = k$ since it is related to the $\widehat{K}_3$ eigenvalue of a BPS solution Φ_k. As this speculation implies, more careful study is certainly required to understand the topological nature of fuzzy sphere. The topological properties of fuzzy sphere have been studied in [23] based on the boson realization of $SU(2)$ algebra, Schwinger vs. Holstein–Primakoff.

The outstanding properties of (supersymmetric) $\mathbf{CP}^n$ model with quarks are asymptotic freedom, confinement of quarks, and spontaneous chiral symmetry breaking, which are very similar to QCD_4 [21,22]. Since the $\mathbf{CP}^n$ model can be $1/n$ expanded, all these properties can be explored based on the $1/n$ expansion. A natural way to couple quarks to $\mathbf{CP}^n$ fields is to introduce the supersymmetric $\mathbf{CP}^n$ model. The $\mathbf{CP}^n$ model on noncommutative spaces presented here and in [18] can be generalized to the supersymmetric model. It is interesting to study low-energy dynamics of quarks in the context of supersymmetric $\mathbf{CP}^n$ model on noncommutative spaces (plane and sphere) since it is more similar to QCD_4 due to the non-Abelian nature from noncommutative space. As indicated in Section 5, axial anomaly and related $U(1)_A$ problem can also be studied along this line.

The (noncommutative) $\mathbf{CP}^n$ model can be understood as a formal limit of (noncommutative) Maxwell–Higgs theory,

$$S = \operatorname{Tr}\left[\frac{1}{4g^2}\,\widehat{F}_{ab}\widehat{F}^{ab} + \left(\widehat{D}_a\Phi\right)^\dagger\left(\widehat{D}_a\Phi\right) + \frac{1}{2}\lambda\left(\Phi^\dagger\Phi - 1\right)^2\right],$$

namely, $\lambda \to \infty$ and $g^2 \to \infty$. Thus this theory defined on fuzzy sphere may also enjoy the same properties as the $\mathbf{CP}^n$ model even though the dynamics of gauge fields is considered. This model, more generally, Maxwell–Chern–Simons–Higgs theory on fuzzy sphere, can be of interest itself since it is related to the world-volume theory of the spherical D2-brane formed by the bound state of N D0-branes [9].

It would be desirable to extend the analysis in this paper to four-dimensional case, especially instantons on fuzzy $\mathbf{S}^4$. On the noncommutative $\mathbf{R}^4$, it was shown that the noncommutative instantons are well defined where small instanton singularities are resolved [27] and the topological charge of instantons is always an integer [28]. On fuzzy $\mathbf{S}^4$, the topological properties of instanton solutions may appear with more elegant structures.

Acknowledgements

We are grateful to Pei-Ming Ho and Miao Li for helpful discussions. F.L.L. likes to thank Bin Chen for discussions. H.S.Y. also thanks Bum-Hoon Lee and Kimyeong Lee for discussions. Two of us are supported by NSC (C.T.C.: NCS89-2112-M-009-006 and H.S.Y.: NCS89-2811-M-002-0095). C.M.C. is supported by the CosPA project of the Ministry of Education, Taiwan. F.L.L. is supported by BK-21 Initiative in Physics (SNU-Project 2). We also acknowledge NCTS as well as CTP at Taida for partial support.

Appendix A. Fuzzy spherical harmonics

First we will briefly review quantum mechanics on the addition of angular momentum in order to fix the notations and to illustrate how to generalize it to fuzzy spherical harmonics. We find the most useful reference on this is [29].

The space $\mathrm{Fun}(\mathbf{S}^2)$ of functions on $\mathbf{S}^2$ is spanned by spherical harmonics $Y_m^J \in \mathrm{Fun}(\mathbf{S}^2)$, where J runs through all integer spins. A product of any two spherical harmonics is again a function on $\mathbf{S}^2$ and hence it can be written as a linear combination of spherical harmonics,

$$Y_l^I Y_m^J = \sum_{K,n} \sqrt{\frac{(2I+1)(2J+1)}{4\pi(2K+1)}}\, c_{IJK} C_{lmn}^{IJK} Y_n^K \tag{A.1}$$

with C_{lmn}^{IJK} denoting the Clebsch–Gordan coefficients of $su(2)$. The explicit form of structure constants $c_{IJK} = C_{000}^{IJK}$ is given by [29]

$$c_{IJK} = \begin{cases} 0, & \text{if } I+J+K = 2g+1, \\ \frac{(-1)^{g-K}\sqrt{2K+1}\,g!}{(g-I)!(g-J)!(g-K)!}\left[\frac{(2g-2I)!(2g-2J)!(2g-2K)!}{(2g+1)!}\right]^{1/2}, & \text{if } I+J+K = 2g, \end{cases} \tag{A.2}$$

where g is a positive integer.

The spherical harmonics Y_m^J form multiplets with respect to the $su(2)$ action on $\mathrm{Fun}(\mathbf{S}^2)$. More generally, the direct product of two irreducible tensors $\mathcal{M}_l^I$ and $\mathcal{N}_m^J$ may

be decomposed into irreducible tensors. The coefficients of this decomposition are just the Clebsch–Gordan coefficients:

$$\mathcal{M}_l^I \mathcal{N}_m^J = \sum_{K,n} C_{lmn}^{IJK} \{\mathcal{M} \otimes \mathcal{N}\}_n^K. \tag{A.3}$$

The inverse relation is

$$\{\mathcal{M} \otimes \mathcal{N}\}_n^K = \sum_{l,m} C_{lmn}^{IJK} \mathcal{M}_l^I \mathcal{N}_m^J. \tag{A.4}$$

The fuzzy sphere is constructed replacing the algebra of function on $\mathbf{S}^2$, Fun($\mathbf{S}^2$), by the noncommutative algebra taken in an irreducible representation of $SU(2)$. This is full matrix algebra Mat$(N+1)$ which is generated by the fuzzy spherical harmonics $\widehat{Y}_m^J$ with $-J \leqslant m \leqslant J$, $J = 0, 1, \ldots, N$, a complete basis of the space Mat$(N+1)$. The explicit form of $\widehat{Y}_m^J$ in $(N+1)$-dimensional representation is given by [29]

$$\big[\widehat{Y}_m^J\big]_{s's} = \sqrt{\frac{2J+1}{N+1}}\, C_{sms'}^{NJN}, \tag{A.5}$$

where $s, s' = -N/2, \ldots, 0, \ldots, N/2$. The operators $\widehat{Y}_m^J$ transform under $su(2)$ according to the representation D^J, so they are irreducible tensors of rank J. Thus an arbitrary matrix $\widehat{A} \in \text{Mat}(N+1)$ may be written in the form

$$\widehat{A} = \sum_{J=0}^{N} \sum_{m=-J}^{J} A_{Jm} \widehat{Y}_m^J, \tag{A.6}$$

where the expansion coefficients A_{Jm} are given by

$$A_{Jm} = \text{Tr}\big(\widehat{Y}_m^{\dagger J} \widehat{A}\big). \tag{A.7}$$

The derivative of $\widehat{A} \in \text{Mat}(N+1)$ is defined by the adjoint action of $\widehat{S}_a \in su(2)$

$$\widehat{\nabla}_a \widehat{A} = i\big[\widehat{S}_a, \widehat{A}\big] = i \sum_{J=0}^{N} \sum_{m=-J}^{J} A_{Jm} \big[\widehat{S}_a, \widehat{Y}_m^J\big]. \tag{A.8}$$

The commutators in (A.8) can be calculated by the following commutation relations [29]

$$\big[\widehat{S}_\mu, \widehat{Y}_m^J\big] = \sqrt{J(J+1)}\, C_{m\mu(m+\mu)}^{J1J} \widehat{Y}_{m+\mu}^J \quad \big(\mu = \pm, 0,\ \widehat{S}_0 \equiv \widehat{S}_3\big). \tag{A.9}$$

The Laplacian on the fuzzy sphere is given by

$$\widehat{\nabla}_a \widehat{\nabla}_a \widehat{A} = -\sum_{J=0}^{N} \sum_{m=-J}^{J} A_{Jm} \big[\widehat{S}_a, \big[\widehat{S}_a, \widehat{Y}_m^J\big]\big] = -\sum_{J=0}^{N} \sum_{m=-J}^{J} J(J+1) A_{Jm} \widehat{Y}_m^J. \tag{A.10}$$

The product of any two such matrices can be expressed as a linear combination of matrices $\widehat{Y}_n^K$ [29],

$$\widehat{Y}_l^I \widehat{Y}_m^J = \sum_{K=0}^{N} (-)^{N+K} \sqrt{(2I+1)(2J+1)} \begin{Bmatrix} I & J & K \\ \frac{N}{2} & \frac{N}{2} & \frac{N}{2} \end{Bmatrix} C_{lmn}^{IJK} \widehat{Y}_n^K, \tag{A.11}$$

where $\left\{\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\right\}$ denotes the recoupling coefficients ($6J$-symbols) of $su(2)$. The inverse relation is

$$\widehat{Y}_n^K = \sum_{IlJm} (-)^{N+I+J} \sqrt{(2I+1)(2J+1)} \begin{Bmatrix} I & J & K \\ \frac{N}{2} & \frac{N}{2} & \frac{N}{2} \end{Bmatrix} C_{lmn}^{IJK} \widehat{Y}_l^I \widehat{Y}_m^J. \tag{A.12}$$

More generally, the product module $V^I \otimes V^J$ can be expanded into the irreducible module V^K:

$$V^I \otimes V^J = \bigoplus_{K=0}^{N} V^K. \tag{A.13}$$

Note that the $SU(2)$ operators $\widehat{K}$ in (2.18) and (2.20) are a sum of two spin operators $\widehat{S}_1$ and $\widehat{S}_2$:

$$\widehat{K} = \widehat{S}_1 + \widehat{S}_2, \tag{A.14}$$

where

$$\left[\widehat{S}_{1a}, \widehat{S}_{1b}\right] = i\epsilon_{abc}\widehat{S}_{1c}, \qquad \left[\widehat{S}_{2a}, \widehat{S}_{2b}\right] = i\epsilon_{abc}\widehat{S}_{2c}, \qquad \left[\widehat{S}_{1a}, \widehat{S}_{2b}\right] = 0. \tag{A.15}$$

The $(N+1)$-dimensional unitary representations of $\widehat{S}_1$ and $\widehat{S}_2$ in the Holstein–Primakoff realization [16], denoted as $\mathcal{H}_N^1$ and $\mathcal{H}_N^2$, respectively, can be given by the following orthonormal basis

$$\begin{aligned} \mathcal{H}_N^1 &= \left\{ |n\rangle_1 = \left|\frac{N}{2}, n - \frac{N}{2}\right\rangle_1 = \left(a_1^\dagger\right)^n |0\rangle_1,\ n = 0, 1, \ldots, N \right\}, \\ \mathcal{H}_N^2 &= \left\{ |m\rangle_2 = \left|\frac{N}{2}, m - \frac{N}{2}\right\rangle_2 = \left(a_2^\dagger\right)^m |0\rangle_2,\ m = 0, 1, \ldots, N \right\}, \end{aligned} \tag{A.16}$$

where $|l, m\rangle_{1,2}$ is a spherical harmonics for each spin operator and $|0\rangle_{1,2}$ is the vacuum defined by $a_1|0\rangle_1 = a_2|0\rangle_2 = 0$. Then the basis (2.14) is a tensor product of $\mathcal{H}_N^1$ and $\mathcal{H}_N^2$ and can be expanded in the basis of total spin operator, that is,

$$\mathcal{H}_N^1 \otimes \mathcal{H}_N^2 = \bigoplus_{J=0}^{N} \mathcal{H}^{(J)} \tag{A.17}$$

and

$$\bigoplus_J \mathcal{H}^{(J)} = \left\{ |J, 0\rangle,\ J = 0, 1, \ldots, N \right\}, \tag{A.18}$$

where the spherical harmonics $|J, 0\rangle$ is an irreducible basis of the total spin operator $\widehat{K}$:

$$\widehat{K}^2 |J, 0\rangle = J(J+1)|J, 0\rangle, \qquad \widehat{K}_3 |J, 0\rangle = 0. \tag{A.19}$$

The second condition in (A.19) is coming from the usual rule of the addition of angular momentum, $m = m_1 + m_2$, which is zero for the basis (2.14), where $\widehat{S}_{13}|N/2, m_1\rangle_1 = m_1|N/2, m_1\rangle_1$, $\widehat{S}_{23}|N/2, m_2\rangle_2 = m_2|N/2, m_2\rangle_2$, and $\widehat{K}_3|J, m\rangle = m|J, m\rangle$. Thus the states in (A.18) can serve as $(N+1)$-dimensional basis of $\widehat{K}$.

Let $\widehat{Y}^I_{1l}$, $\widehat{Y}^J_{2m}$, and $\widehat{Y}^K_n$ be the fuzzy spherical harmonics for $\widehat{S}_1$, $\widehat{S}_2$, and $\widehat{K}$, respectively. They are complete and irreducible basis of the space $\mathrm{Mat}(N+1)$ whose matrix elements can be represented in the corresponding basis $\mathcal{H}^1_N$, $\mathcal{H}^2_N$ and $\mathcal{H}^{(J)}$ and are given by (A.5). Furthermore they satisfy the Clebsch–Gordan decomposition (A.11) and (A.12). Thus, using these relations, the Casimir operator of $\widehat{K}$ as in (A.10) can be calculated based on these basis:

$$\begin{aligned}
&\big[\widehat{K}_a,\big[\widehat{K}_a,\widehat{Y}^K_n\big]\big]\\
&\quad= K(K+1)\widehat{Y}^K_n\\
&\quad= \sum_{IlJm}(-)^{N+I+J}\sqrt{(2I+1)(2J+1)}\begin{Bmatrix} I & J & K\\ \frac{N}{2} & \frac{N}{2} & \frac{N}{2}\end{Bmatrix} C^{IJK}_{lmn}\\
&\qquad\times\Big(\big(I(I+1)+J(J+1)\big)\widehat{Y}^I_{1l}\widehat{Y}^J_{2m}+2\big[\widehat{S}_{1a},\widehat{Y}^I_{1l}\big]\big[\widehat{S}_{2a},\widehat{Y}^J_{2m}\big]\Big),
\end{aligned}\tag{A.20}$$

where $[\widehat{S}_1,\widehat{Y}^J_{2m}]=[\widehat{S}_2,\widehat{Y}^I_{1l}]=0$ are used. The commutators in (A.20) can be calculated by using (A.9).

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